Proc. Indian Acad. Sci. (Math. Sci.), Vol. 111, No. 4, November 2001, pp. 465–470.


# A variational principle for vector equilibrium problems

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MS received 1 December 2000; revised 17 May 2001

**Abstract.** A variational principle is described and analysed for the solutions of vector equilibrium problems.

**Keywords.** Vector equlibrium problem; variational principle; $P$-convexity; $P-\psi$-monotonicity.

## 1. Introduction

Throughout this paper, $X$ is a real topological vector space; $K \subset X$ be a nonempty, closed and convex set; $(Y, P)$ be a real ordered topological vector space with a partial order $\leq_P$ induced by a solid, pointed, closed and convex cone $P$ with apex at origin, thus

$$x \leq_P y \iff y - x \in P \quad \forall x, y \in Y.$$

If int$P$ denotes the topological interior of the cone $P$, then weak ordering, say $\not\leq_{\mathrm{int}P}$ (or $\not\geq_{\mathrm{int}P}$), on $Y$ is defined by

$$x \not\leq_{\mathrm{int}P} y \;\text{ or }\; y \not\geq_{\mathrm{int}P} x \iff y - x \notin \mathrm{int}P \quad \forall x, y \in Y.$$

Let $f : X \times X \longrightarrow Y$ be a mapping with $f(x, x) = 0 \;\; \forall x \in X$, then vector equilibrium problem (for short, VEP($f$, $K$)) is to find $x \in K$ such that

$$f(x, y) \not\leq_{\mathrm{int}P} 0, \quad \forall x, y \in K.$$

VEP($f$, $K$) has been studied by Kazmi [K2]. VEP($f$, $K$) includes as special cases, vector optimization problems, vector variational inequalities, vector variational-like inequalities, vector complementarity problems, etc., see Kazmi [K2] and the references therein.

If $Y = R,\; P = R_+$, then VEP($f$, $K$) reduces to the scalar equilibrium problem [B-O1, B-O2] of finding $x \in K$ such that

$$f(x, y) \geq 0, \quad \forall y \in K.$$

In this paper, we shall describe and analyse a variational principle for the solutions of VEP($f$, $K$).

The construction of variational principles is of interest both theoretically and in practice. Conceptually, it is of singnificance to know that there is a mapping defined on $X$ which is optimized precisely at the solutions of VEP($f$, $K$). In practice, it is of importance because it allows one to use the highly developed theory of numerical optimization to numerically approximate, and compute solutions of these problems.

More precisely, following the terminology of Auchmuty [A], we say that a variational principle holds for VEP($f, K$), if there exists a mapping $F : K \longrightarrow Y$ depending on the data of VEP($f, K$) but not on its solution set, such that the solution set of VEP($f, K$) coincides with the solution set of the vector maximization problem (for short, VMP($f, K$))

$$\max_{\mathrm{int}P} F(x), \quad \text{subject to } x \in K.$$

If $f(x, y) = \langle \phi'(x), \eta(y, x)\rangle$, where $\eta : K \times K \longrightarrow X$ is a continuous function and $\phi : K \longrightarrow Y$ is Fréchet (or linear Gateaux) differentiable and $P$-convex mapping, the $x$ is a solution of VEP($f, K$) if and only if $x$ is a solution of

$$\min_{\mathrm{int}P} \phi(x), \quad \text{subject to } x \in K,$$

see Kazmi [K1]. For related work, see [K3, K-A].

Thus, setting $F = -\phi$, a variational principle for VEP($f, K$) holds.

Now, consider the case:

$$f(x, y) := g(x, y) + h(x, y),$$

where $g, h : K \times K \longrightarrow Y$ are nonlinear mappings, then VEP($f, K$) becomes:

(VEP($g + h, K$)), find $x \in K$ such that $g(x, y) + h(x, y) \not\leq_{\mathrm{int}P} 0, \ \ \forall y \in K$.

We shall use the following concepts and result:

The mapping $g$ is called $P$-monotone if and only if

$$g(x, y) \leq_P -g(y, x), \quad \forall x, y \in K.$$

A mapping $T : X \longrightarrow Y$ is called $P$-convex if and only if for each pair $x, y \in K$ and $\lambda \in [0, 1]$,

$$T(\lambda x + (1 - \lambda)y) \leq_P \lambda T(x) + (1 - \lambda)T(y).$$

Note that if $g(x, y) = \langle \phi^{'}(x), y - x\rangle$ where $\phi : X \longrightarrow Y$ be $P$-convex and linear Gateaux differentiable, then $g$ is $P$-monotone since $\phi^{'}(\cdot)$ is $P$-monotone.

*Lemma* 1 [C]. *Let* $(Y, P)$ *be an ordered topological vector space with a solid, pointed, closed and convex cone* $P$*. Then* $\forall x, y \in X$*, we have*

$$y \leq_P x \ \ and \ \ y \not\leq_{\mathrm{int}P} 0 \ \ imply \ \ x \not\leq_{\mathrm{int}P} 0.$$

Finally, in order to formulate our variational principle we introduce a perturbation mapping $\psi(\cdot, \cdot) : K \times K \longrightarrow Y$ which satisfies for all $x, y \in K$:

(i) $0 \leq_P \psi(x, y)$,

(ii) $\psi(x, x) = 0$,

(iii) $\psi(x, \lambda y + (1 - \lambda)x) = o(\lambda), \quad \lambda \in [0, 1]$.

