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On a generalized Hankel type convolution of generalized functions

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Abstract. The classical generalized Hankel type convolution are defined and extended to a class of generalized functions. Algebraic properties of the convolution are explained and the existence and significance of an identity element are discussed.

Keywords. Generalized Hankel type transformation; Parserval relation; generalized functions (distributions); convolution.

1. Introduction

The fact that there is no simple expression for the product $J_\mu[x^\nu]J_\mu[y^\nu]$ in the sense that there is a simple expression $\mathrm{e}^{i(\nu x+y)}$ for the product $\mathrm{e}^{i\nu x}\mathrm{e}^{iy}$ means that there is no simple Faltung or convolution theorem for the Hankel transform corresponding to well known transforms like Laplace, Fourier transform and so on.

Hankel-convolution operation has been defined in the classical sense by [4] and [2]. We consider here the generalized Hankel type convolution and an extension of that definition to a class of generalized functions analogous to that introduced by [11, 5] and [6]. This extension has useful applications, when dealing with continuous linear systems which can be characterized by a Hankel convolutional representation; such systems, which we may call 'generalized Hankel translation invariant continuous linear systems', may thereafter be considered when developing sampling expansions for inverse generalized Hankel type transforms of distributions of compact support on the positive half line of which the work of [8] is a particular case.

2. Notation and preliminary results

We use the following definition for the classical generalized Hankel type transform of order $\mu \geq -1/2$.

$$(h_{\mu,\nu}f)(\tau) = F(\tau) = \nu\tau^{-1}\int_0^\infty (x\tau)^\nu J_\mu[(x\tau)^\nu]f(x)\mathrm{d}x, \tag{1}$$

$$f(x) = h_{\mu,\nu}^{-1}[F](x) = \nu x^{-1}\int_0^\infty (x\tau)^\nu J_\mu[(x\tau)^\nu]F(\tau)\mathrm{d}\tau. \tag{2}$$

The transform pair (1) and (2) has been extended to certain spaces of generalized functions in [6] by kernel method and in [5] by mixed Parseval equation (a new adjoint method).

We begin with a brief review of the essential results obtained by [5] for the generalized Hankel type transform of generalized functions.

Lemma 2.1. *If $f(x)$ is of bounded variation and $x^{\nu/2} f(x) \in L^1(0, \infty)$ then the direct transform is well defined by* (1), *and the inversion formula* (2) *holds almost everywhere in a neighbourhood of every point $y = x > 0$.*

Lemma 2.2. *For $f(x)$ and $G(x)$ satisfying the conditions of Lemma* 2.1 *we have the Parseval relation*

$$\int_0^\infty x f(x) g(x) \mathrm{d}x = \int_0^\infty \tau F(\tau) G(\tau) \mathrm{d}\tau. \tag{3}$$

Finally we shall need results involving the linear differential operator $N_{\mu,\nu}$, $\mu \geq -1/2$, defined by

$$N_{\mu,\nu}[f(x)] = x^{\nu\mu} D x^{-\nu\mu-\nu+1} f(x) \tag{4}$$

and the Bessel type differential operator of order μ, Δ, defined by

$$\Delta[f(x)] = x^{\nu-1} \Delta_1 x^{-\nu+1} = x^{-\nu-\nu\mu} D x^{2\nu\mu+1} D x^{-\nu\mu-\nu+1}, \tag{5}$$

where $\Delta_1 = x^{-\nu\mu-2\nu+1} D x^{2\nu\mu+1} D x^{-\nu\mu} = \Delta_{1,x}$ and D stands for the usual differential operator.

PROPOSITION 2.3

If $x f(x) \to 0$ as $x \to \infty$ where $f(x)$ is sufficiently smooth $h_{\mu,\nu}$-transformable function, then integration by parts shows that

$$h_{\mu+1,\nu} N_{\mu,\nu}[f](\tau) = -\nu\tau^\nu h_{\mu,\nu}[f](\tau) \tag{6}$$

or, setting $g = h_{\mu,\nu}[f]$ and changing τ into x,

$$\nu h_{\mu+1,\nu}[-x^\nu g(x)](\tau) = N_{\mu,\nu} h_{\mu,\nu}[g](\tau). \tag{7}$$

PROPOSITION 2.4

In general, for sufficiently well behaved $\phi(x)$ and non-negative i, j we can obtain from (6) *and* (7)

$$h_{\mu+i+j,\nu} N_{\mu+i+j,\nu} \ldots N_{\mu+i,\nu}[(-\nu x^\nu)^i](\tau) = (-\nu\tau^\nu)^j N_{\mu+i,\nu} \ldots N_{\mu,\nu}[\Phi(\tau)] \tag{8}$$

or taking the defining formula (4) *into consideration*,

$$h_{\mu+i+j,\nu}[x^{\nu(\mu+i+j)}(x^{1-2\nu} D)^j \phi(x)](\tau) = (-\nu\tau^\nu)^{i+j} (\tau^{1-2\nu} D_\tau)^i \Phi(\tau)]. \tag{9}$$

PROPOSITION 2.5

For any sufficiently smooth function $f(x)$ *on* $(0, \infty)$ *it can be shown that*

$$h_{\mu,\nu}[\Delta f(x)](\tau) = -\nu^2\tau^{2\nu}h_{\mu,\nu}[f](\tau) \tag{10}$$

provided that f *is* $h_{\mu,\nu}$*-transformable and that* $xf(x)$ *and* $xN_{\mu,\nu}f(x)$ *both tend to zero as* $x \to \infty$.

3. Spaces of fundamental and generalized functions

3.1 *Testing function spaces*

A complex valued function ϕ, defined and infinitely differentiable on $(0, \infty)$, is said to belong to the space $H_{\mu,\nu}(I)$ if and only if the numbers $\gamma_{m,k}^{\mu,\nu}(\phi)$ defined by

$$\gamma_{m,k}^{\mu,\nu}(\phi) = \sup_{0<x<\infty}\left|x^m(x^{1-2\nu}D)^k x^{-\nu\mu-\nu+1}\phi\right| \tag{11}$$

are finite for every pair m, k of non-negative integers where ν is real number and $\mu \geq -1/2$. $H_{\mu,\nu}(I)$ is a testing function space with the topology generated by the multinorm $\{\gamma_{m,k}^{\mu,\nu}(\phi)\}_{m,k=0}^{\infty}$ and we have

$$D(0,\infty) \subset H_{\mu,\nu}(0,\infty) \subset E(0,\infty), \tag{12}$$

where $D(0, \infty)$ and $E(0, \infty)$ denote respectively the restrictions of $D(R)$ and $E(R)$ to the positive real axis. Using (9) and following the same lines of [5], it can be readily shown that the $h_{\mu,\nu}$ transformation is a topological isomorphism of $H_{\mu,\nu}(0, \infty)$ onto itself.

3.2 *The space* $M(0, \infty)$ *of multipliers*

Denote by $M(0, \infty)$ the linear space of all infinitely smooth functions $\theta(x)$, $0 < x < \infty$ such that for each non-negative integer l there exists non-negative integer $l = l(i)$ for which

$$(1 + x^l)^{-1}(x^{1-2\nu}D)^i\theta(x) \tag{13}$$

is bounded on $(0, \infty)$. By using the generalized Leibnitz formula it can be shown that the map $\theta \to \theta\phi$ is an isomorphism of $H_{\mu,\nu}(0, \infty)$ for each $\theta \in M(0, \infty)$; $M(0, \infty)$ is the space of multipliers on $H_{\mu,\nu}(0, \infty)$.