Let us indicate some possible choices for $\psi(\cdot, \cdot)$: $K \times K \longrightarrow Y$ satisfying properties (i)–(iii) above. Clearly, the choice $\psi(\cdot, \cdot) = 0$ is always possible. Next let $\phi(\cdot, \cdot)$: $K \times K \longrightarrow Y$ be $P$-convex in the second argument, and $\forall x \in K$, let $\phi(\cdot, \cdot)$ be Gateaux differentiable at $x$ with Gateaux differential $\phi^{'}(x, \cdot) \in L(X, Y)$ where $L(X, Y)$ is a space of all linear bounded functionals from $X$ to $Y$. Set

$$\psi(x, y) = \phi(x, y) - \phi(x, x) - \langle \phi^{'}(x, x), y - x\rangle,$$

then $\psi(\cdot\,,\cdot)$ satisfies properties (i)–(iii). In particular if $\phi(\cdot)\colon K \longrightarrow Y$ be $P$-convex, and Gateaux differentiable, then we may choose

$$\psi(x, y) = \phi(y) - \phi(x) - \langle \phi^{'}(x), y - x\rangle.$$

Finally, if $\psi(\cdot\,,\cdot)\colon K \times K \longrightarrow R\bigcup\{\infty\}$, where $K$ is a subset of normed linear space $X$, then we may choose $\psi(x, y) = \alpha\|y - x\|^2$, for $\alpha > 0$, which satisfies (i)–(iii).

Now, we define a mapping $G : K \longrightarrow Y$ by means of

$$G(x) := \inf\{-g(y, x) + h(x, y) + \psi(x, y) : y \in K\}, \tag{1}$$

and we associate to VEP($g + h, K$) the following vector maximization problem:

$$\text{VMP}(g + h, \psi, K):\ \max_{\text{int}\,P}\{G(x) : x \in K\}.$$

We remark that the mapping $G(\cdot)$ generalizes the gap function used in connection with variational inequalities, see Herker and Pang [H-P], and the references therein.

From $-g(x, x) + h(x, x) + \psi(x, x) = 0$ follows

$$G(x) \leq_P 0 \quad \forall x \in K. \tag{2}$$

We also define the following concept.

Let $\psi$ satisfy (i)–(iii), the mapping $g$ is called $P - \psi$-monotone if and only if

$$g(x, y) \leq_P \psi(x, y) - g(y, x), \quad \forall x, y \in K.$$

If $\psi(x, y) = 0,\ \forall x, y \in K$, then $P - \psi$-monotone mapping becomes $P$-monotone.

## 2. Results

First we prove the following results:

**Theorem 2.** *Let the following assumptions hold:*

(i) *The mapping $g$ satisfies*: $g(x, x) = 0,\ \forall x \in K$; *$g$ is $P$-monotone*; $\forall x, y \in K$, *the mapping $\lambda \in [0, 1] \longrightarrow g(\lambda y + (1 - \lambda)x, y)$ is continuous at $0_+$*; *$g$ is $P$-convex in the second argument*.

(ii) *The mapping $h$ satisfies*: $h(x, x) = 0,\ \forall x \in K$; *$h$ is $P$-convex in the second argument*.

*Then* VEP($g + h, K$) *and the problem of finding $x \in K$ such that*

$$G(x) \not\leq_{\text{int}\,P} 0, \tag{3}$$

*both have the same solution set.*

*Proof*. Let $x$ be a solution of VEP($g + h, K$), that is,

$$g(x, y) + h(x, y) \not\leq_{\text{int}\,P} 0, \quad \forall y \in K. \tag{4}$$

Since $g$ is $P$-monotone, $\forall x, y \in K$, we have

$$(g(x, y) + h(x, y)) - (-g(y, x) + h(x, y)) \leq_P 0. \tag{5}$$

From Lemma 1, eqs (4) and (5), it follows that

$$-g(y, x) + h(x, y) \not\leq_{\text{int}\,P} 0, \quad \forall y \in K$$

or,

$$-g(y,x)+h(x,y)\in W:=Y\setminus(-\mathrm{int}P),\quad \forall y\in K. \tag{6}$$

Since $\psi(x,y)\in P$, we have

$$-g(y,x)+h(x,y)+\psi(x,y)\in W+P\subset W,\quad \forall y\in K,$$

which implies

$$\inf\{-g(y,x)+h(x,y)+\psi(x,y)\ :\ y\in K\}\not\leq_{\mathrm{int}P} 0,$$

that is,

$$G(x)\not\leq_{\mathrm{int}P} 0.$$

Conversly, let $x$ be a solution of problem (3). Then by the definition of $G(\cdot)$, we have

$$-g(y,x)+h(x,y)+\psi(x,y)\not\leq_{\mathrm{int}P} 0,\quad \forall y\in K.$$

Fix $y\in K$ arbitrarily, let $x_\lambda:=\lambda y+(1-\lambda)x,\ \lambda\in]0,1],\ x_\lambda\in K$ as $K$ is convex, and hence the above inequality becomes

$$-g(x_\lambda,x)+h(x,x_\lambda)+\psi(x,x_\lambda)\not\leq_{\mathrm{int}P} 0. \tag{7}$$