3.3 *Duals of testing function spaces*

We denote $H_{\mu,\nu}^*(0, \infty)$ the space of all complex valued functions ψ, defined and infinitely smooth on $(0, \infty)$ which are of the form

$$\psi(x) = x\phi(x). \tag{14}$$

$H_{\mu,\nu}^*(0, \infty)$ is again a (complete) testing function space, with the topology generated by the sequence of multinorms

$$\gamma_{m,k}^{*,\mu,\nu}(\psi) = \gamma_{m,k}^{\mu,\nu}(x^{-1}\psi). \tag{15}$$

As usual we denote the dual of $H_{\mu,\nu}^*(0, \infty)$ by $H_{\mu,\nu}^{*\prime}(0, \infty)$.

For any $\psi(x) = x\phi(x) \in H^*_{\mu,\nu}(0,\infty)$, and any non-negative integer r, set

$$\varsigma^*_r(\psi) = \max_{0\le m,k\le r} \gamma^{*,\mu,\nu}_{m,k}(\psi) = \max_{0\le m,k\le r} \gamma^{\mu,\nu}_{m,k}(\phi) = \varsigma_r(\psi).$$

Then, for each $f \in H^*_{\mu,\nu}(0,\infty)$ there will exist constants c and r such that

$$\phi \in H_{\mu,\nu}(0,\infty) \Rightarrow |\langle f(x), x\phi(x)\rangle| \le C\varsigma_r(\phi). \tag{16}$$

In particular, let $f(x)$ be any locally integrable function on $(0,\infty)$ which is such that $xf(x) \in L^1(0,\infty)$ and $f(x)$ does not grow more rapidly than a polynomial when $x \to \infty$. Then $f(x)$ generates a regular generalized function in $H^*_{\mu,\nu}(0,\infty)$ by the formula

$$\langle f(x), x\phi(x)\rangle = \int_0^\infty xf(x)\phi(x)\mathrm{d}x. \tag{17}$$

Any generalized function in $H^*_{\mu,\nu}(0,\infty)$ not generated by the formula of the type (17) will be described as singular.

In general, the derivative of a generalized function in $H^*_{\mu,\nu}(0,\infty)$ (defined in the usual sense of Schwartz), is not in $H^*_{\mu,\nu}(0,\infty)$. However, in certain cases the result of applying a differential operator to a generalized function in $H^*_{\mu,\nu}(0,\infty)$ does yield a generalized function in $H^*_{\mu,\nu}(0,\infty)$. In particular, using for differential operators in a generalized sense the same notation as the one used for the corresponding operators which applied in a classical sense, we have the following results:

(i) $f \in E'(0,\infty) \subset H^{*\prime}_{\mu,\nu}(0,\infty) \Rightarrow Df \in H^{*\prime}_{\mu,\nu}(0,\infty)$;

(ii) $f \in H^{*\prime}_{\mu,\nu}(0,\infty) \Rightarrow (x^{1-2\nu}D)^k f \in H^{*\prime}_{\mu,\nu}(0,\infty)$;

(iii) $f \in H^{*\prime}_{\mu,\nu}(0,\infty) \Rightarrow \Delta^k f \in H^{*\prime}_{\mu,\nu}(0,\infty)$; for any non-negative integer k.

3.4 *Distributional generalized $h_{\mu,\nu}$-transform*

We can now define the generalized $h_{\mu,\nu}$-transform of any $f \in H^{*\prime}_{\mu,\nu}(0,\infty)$ by the analogue of the Parseval relation:

$$\langle f(x), x\phi(x)\rangle = \left\langle h_{\mu,\nu}[f](\tau), \tau\Phi(\tau)\right\rangle \tag{18}$$

and clearly we have that $f \in H^{*\prime}_{\mu,\nu}(0,\infty) \Rightarrow h_{\mu,\nu}[f] \in H^{*\prime}_{\mu,\nu}(0,\infty)$. Moreover, we can establish that

$$h_{\mu,\nu}[\Delta^k f(x)](\tau) = (-\nu^2\tau^{2\nu})^k h_{\mu,\nu}[f](\tau) \tag{19}$$

for any non-negative integer k.

The generalized $h_{\mu,\nu}$-transform of any distribution $\sigma \in E'(0,\infty)$, in the sense of (18), is a regular generalized function in $H^{*\prime}_{\mu,\nu}(0,\infty)$ generated by a smooth function $f(x)$ defined on $(0,\infty)$ by

$$f(x) = \left\langle \sigma(\tau), \tau^\nu J_\mu[(x\tau)^\nu]\right\rangle = \left\langle \sigma(\tau), \tau^\nu \Lambda(\tau) J_\mu[(x\tau)^\nu]\right\rangle, \tag{20}$$

where $\Lambda \in D(0,\infty)$ is such that $\Lambda(\tau) = 1$ on the support of σ. The function extend into the finite complex-plane as an entire function of exponential type which grows no faster than a polynomial on the positive real axis; it is easy to show that $f(x) \in M(0,\infty)$.

4. Classical generalized Hankel type convolution

DEFINITION 4.1

Let us define $L^p_{\mu,\nu}[0,\infty)$, $1 \le p < \infty$, the space of Lebesgue measurable functions on $(0,\infty)$ such that

$$\left\| f \right\|_{\mu,\nu,p} = \left[\int_0^\infty x^{\mu+2\nu-1} \left| f(x) \right|^p \right]^{1/p} < \infty.$$

We consider the kernel $D_{\mu,\nu}(x,\ y,\ z)$, $0 < x,\ y,\ z < \infty$ defined by

$$D_{\mu,\nu}(x,\ y,\ z) = \int_0^\infty \nu^2 \tau^{2\nu} J_\mu[(x\tau)^\nu] J_\mu[(y\tau)^\nu] J_\mu[(z\tau)^\nu] \mathrm{d}\tau. \tag{21}$$

4.2 *Properties of the kernel* $D_{\mu,\nu}(x,\ y,\ z)$

Following Watson [10], Hirschmann [4] and Cholewinski [2] we can establish the following properties for (21):
(i) For $0 < x,\ y < \infty$ and $0 \le \tau < \infty$, we have

$$\int_0^\infty z^{2\nu-1} J_\mu[(z\tau)^\nu] D_{\mu,\nu}(x,\ y,\ z) \mathrm{d}z = J_\mu[(x\tau)^\nu] J_\mu[(y\tau)^\nu]. \tag{22}$$

Proof.

$$\begin{aligned} D_{\mu,\nu}(x,\ y,\ z) &= \int_0^\infty \nu^2 \tau^{2\nu} J_\mu[(x\tau)^\nu] J_\mu[(y\tau)^\nu] J_\mu[(z\tau)^\nu] \mathrm{d}z \\ &= \nu z^{1-\nu} \left[\nu z^{-1} \int_0^\infty (z\tau)^\nu [\tau^{-1+\nu} J_\mu[(x\tau)^\nu] J_\mu[(y\tau)^\nu]] J_\mu[(z\tau)^\nu \right] \mathrm{d}z \\ &= \nu z^{-1-\nu} h^{-1}_{\mu,\nu} \left[\tau^{-1+\nu} J_\mu[(x\tau)^\nu] J_\mu[(y\tau)^\nu] \right]. \end{aligned}$$

Therefore,

$$h^{-1}_{\mu,\nu}[(\nu)^{-1} z^{-1+\nu} D_{\mu,\nu}(x,\ y,\ z)] = \tau^{-1+\nu} J_\mu[(x\tau)^\nu] J_\mu[(y\tau)^\nu],$$

i.e.

$$\nu\tau^{-1} \int_0^\infty (z\tau)^\nu J_\mu[(z\tau)^\nu] (\nu)^{-1} z^{-1+\nu} D_{\mu,\nu}(x,\ y,\ z) \mathrm{d}z = \tau^{-1+\nu} J_\mu[(x\tau)^\nu] J_\mu[(y\tau)^\nu]$$

and hence the result. In particular, taking $\tau = 0$, gives

(ii) $$\int_0^\infty z^{2\nu-1} D_{\mu,\nu}(x,\ y,\ z) \mathrm{d}z = 1, \tag{23}$$

that is, for which $x,\ y>0,\ D_{\mu,\,\nu}(x,\ y,\ z)$ belongs to $L^1_{0,\,\nu}(0,\infty)$.
(iii) $0<x,\ y,\ z<\infty$, $D_{\mu,\,\nu}(x,\ y,\ z)\geq 0$, and
(iv) $D_{\mu,\,\nu}(x,\ y,\ z)=D_{\mu,\,\nu}(z,\ x,\ y)=D_{\mu,\,\nu}(y,\ z,\ x)=\dots$.

DEFINITION 4.3

We define the classical $h_{\mu,\,\nu}$-convolution, for any two function $f(x)$ and $g(x)$, $0<x<\infty$ as

$$f*g(x)=\nu\int_0^\infty\int_0^\infty (yz)^\nu f(y)g(z)D_{\mu,\,\nu}(x,\ y,\ z)\mathrm{d}y\mathrm{d}z \tag{24}$$

whenever the integral exists. We observe the following properties:
(i) *Commutativity*: For any $x\in I,\ (f*g)(x)=(g*f)(x)$. Proof is obvious from the relation (24).
(ii) *Associativity*: For any $t\in I,\ (f*g)*h(t)=f*(g*h)(t)$,

$$(f*g)*h=\left[\nu\int_0^\infty\int_0^\infty (yz)^\nu f(y)g(z)D_{\mu,\,\nu}(x,\ y,\ z)\mathrm{d}y\mathrm{d}z\right]*h(t)$$

$$=\nu^2\int_0^\infty\int_0^\infty (xs)^\nu\left[\int_0^\infty\int_0^\infty (yz)^\nu f(y)g(z)D_{\mu,\,\nu}(x,\ y,\ z)\mathrm{d}y\mathrm{d}z\right]h(s)D_{\mu,\,\nu}(x,\ s,\ t)\mathrm{d}x\mathrm{d}s.$$

Since the integral exists due to equation (24), by changing the order of integration and $D_{\mu,\,\nu}(x,y,z)D_{\mu,\,\nu}(x,s,t)=D_{\mu,\,\nu}(z,s,x)D_{\mu,\,\nu}(x,y,t)$ we have the result. While f and g are such that both $h_{\mu,\,\nu}(f)$ and $h_{\mu,\,\nu}(g)$ exists, we have the convolution product properties.
(iii)

$$h_{\mu,\,\nu}[x^{1+\nu}(f*g)(x)](\tau)=\tau^{1-\nu}h_{\mu,\,\nu}(f)h_{\mu,\,\nu}(g). \tag{25}$$

Proof.

$$\begin{aligned}\text{LHS}&=\nu\tau^{-1}\int_0^\infty (x\tau)^\nu J_\mu[(x\tau)^\nu]x^{-1+\nu}(f*g)(x)\mathrm{d}x\\ &=\nu\tau^{-1+\nu}\int_0^\infty \nu\left[\int_0^\infty\int_0^\infty (yz)^\nu f(y)g(z)D_{\mu,\,\nu}(x,y,z)\mathrm{d}z\mathrm{d}y\right]x^{2\nu-1}J_\mu[(xz)^\nu]\mathrm{d}x.\end{aligned}$$

Changing the order of integration we get

$$=\nu^2\tau^{-1+\nu}\int_0^\infty\int_0^\infty (yz)^\nu f(y)g(z)\left[\int_0^\infty x^{2\nu-1}J_\mu[(x\tau)^\nu]D_{\mu,\,\nu}(x,y,z)\mathrm{d}x\right]\mathrm{d}y\mathrm{d}z.$$

Using (22) we get

$$=\nu^2\tau^{-1+\nu}\int_0^\infty\int_0^\infty (yz)^\nu f(y)g(z)J_\mu[(y\tau)^\nu]J_\mu[(z\tau)^\nu]\mathrm{d}y\mathrm{d}z$$

$$= \tau^{-1-\nu}\left[\nu\tau^{-1+\nu}\int_0^\infty (y\tau)^\nu J_\mu[(y\tau)^\nu]f(y)\mathrm{d}y\right]\left[\nu\tau^{-1}\int_0^\infty (z\tau)^\nu J_\mu[(z\tau)^\nu]g(z)\mathrm{d}z\right]$$

$$= \tau^{1-\nu}h_{\mu,\nu}[f](\tau)h_{\mu,\nu}[g](\tau).$$

4.4 *$h_{\mu,\nu}$-translation*

If the $h_{\mu,\nu}$-convolution $f * g$ exists, then using Fubinis theorem we can write it in the form

$$f * g(x) = \nu\int_0^\infty y^\nu f(y)\left[\int_0^\infty z^\nu g(z)D_{\mu,\nu}(x,y,z)\mathrm{d}z\right]\mathrm{d}y = \nu\int_0^\infty y^\nu f(y)g(x\circ y)\mathrm{d}y, \tag{26}$$

where we write

$$g(x\circ y) = \left[\int_0^\infty z^\nu g(z)D_{\mu,\nu}(x,y,z)\mathrm{d}z\right] \tag{27}$$

with $x \circ y$ denoting the $h_{\mu,\nu}$-translation on the positive real line. (The analogue of the translation consider for the definition of the usual convolution *.)

The function $g(x\circ y)$ will be called the $h_{\mu,\nu}$ translate of $g(x)$; provided $g(x)$ is locally bounded on $0 < x < \infty$, $g(x\circ y)$ is well-defined and continuous on $(0,\infty)\times(0,\infty)$, (Nussbaum [7]). The $h_{\mu,\nu}$-translation is a particular case of the translations of Delsarte [3], subsequently studied by Braaksma [1].