Since $g$ is $P$-convex in the second argument, we have

$$\begin{aligned} 0 &= g(x_\lambda,x_\lambda)\\ &\leq_P \lambda g(x_\lambda,y)+(1-\lambda)g(x_\lambda,x)\\ -(1-\lambda)g(x_\lambda,x) &\leq_P \lambda g(x_\lambda,y). \end{aligned}$$

By using preceding inequality and the properties of cone $P$, we have

$$\begin{aligned} &(1-\lambda)(-g(x_\lambda,x)+h(x,x_\lambda)+\psi(x,x_\lambda))\\ &\quad\leq_P \lambda g(x_\lambda,y)+(1-\lambda)(h(x,x_\lambda)+\psi(x,x_\lambda))\\ &\quad\leq_P \lambda g(x_\lambda,y)+(1-\lambda)(h(x,y)+o(\lambda)), \end{aligned}$$

using the properties of $h$ and $\psi$.

Since $(1-\lambda)>0$, after dividing the preceding inequality by $(1-\lambda)>0$, we have, from Lemma 1, (7) and the resultant inequality,

$$\frac{\lambda}{(1-\lambda)}g(x_\lambda,y)+\lambda h(x,y)+o(\lambda)\not\leq_{\mathrm{int}P} 0\in W.$$

After dividing the preceding inclusion by $\lambda>0$, letting $\lambda\downarrow 0$ and hence $x_\lambda\longrightarrow x\in K$, and then by hemicontinuity of $g$ and closedness of $W$, we have

$$g(x,y)+h(x,y)\not\leq_{\mathrm{int}P} 0,\quad \forall y\in K.$$

This completes the proof.

**Theorem 3.** *Let all the assumptions of Theorem 2 except $P$-monotonicity of $g$ hold. Let $g$ be $P-\psi$-monotone, then* VEP*$(g+h,K)$ and problem* (3) *both have the same solution set.*

*Proof.* Let $x$ be a solution of VEP$(g+h, K)$, that is,

$$g(x, y) + h(x, y) \not\leq_{\operatorname{int} P} 0, \quad \forall y \in K. \tag{8}$$

Since $g$ is $P - \psi$-monotone, $\forall x, y \in K$, we have

$$(g(x, y) + h(x, y)) - (-g(y, x) + h(x, y) + \psi(x, y)) \leq_P 0. \tag{9}$$

From Lemma 1, (8) and (9), it follows that

$$-g(y, x) + h(x, y) + \psi(x, y) \not\leq_{\operatorname{int} P} 0, \quad \forall y \in K,$$

which implies

$$\inf\{-g(y, x) + h(x, y) + \psi(x, y) \ : \ y \in K\} \not\leq_{\operatorname{int} P} 0.$$

Converse part of theorem is just same as the converse part of Theorem 2. This completes the proof.

Now, on combining Theorem 2 (or Theorem 3) with inequality (2), we have the following variational principle for VEP$(g+h, K)$.

**Theorem 4.** *Let the assumptions of Theorem* 2 *and inequality* (2) *hold.* $x$ *is a solution of* VEP$(g+h, K)$ *if and only if* $G(x) = 0$. *If the solution set of* VEP$(g+h, K)$ *is nonempty, then the solution sets of* VEP$(g+h, K)$ *and* VMP$(g+h, \psi, K)$ *coincide.*

*Proof.* If $x$ is a solution of VEP$(g+h, K)$ then, by Theorem 2,

$$G(x) \not\leq_{\operatorname{int} P} 0.$$

From (2),

$$G(x) \leq_P 0.$$

These above inequalities imply that $G(x) = 0$. Next, if $G(x) = 0$ then by definition of $G(\cdot)$, we have

$$0 \not\geq_{\operatorname{int} P} 0 \ \leq_P \ -g(y, x) + h(x, y) + \psi(x, y), \quad \forall y \in K.$$

By Lemma 1, it follows that

$$-g(y, x) + h(x, y) + \psi(x, y) \not\leq_{\operatorname{int} P} 0 \quad \forall y \in K.$$

Follow the same lines of converse part of Theorem 2, we can have that $x$ is a solution of VEP$(g+h, K)$. This proves the first part of the theorem. If $x$ is a solution of VEP$(g+h, K)$, then $G(x) = 0$, and from inequality (2) follows that $x$ is a solution of VMP$(g+h, \psi, K)$. Then all solutions of VMP$(g+h, \psi, K)$ must satisfy $G(x) = 0$, and therefore are in the solution set of VEP$(g+h, K)$. This completes the proof.

We remark that the variational principle described in this paper is a generalization of variational principles described by Blum and Oettli [B-O1, B-O2], and Auchmuty [A].

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