Theorem 4.5. *If $g \in L^1_{0,\nu}(0,\infty)\cap L^\infty(0,\infty)$ and $a\in[0,\infty)$, then a simple calculation using Fubinis theorem shows that*

$$h_{\mu,\nu}[x^{-\nu}g(x\circ a)](\tau) = J_\mu[(a\tau)^\nu]h_{\mu,\nu}[g(z)](\tau). \tag{28}$$

Proof.

$$\begin{aligned}\text{LHS} &= \nu\tau^{-1}\int_0^\infty (x\tau)^\nu J_\mu[(x\tau)^\nu]x^{-\nu}g(x\circ a)\mathrm{d}x\\ &= \nu\tau^{-1+\nu}\int_0^\infty J_\mu[(x\tau)^\nu]\int_0^\infty z^\nu g(z)D_{\mu,\nu}(x,a,z)\mathrm{d}z\mathrm{d}x.\end{aligned}$$

Using Fubinis theorem,

$$= \nu\tau^{-1+\nu}\int_0^\infty z^\nu g(z)\mathrm{d}z\int_0^\infty J_\mu[(x\tau)^\nu]D_{\mu,\nu}(x,a,z)\mathrm{d}x.$$

Using (22) we get

$$= \nu\tau^{-1+\nu}\int_0^\infty z^\nu g(z)J_\mu[(z\tau)^\nu]J_\mu[(a\tau)^\nu]\mathrm{d}z$$

$$= J_\mu[(a\tau)^\nu]\{\nu z^{-1}\int_0^\infty (z\tau)^\nu J_\mu[(z\tau)^\nu]g(z)\mathrm{d}z\}$$
$$= J_\mu[(a\tau)^\nu]h_{\mu,\nu}[g(z](\tau).$$

5. Generalized Hankel type convolution of generalized functions

5.1

For fixed $x, y \in (0,\infty)$ then the function $D_{\mu,\nu}(x,y,z), 0 < z < \infty$ defines a regular generalized function in $H^*_{\mu,\nu}(0,\infty)$ which we denote by $D_{\mu,\nu}(x \circ y, z)$. In fact for fixed $x, y \in (0,\infty)$ and $\phi \in H_{\mu,\nu}(0,\infty)$ we have that

$$\left\langle D_{\mu,\nu}(x\circ y, z), z^\nu\phi(z)\right\rangle = \left\langle D_{\mu,\nu}(x,y,z), z^\nu\phi(z)\right\rangle$$
$$= \int_0^\infty z^\nu\phi(z)D_{\mu,\nu}(x,y,z)\mathrm{d}z = \phi(x\circ y) \tag{29}$$

and since

$$|\phi(x\circ y)| = \int_0^\infty \left|z^{1-\nu}z^{2\nu-1}D_{\mu,\nu}(x,y,z)\phi(z)\right|\mathrm{d}z$$
$$\le \int_0^\infty \left|z^{1-\nu}\phi(z)\right|z^{2\nu-1}D_{\mu,\nu}(x,y,z)\mathrm{d}z$$
$$\le \gamma^{0,\nu}_{0,0}(\phi)\int_0^\infty z^{2\nu-1}D_{\mu,\nu}(x,y,z)\mathrm{d}z$$
$$\le \gamma^{0,\nu}_{0,0}(\phi), \quad \text{(by (23))} \tag{30}$$

then $D_{\mu,\nu}(x\circ y, z), 0 < z < \infty$ truly generates a continuous linear functional on $H^*_{\mu,\nu}(0,\infty)$ through (29). Moreover, since

$$\begin{aligned}\phi(x\circ y) = \left\langle D_{\mu,\nu}(x,y,z), z^\nu\phi(z)\right\rangle &= \left\langle z^{-1+\nu}D_{\mu,\nu}(x,y,z), z\phi(z)\right\rangle \\ &= \left\langle h_{\mu,\nu}[z^{-1+\nu}D_{\mu,\nu}(x,y,z)], \tau\Phi(\tau)\right\rangle \\ &= \left\langle \nu\tau^{-1+\nu}J_\mu[(x\tau)^\nu]J_\mu[(y\tau)^\nu], \tau\Phi(\tau)\right\rangle \\ &= \left\langle J_\mu[(x\tau)^\nu]J_\mu[(y\tau)^\nu], \tau^\nu\Phi(\tau)\right\rangle,\end{aligned}$$

then we can write

$$h_{\mu,\nu}[D_{\mu,\nu}(x\circ y, z)] = J_\mu[(x\tau)^\nu]J_\mu[(y\tau)^\nu],\ 0 < x, y < \infty \tag{31}$$

in the sense of $H^{*\prime}_{\mu,\nu}(0,\infty)$ and even in the classical sense.

We now show that, for every fixed $y > 0$, the following implication

$$\phi(x) \in H_{\mu,\nu}(0,\infty) \Rightarrow x^{-1+\nu}\phi(x \circ y) \in H_{\mu,\nu}(0,\infty) \tag{32}$$

holds.

In fact, since $\phi(x) \in H_{\mu,\nu}(0,\infty)$, then, $\Phi = h_{\mu,\nu}[\phi] \in H_{\mu,\nu}(0,\infty)$. On the other hand,

$$h_{\mu,\nu}[x^{-1+\nu}\phi(x \circ y)] = J_\mu[(y\tau)^\nu]\Phi(\tau);$$

but $J_\mu[(y\tau)^\nu] \in M_\tau(0,\infty)$ and so $J_\mu[(y\tau)^\nu]\Phi(\tau) \in H_{\mu,\nu}(0,\infty)$.

Hence, since $h_{\mu,\nu}$ transformation is an automorphism on $H_{\mu,\nu}(0,\infty)$, the function of x given by $h^{-1}_{\mu,\nu}[J_\mu[(y\tau)^\nu]\Phi(\tau)] = x^{-1+\nu}\phi(x \circ y$ also belongs to $H_{\mu,\nu}(0,\infty)$.

5.2 *Delsarte translation*

For any $\phi(x) \in H_{\mu,\nu}(0,\infty)$ and $0 < x, y < \infty$, following from well-known property of the Delsarte translation, we also have that

$$\Delta^m_{1,x}\phi(x \circ y) = \Delta^m_{1,y}\phi(x \circ y) \tag{33}$$

for any non-negative integer m, by

$$\Delta^m_{1,x} J_\mu[(x\tau)^\nu] = (-\nu^2\tau^{2\nu})^m J_\mu[(x\tau)^\nu].$$

PROPOSITION 5.3

If $\phi_1, \phi_2 \in H_{\mu,\nu}(0,\infty)$ *and* m *is a non-negative integer, then*
(i) $\phi_1 * \phi_2$ *exists for all* $0 < x < \infty$;
(ii) $x^{-1+\nu}\phi_1 * \phi_2 \in H_{\mu,\nu}(0,\infty)$;
(iii) $\Delta^m_1[\phi_1 * \phi_2] = (\Delta^m_1\phi_1) * \phi_2 = [\phi_1 * (\Delta^m_1\phi_2)$ (34)
(iv) $\phi_1 * \phi_2(x) = \phi_2 * \phi_1(x)$.

Proof. In fact (34)(i) follows since $\phi_1, \phi_2 \in L^p_{0,\nu}(0,\infty)$ for any p such that $1 \le p < \infty$; (34)(ii) is justified by the fact that the function

$$h_{\mu,\nu}[x^{-1+\nu}(\phi_1 * \phi_2)] = \tau^{1-\nu}\Phi_1(\tau).\Phi_2(\tau)$$

belongs to $H_{\mu,\nu}(0,\infty)$ and similarly for its $h^{-1}_{\mu,\nu}$-transform; (34)(iii) follows from (33) and differentiation under integral sign. Note finally that for any $\phi_1, \phi_2 \in H_{\mu,\nu}(0,\infty)$,

$$\phi_1 * \phi_2(x) = \nu\langle\phi_1(y), y^\nu\phi_2(x \circ y)\rangle = \nu\int_0^\infty y^\nu\phi_1(y)\phi_2(x \circ y)\mathrm{d}y$$

$$= \nu\int_0^\infty y^\nu\phi_1(y)\int_0^\infty z^\nu\phi_2(z)D_{\mu,\nu}(x,y,z)\mathrm{d}z\mathrm{d}y$$

Since $\phi_1, \phi_2 \in L^p_{0,\nu}(0,\infty)$ we can make use of Fubini's theorem to get

$$= \nu\int_0^\infty z^\nu\phi_2(z)\int_0^\infty y^\nu\phi_1(y)D_{\mu,\nu}(x,y,z)\mathrm{d}y\mathrm{d}z$$

$$= \nu \int_0^\infty z^\nu \phi_2(y)\phi_1(x \circ z)\mathrm{d}z = \nu \left\langle \phi_2, z^\nu \phi_1(x \circ z)\right\rangle = \phi_2 * \phi_1(x).$$

This proves (34)(iv).

5.4

If $\lambda \in H_{\mu,\nu}(0,\infty)$, then for each fixed $x \in (0,\infty)$; we have

$$x^{-1+\nu}\lambda(x \circ y) \in H_{\mu,\nu}(0,\infty)$$

it follows that for any $\sigma \in E'(0,\infty)$ the convolution $\sigma * \lambda(x)$ is well-defined by

$$x^{-1+\nu}\sigma * \lambda(x) = \left\langle \sigma(y), \nu y^\nu x^{-1+\nu}\lambda(x \circ y)\right\rangle. \tag{35}$$

Further,

$$\begin{aligned} h_{\mu,\nu}[x^{-1+\nu}\sigma * \lambda(x)](\tau) &= \nu\left\langle \sigma(y), y^\nu h_{\mu,\nu}[x^{-1+\nu}\lambda(x \circ y)]\right\rangle \\ &= \nu\left\langle \sigma(y), y^\nu J_\mu[(y\tau)^\nu]\Lambda(\tau)\right\rangle \\ &= \tau^{1-\nu}\left\langle \sigma(y), \nu\tau^{-1}(y\tau)^\nu J_\mu[(y\tau)^\nu]\right\rangle\Lambda(\tau) \\ &= \tau^{1-\nu}h_{\mu,\nu}[\sigma](\tau)\Lambda(\tau), \end{aligned} \tag{36}$$

where $\Lambda(\tau) = h_{\mu,\nu}[\lambda]$.

Now $h_{\mu,\nu}[x^{-1+\nu}\sigma*\lambda(x)](\tau) \in H_{\mu,\nu}(0,\infty)$, and therefore $x^{-1+\nu}\sigma*\lambda(x) \in H_{\mu,\nu}(0,\infty)$. Hence $x^{-1+\nu}\sigma * \lambda(x)$ generates a regular generalized function in $H^{*\prime}_{\mu,\nu}(0,\infty)$, and for any $\phi \in H_{\mu,\nu}(0,\infty)$ we get

$$\begin{aligned} &\left\langle x^{-1+\nu}\sigma * \lambda(x), x\phi(x)\right\rangle = \left\langle h_{\mu,\nu}[x^{-1+\nu}\sigma * \lambda(x)](\tau), \tau\Phi(\tau)\right\rangle \\ &= \left\langle \tau^{1-\nu}h_{\mu,\nu}[\sigma](\tau)h_{\mu,\nu}[\lambda](\tau), \tau\Phi(\tau)\right\rangle \\ &= \left\langle h_{\mu,\nu}[\sigma](\tau), \tau^{2-\nu}h_{\mu,\nu}[\lambda](\tau)\Phi(\tau)\right\rangle \\ &= \left\langle h_{\mu,\nu}[\sigma](\tau), \tau h_{\mu,\nu}[x^{-1+\nu}\lambda * \Phi(x)](\tau)\right\rangle \\ &= \left\langle \sigma(x), x^\nu\lambda * \Phi(x)](\tau)\right\rangle = \left\langle \sigma(x), x^\nu\lambda * \Phi(x)\right\rangle. \end{aligned} \tag{37}$$

This could be taken as the definition of the generalized Hankel type convolution of generalized function (or generalized $h_{\mu,\nu}$-convolution), and this in turn allows another form analogous to the direct product definition of the generalized Hankel type ordinary convolution:

$$\begin{aligned} &\left\langle x^{-1+\nu}\sigma * \lambda(x), x\phi(x)\right\rangle = \left\langle \sigma(x), x^\nu\lambda * \phi(x)\right\rangle \\ &= \left\langle \sigma(x), x\left\langle \lambda(y), \nu y^\nu x^{-1+\nu}\phi(x \circ y)\right\rangle\right\rangle \\ &= \left\langle \sigma(x), \nu x^\nu\left\langle \lambda(y), y^\nu\phi(x \circ y)\right\rangle\right\rangle = \left\langle \sigma(x) \otimes \lambda(y), \nu(xy)^\nu\phi(x \circ y)\right\rangle. \end{aligned} \tag{38}$$

5.5

For $f \in H^{*\prime}_{\mu,\nu}(0,\infty)$ and $\lambda \in H_{\mu,\nu}(0,\infty)$. The convolution is again well-defined as a generalized function in $H^{*\prime}_{\mu,\nu}(0,\infty)$. By

$$\left\langle x^{-1+\nu} f * \lambda(x), x\phi(x)\right\rangle = \left\langle f(x), x^\nu\lambda * \phi(x)\right\rangle$$

since $x^{\nu}\lambda * \phi(x) \in H_{\mu,\nu}(0,\infty)$ by (34)(ii). Using (18), we get

$$\begin{aligned}
&\big\langle h_{\mu,\nu}[x^{-1+\nu} f * \lambda(x)](\tau), \tau\Phi(\tau)\big\rangle = \big\langle x^{-1+\nu} f * \lambda(x), x\phi(x)\big\rangle \\
&= \big\langle f(x), x^{\nu}\lambda * \phi(x)\big\rangle = \big\langle f(x), x^{\nu}\lambda * \phi(x)\big\rangle \\
&= \big\langle h_{\mu,\nu}[f](\tau), \tau[\tau^{1-\nu} h_{\mu,\nu}[\lambda](\tau)\Phi(\tau)\big\rangle = \big\langle \tau^{1-\nu} h_{\mu,\nu}[f](\tau)\Lambda(\tau), \tau\Phi(\tau)\big\rangle.
\end{aligned}$$

So that, in the sense of $H^{*\prime}_{\mu,\nu}(0,\infty)$,

$$h_{\mu,\nu}[x^{-1+\nu} f * \lambda(x)] = \tau^{1-\nu} h_{\mu,\nu}[f] h_{\mu,\nu}[\lambda]. \tag{39}$$

5.6

Finally, let $f \in H^{*\prime}_{\mu,\nu}(0,\infty)$ and $\sigma \in E'(0,\infty)$. Since, for any $\phi \in H_{\mu,\nu}(0,\infty)$ we have $\sigma * \phi(x) \in H_{\mu,\nu}(0,\infty)$, it follows that $x^{-1+\nu} f * \sigma$ is well-defined as a generalized function in $H^{*\prime}_{\mu,\nu}(0,\infty)$ by

$$\big\langle x^{-1+\nu} f * \sigma(x), x\phi(x)\big\rangle = \big\langle f(x), x^{\nu}\sigma * \phi(x)\big\rangle. \tag{40}$$

As before, this may also be expressed in the form

$$\big\langle x^{-1+\nu} f * \sigma(x), x\phi(x)\big\rangle = \big\langle f(x) \otimes \sigma(x), \nu(xy)^{\nu}\phi(x \circ y)\big\rangle \tag{41}$$

and, using (18) again, we can derive the analogue of (39)

$$h_{\mu,\nu}[x^{-1+\nu} f * \sigma] = \tau^{1-\nu} h_{\mu,\nu}[f] h_{\mu,\nu}[\sigma]. \tag{42}$$

Note that $h_{\mu,\nu}[\sigma] \in M(0,\infty)$, so that the product in (42) makes sense in $H^{*\prime}_{\mu,\nu}(0,\infty)$.

6. Algebraic properties of the generalized $h_{\mu,\nu}$- convolution

As already remarked, the classical $h_{\mu,\nu}$-convolution defined in $L^1_{0,\nu}(0,\infty)$ is commutative and associative; however, it possesses no identity element. We consider in turn these properties with respect to generalized $h_{\mu,\nu}$-convolution.

6.1 *Commutativity*

(i) $\sigma \in E'(0,\infty)$, $\lambda \in H_{\mu,\nu}(0,\infty)$. We have

$$\begin{aligned}
&\big\langle x^{-1+\nu}\sigma * \lambda(x), x\phi(x)\big\rangle = \big\langle \sigma(x), x^{\nu}\lambda * \phi(x)\big\rangle \\
&= \big\langle h_{\mu,\nu}[\sigma](\tau), \tau h_{\mu,\nu}[x^{-1+\nu}\lambda * \phi(x)]\big\rangle \\
&= \big\langle h_{\mu,\nu}[\sigma](\tau), \tau[\tau^{1-\nu}\Lambda(\tau)\Phi(\tau)]\big\rangle \\
&= \big\langle \tau h_{\mu,\nu}[\sigma](\tau)\Lambda(\tau), \tau\Phi(\tau)\big\rangle \\
&= \big\langle h_{\mu,\nu}[x^{-1+\nu}\lambda * \sigma](\tau), \tau\Phi(\tau)\big\rangle \\
&= \big\langle x^{-1+\nu}\lambda * \sigma(x), x\phi(x)\big\rangle,
\end{aligned}$$

where the last manipulation make sense since $h_{\mu,\nu}[\sigma] \in M(0,\infty)$ and see (37) and Pinto's paper for other type proof .

(ii) If $f \in H^{*\prime}_{\mu,\nu}(0,\infty)$, $\lambda \in H_{\mu,\nu}(0,\infty)$ then

$$\begin{aligned}
&\left\langle x^{-1+\nu} f * \lambda(x), x\phi(x)\right\rangle = \left\langle f(x), x^{\nu}\lambda * \phi(x)\right\rangle\\
&= \left\langle f(x), x[x^{-1+\nu}\lambda * \phi(x)]\right\rangle\\
&= \left\langle h_{\mu,\nu}[f(x)](\tau), \tau h_{\mu,\nu}[x^{-1+\nu}\lambda * \phi(x)]\right\rangle\\
&= \left\langle h_{\mu,\nu}[f(x)](\tau), \tau[\tau^{1-\nu} h_{\mu,\nu}[\lambda] h_{\mu,\nu}[\phi]\right\rangle\\
&= \left\langle \tau^{1-\nu}\Lambda(\tau) h_{\mu,\nu}[f](\tau), \tau\Phi(\tau)\right\rangle\\
&= \left\langle h_{\mu,\nu}[x^{-1+\nu}\lambda * f](\tau), \tau\Phi(\tau)\right\rangle\\
&= \left\langle x^{-1+\nu}\lambda * f(x), x\phi(x)\right\rangle.
\end{aligned}$$

This is justified because every function in $H_{\mu,\nu}(0,\infty)$ is also a multiplier in $H^{*\prime}_{\mu,\nu}(0,\infty)$ whenever $x\phi(x) \in H_{\mu,\nu}(0,\infty)$.
(iii) If $f \in H^{*\prime}_{\mu,\nu}(0,\infty)$, $\sigma \in E'(0,\infty)$ then the same kind of argument gives

$$\begin{aligned}
&\left\langle x^{-1+\nu} f * \sigma(x), x\phi(x)\right\rangle = \left\langle f(x), x^{\nu}\sigma * \phi(x)\right\rangle\\
&= \left\langle f(x), x[x^{-1+\nu}\sigma * \phi(x)]\right\rangle\\
&= \left\langle h_{\mu,\nu}[f(x)](\tau), \tau h_{\mu,\nu}[x^{-1+\nu}\sigma * \phi(x)]\right\rangle\\
&= \left\langle h_{\mu,\nu}[f(x)](\tau), \tau[\tau^{1-\nu} h_{\mu,\nu}[\sigma] h_{\mu,\nu}[\phi]\right\rangle\\
&= \left\langle \tau^{1-\nu} h_{\mu,\nu}[f](\tau) h_{\mu,\nu}[\sigma](\tau), \tau\Phi(\tau)\right\rangle\\
&= \left\langle h_{\mu,\nu}[x^{-1+\nu}\sigma * f](\tau), \tau\Phi(\tau)\right\rangle\\
&= \left\langle x^{-1+\nu}\sigma * f(x), x\phi(x)\right\rangle.
\end{aligned}$$

But since $h_{\mu,\nu}[f]$ does not belong to $M(0,\infty)$, no general commutativity property can be deduced. If, in addition, we have $f \in E'(0,\infty)$, then $h_{\mu,\nu}[f] \in M(0,\infty)$, and the argument to establish commutativity proceed as before.

6.2 *Associativity*

(i) $\sigma \in E'(0,\infty)$, $\lambda_1, \lambda_2 \in H_{\mu,\nu}(0,\infty)$. We can establish the result

$$x^{-1+\nu}[x^{-1+\nu}\sigma * \lambda_1] * \lambda_2 = x^{-1+\nu}\sigma * [x^{-1+\nu}\lambda_1 * \lambda_2] \tag{43}$$

in the following sense, for any $\phi \in H_{\mu,\nu}(0,\infty)$.

$$\begin{aligned}
&\left\langle x^{-1+\nu}(x^{-1+\nu}\sigma * \lambda) * \lambda_2, x\phi(x)\right\rangle\\
&= \left\langle x^{-1+\nu}\sigma * \lambda_1, x^{\nu}\lambda_2 * \phi(x)\right\rangle\\
&= \left\langle x^{-1+\nu}\sigma * \lambda_1, x(x^{-1+\nu}\lambda_2 * \phi(x))\right\rangle\\
&= \left\langle \sigma(x), x^{\nu}\lambda_1 * [x^{-1+\nu}\lambda_2 * \phi(x)]\right\rangle\\
&= \left\langle \sigma(x), x[x^{-1+\nu}\lambda_1 * (x^{-1+\nu}\lambda_2 * \phi(x)])\right\rangle\\
&= \left\langle x^{-1+\nu}\sigma * [x^{1-\nu}\lambda_1 * \lambda_2(x)], x\phi(x)\right\rangle.
\end{aligned}$$

The equality $x^{\nu}\lambda_1 * (x^{-1+\nu}\lambda_2 * \phi) = ((x^{\nu}\lambda_1 * x^{-1+\nu}\lambda_2) * \phi)$ is justified by the fact that λ_1, λ_2 and ϕ belong to $L^1_{0,\nu}(0,\infty)$. (ii) $f \in H^{*\prime}_{\mu,\nu}(0,\infty)$, $\sigma \in E'(0,\infty)$, $\lambda \in H_{\mu,\nu}(0,\infty)$. We have that

$$x^{-1+\nu}[x^{-1+\nu} f * \sigma] * \lambda(x) = x^{-1+\nu} f * [x^{-1+\nu}\sigma * \lambda](x). \tag{44}$$

Proof.

$$\begin{aligned}
&\langle x^{-1+\nu}[x^{-1+\nu} f * \sigma] * \lambda(x), x\phi(x)\rangle \\
&= \langle x^{-1+\nu} f * \sigma(x), x^{\nu}\lambda * \phi(x)\rangle \\
&= \langle x^{-1+\nu} f * \sigma(x), x[x^{-1+\nu}\lambda * \phi(x)]\rangle \\
&= \langle f(x), x^{\nu}\sigma * [x^{-1+\nu}\lambda * \phi(x)]\rangle \\
&= \langle f(x), x[x^{-1+\nu}\sigma * [x^{-1+\nu}\lambda * \phi(x)]\rangle \\
&= \langle x^{-1+\nu} f * [x^{-1+\nu}\sigma * \lambda(x)], x\phi(x)\rangle.
\end{aligned}$$

(iii) If $f \in H^{*\prime}_{\mu,\nu}(0,\infty)$, $\sigma_1, \sigma_2 \in E'(0,\infty)$. We show, finally, that

$$x^{-1+\nu}[x^{-1+\nu} f * \sigma_1] * \sigma_2(x) = x^{-1+\nu} f * [x^{-1+\nu}\sigma_1 * \sigma_2](x). \tag{45}$$

Proof.

$$\begin{aligned}
&\langle x^{-1+\nu}[x^{-1+\nu} f * \sigma_1] * \sigma_2(x), x\phi(x)\rangle \\
&= \langle x^{-1+\nu} f * \sigma_1(x), x^{\nu}\sigma_2 * \phi(x)\rangle \\
&= \langle x^{-1+\nu} f * \sigma_1(x), x[x^{-1+\nu}\sigma_2 * \phi(x)\rangle \\
&= \langle f(x), x^{\nu}\sigma_1 * [x^{-1+\nu}\sigma_2 * \phi(x)]\rangle \\
&= \langle f(x), x[x^{-1+\nu}\sigma_1 * (x^{-1+\nu}\sigma_2 * \phi(x))]\rangle \\
&= \langle x^{-1+\nu} f * [x^{-1+\nu}\sigma_1 * \sigma_2(x)], x\phi(x)\rangle.
\end{aligned}$$

6.3 *Identity element*

For a, b strictly positive we know that $D_{\mu,\nu}(a,b,z)$ defines a regular generalized function $D_{\mu,\nu}(a,b,z)$ in $H^{*\prime}_{\mu,\nu}(0,\infty)$ If either of a, b takes the value zero then $D_{\mu,\nu}(a,b,z)$ is no longer defined as an ordinary function since

$$D_{\mu,\nu}(a,0,z) = \int_0^\infty \nu^2\tau^{2\nu-1} J_\mu[(a\tau)^\nu] J_\mu[(z\tau)^\nu]\mathrm{d}\tau, \; a > 0,$$

is only a formal identity because the integral fails to converge for any z.

Instead, for any fixed $a > 0$, we consider the integral

$$\int_0^R \nu^2\tau^{2\nu-1} J_\mu[(a\tau)^\nu] J_\mu[(z\tau)^\nu]\mathrm{d}\tau \tag{46}$$

which for each $R > 0$ is uniformly convergent on $0 < z < \infty$.

DEFINITION 6.4

Define the generalized function $D_{\mu,\nu}(a,z)$ in $H^{*\prime}_{\mu,\nu}(0,\infty)$ by

$$D_{\mu,\nu}(a,z) = \lim_{R\to\infty} \int_0^R \nu^2\tau^{2\nu} J_\mu[(a\tau)^\nu] J_\mu[(z\tau)^\nu]\mathrm{d}\tau$$

in the sense that for any $\phi \in H_{\mu,\nu}(0,\infty)$,

$$\left\langle D_{\mu,\nu}(a,z), z^{\nu}\phi(z)\right\rangle = \lim_{R\to\infty}\left\langle \int_0^R \nu^2\tau^{2\nu-1}J_\mu[(a\tau)^\nu]J_\mu[(z\tau)^\nu]\mathrm{d}\tau\mathrm{d}z, z^\nu\phi(z)\right\rangle. \tag{47}$$

For each finite $R > 0$ the integral (44) defines a function which generates a regular generalized function in $H^{*\prime}_{\mu,\nu}(0,\infty)$ (Sneddon [9]), Therefore,

$$\left\langle \int_0^R \nu^2\tau^{2\nu-1}J_\mu[(a\tau)^\nu]J_\mu[(z\tau)^\nu]\mathrm{d}\tau, z^\nu\phi(z)\right\rangle = \int_0^\infty z^\nu\phi(z)\int_0^R \nu^2\tau^{2\nu-1}J_\mu [(a\tau)^\nu]J_\mu[(z\tau)^\nu]\mathrm{d}\tau\mathrm{d}z$$

or by Fubini's theorem

$$\left\langle \int_0^R \nu^2\tau^{2\nu-1}J_\mu[(a\tau)^\nu]J_\mu[(z\tau)^\nu]\mathrm{d}\tau, z^\nu\phi(z)\right\rangle = \int_0^R \nu^2 J_\mu[(a\tau)^\nu]\int_0^\infty z^\nu\phi(z)J_\mu[(z\tau)^\nu]\mathrm{d}z\mathrm{d}\tau$$

$$= \nu\int_0^R \tau^\nu J_\mu[(a\tau)^\nu]\nu\tau^{-1}\int_0^\infty (z\tau)^\nu\phi(z)J_\mu[(a\tau)^\nu]\mathrm{d}z\mathrm{d}\tau$$

$$= \nu\int_0^R \tau^\nu J_\mu[(a\tau)^\nu]\Phi(\tau)\mathrm{d}\tau.$$

Thus

$$\begin{aligned}
\left\langle D_{\mu,\nu}(a,z), z^\nu\phi(z)\right\rangle &= \lim_{R\to\infty}\int_0^R \nu\tau^\nu J_\mu[(a\tau)^\nu]\Phi(\tau)\mathrm{d}\tau\\
&= a^{1-\nu}\left[\nu a^{-1}\int_0^R (a\tau)^\nu\phi(z)J_\mu[(a\tau)^\nu]\Phi(\tau)\mathrm{d}\tau\right]\\
&= \lim_{R\to\infty} a^{1-\nu}\left[\nu a^{-1}\int_0^R (a\tau)^\nu\phi(z)J_\mu[(a\tau)^\nu]\Phi(\tau)\mathrm{d}\tau\right]\\
&= a^{1-\nu}\phi(a)
\end{aligned} \tag{48}$$

and so

$$\begin{aligned}
\left|\left\langle D_{\mu,\nu}(a,z), z^\nu\phi(z)\right\rangle\right| &= \lim_{R\to\infty} a^{1-\nu}\left[\nu a^{-1}\int_0^R (a\tau)^\nu\phi(z)J_\mu[(a\tau)^\nu]\Phi(\tau)\mathrm{d}\tau\right]\\
&= a^{1-\nu}\phi(a)\\
&\le \gamma^{a,\nu}_{0,0}(\phi)
\end{aligned}$$

which shows that $D_{\mu,\nu}(a,z) \in H^{*\prime}_{\mu,\nu}(0,\infty)$. Moreover, since

$$\left\langle D_{\mu,\nu}(a,z), z^{\nu}\phi(z)\right\rangle = a^{1-\nu}\phi(a) = \left\langle J_{\mu}[(a\tau)^{\nu}], \nu\tau^{\nu}\Phi(\tau)\right\rangle,$$

we obtain

$$h_{\mu,\nu}[D_{\mu,\nu}(a,z)](\tau) = J_{\mu}[(a\tau)^{\nu}]. \tag{49}$$

Now let $(a_n)_{n=1}^{\infty}$ be a monotone decreasing sequence of positive real numbers, tending to zero as $n \to \infty$, and consider the sequence of generalized functions $(D_{\mu,\nu}(a_n,z))_{n=1}^{\infty}$ in $H^{*\prime}_{\mu,\nu}(0,\infty)$. Since $H^{*\prime}_{\mu,\nu}(0,\infty)$ is complete, this limit is again a generalized function in $H^{*\prime}_{\mu,\nu}(0,\infty)$. For each n and any $\phi \in H_{\mu,\nu}(0,\infty)$,

$$\left\langle D_{\mu,\nu}(a_n,z), z^{\nu}\phi(z)\right\rangle = a^{1-\nu}\phi(a_n).$$

And therefore we define the generalized function $D_{\mu,\nu}(z)$ by

$$\begin{aligned}\left\langle D_{\mu,\nu}(a_n,z), z^{\nu}\phi(z)\right\rangle &= \lim_{n\to\infty}\left\langle D_{\mu,\nu}(a_n,z), z^{\nu}\phi(z)\right\rangle \\ &= \lim_{n\to\infty} a_n^{1-\nu}\phi(a_n) = \nu\phi(0+)\end{aligned} \tag{50}$$

(independently of the particular sequence $(a_n)_{n=1}^{\infty}$ chosen). Moreover, since

$$\left\langle D_{\mu,\nu}(z), z^{\nu}\phi(z)\right\rangle = \nu\phi(0+) = \left\langle 1, \nu\tau^{\nu}\Phi(\tau)\right\rangle = \nu\left\langle 1, \nu\tau^{\nu}\Phi(\tau)\right\rangle,$$

we have

$$h_{\mu,\nu}[D_{\mu,\nu}(z)](\tau) = \nu.1 = \nu \tag{51}$$

the equality being understood in the sense of $H^{*\prime}_{\mu,\nu}(0,\infty)$. The generalized function $D_{\mu,\nu,z}(x) \in H^{*\prime}_{\mu,\nu}(0,\infty)$ is the required identity element with respect to the generalized $h_{\mu,\nu}$-convolution. In fact, it is easy to show that $D_{\mu,\nu,z}(x) \in E'(0,\infty)$ and therefore for any $f \in H^{*\prime}_{\mu,\nu}(0,\infty)$ and every $\phi \in H_{\mu,\nu}(0,\infty)$, by using the results in (40), (41) and (42) we obtain

$$\begin{aligned}&\left\langle f * \frac{D_{\mu,\nu}(x)}{\nu}, x^{\nu}\phi(x)\right\rangle \\ &= \left\langle f(x) \otimes \frac{D_{\mu,\nu}(y)}{\nu}, \nu(xy)^{\nu}\phi(x\circ y)\right\rangle \\ &= \left\langle f(x), x^{\nu}\left\langle D_{\mu,\nu}(y), y^{\nu}\phi(x\circ y)\right\rangle\right\rangle \\ &= \left\langle f(x), x^{\nu}\phi(x)\right\rangle\end{aligned}$$

which shows that

$$f * \frac{D_{\mu,\nu}(x)}{\nu} = f(x) \tag{52}$$

in the sense of $H^{*\prime}_{\mu,\nu}(0,\infty)$, as asserted.

7. Differentiability properties of the $h_{\mu,\nu}$-convolution

We conclude with a brief remark on the differentiability properties of the generalized $h_{\mu,\nu}$-convolution. Let k be any nonnegative integer $f \in H^{*\prime}_{\mu,\nu}(0,\infty)$ and $\lambda \in H_{\mu,\nu}(0,\infty)$.

Then, since for any $\phi \in H_{\mu,\nu}(0,\infty)$,

$$\begin{aligned} A &= \left\langle \Delta^k[f * \lambda(x)], x^\nu \phi(x) \right\rangle \\ &= \int_0^\infty (x^{\nu-1}\Delta_1^k x^{1-\nu})(f*\lambda) x^\nu \phi(x)\mathrm{d}x \\ &= \int_0^\infty x^{2\nu-1}\phi(x)\Delta_1^k x^{1-\nu}(f*\lambda)\mathrm{d}x \\ &= \nu \int_0^\infty x^{2\nu-1}\phi(x)\Delta_1^k x^{1-\nu}\left[\int_0^\infty\int_0^\infty (yz)^\nu f(y)\lambda(z) D_{\mu,z}(x,y,z)\mathrm{d}y\mathrm{d}z\right]\mathrm{d}x. \end{aligned}$$

Differentiating under the integral sign and using Fubini's theorem we get

$$\begin{aligned} A &= \nu \int_0^\infty x^{2\nu-1}\phi(x)\left[\int_0^\infty y^\nu f(y)\left\{\int_0^\infty z^\nu \lambda(z)\Delta_1^k x^{1-\nu} D_{\mu,\nu}(x,y,z)\mathrm{d}z\right\}\mathrm{d}y\right]\mathrm{d}x \\ &= \nu \int_0^\infty x^\nu \phi(x)\int_0^\infty y^\nu f(y)\int_0^\infty z^\nu\lambda(z)[x^{\nu-1}\Delta_1^k x^{1-\nu} D_{\mu,\nu}(x,y,z)]\mathrm{d}z\mathrm{d}y\mathrm{d}x \\ &= \int_0^\infty x^\nu\phi(x) f * [x^{\nu-1}\Delta_1^k x^{1-\nu}]\lambda(x)\mathrm{d}x \\ &= \left\langle f * [x^{\nu-1}\Delta_1^k x^{1-\nu}]\lambda(x), x^\nu\phi(x)\right\rangle = B. \end{aligned}$$

Similarly

$$\begin{aligned} B &= \left\langle [x^{\nu-1}\Delta_1^k x^{1-\nu}] f * \lambda(x), x^\nu\phi(x)\right\rangle \\ &\Rightarrow \ \Delta^k[f*\lambda] = f * [\Delta^k\lambda] = [\Delta^k f]*\lambda \end{aligned} \tag{53}$$

in the sense of $H^{*\prime}_{\mu,\nu}(0,\infty)$.

If now $f \in H^{*\prime}_{\mu,\nu}(0,\infty)$ and $\sigma \in E'(0,\infty)$, then by the same kind of argument, and using (53), we derive the double equality.

$$\Delta^k[f*\sigma] = f*[\Delta^k\sigma] = [\Delta^k f]*\sigma \tag{54}$$

in the sense of $H^{*\prime}_{\mu,\nu}(0,\infty)$.

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