

## A variational proof for the existence of a conformal metric with preassigned negative Gaussian curvature for compact Riemann surfaces of genus $> 1$

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**Abstract.** Given a smooth function  $K < 0$  we prove a result by Berger, Kazhdan and others that in every conformal class there exists a metric which attains this function as its Gaussian curvature for a compact Riemann surface of genus  $g > 1$ . We do so by minimizing an appropriate functional using elementary analysis. In particular for  $K$  a negative constant, this provides an elementary proof of the uniformization theorem for compact Riemann surfaces of genus  $g > 1$ .

**Keywords.** Uniformization theorem; Riemann surfaces; prescribed Gaussian curvature.

### 1. Introduction

In this paper we present a variational proof of a result by Berger [2], Kazhdan and Warner [6] and others, namely given an arbitrary smooth function  $K < 0$  we show that in every conformal class there exists a metric which attains this function as its Gaussian curvature for a compact Riemann surface of genus  $g > 1$ . In particular, this result includes the uniformization theorem of Poincaré [8] when  $K$  is a negative constant. In his proof Berger considers the critical points of a functional subject to the Gauss–Bonnet condition. He shows that the functional is bounded from below and uses the Friedrich’s inequality to complete the proof. The functional we choose is positive definite so that it is automatically bounded from below. Our proof is elementary, using Hodge theory, i.e., the existence of the Green’s operator for the Laplacian. Our proof could be useful for analysing the appropriate condition on  $K$  for a corresponding result for genus  $g = 1$  and  $g = 0$  [6, 10, 3], the two other cases considered by Berger, Kazhdan and Warner. Another variational proof of the uniformization theorem for genus  $g > 1$  can be found in a gauge-theoretic context in [5] which uses Uhlenbeck’s weak compactness theorem for connections with  $L^p$  bounds on curvature [9].

Let  $M$  be a compact Riemann surface of genus  $g > 1$  and let  $ds^2 = h dz \otimes d\bar{z}$  be a metric on  $M$  normalized such that the total area of  $M$  is 1. Let  $K < 0$ . We minimize the functional

$$S(\sigma) = \int_M (K(\sigma) - K)^2 e^{2\sigma} d\mu$$

over  $C^\infty(M, \mathbb{R})$ , where  $K(\sigma)$  stands for the Gaussian curvature of the metric  $e^\sigma ds^2$ , and  $d\mu = \frac{\sqrt{-1}}{2} h dz \wedge d\bar{z}$  is the area form for the metric  $ds^2$ . Using Sobolev embedding theorem

we show that  $S(\sigma)$  takes its absolute minimum on  $C^\infty(M)$  which corresponds to a metric on  $M$  of negative curvature  $K$ .

## 2. The main theorem

### 2.1

All notations are as in §1.

The functional  $S(\sigma) = \int_M (K(\sigma) - K)^2 e^{2\sigma} d\mu$  is non-negative on  $C^\infty(M, \mathbb{R})$ , so that its infimum

$$S_0 = \inf\{S(\sigma), \sigma \in C^\infty(M, \mathbb{R})\}$$

exists and is non-negative. Let  $\{\sigma_n\}_{n=1}^\infty \subset C^\infty(M, \mathbb{R})$  be a corresponding minimizing sequence,

$$\lim_{n \rightarrow \infty} S(\sigma_n) = S_0.$$

Our main result is the following

**Theorem 2.1** *Let  $M$  be a compact Riemann surface of genus  $g > 1$ . The infimum  $S_0$  is attained at  $\sigma \in C^\infty(M, \mathbb{R})$ , i.e. the minimizing sequence  $\{\sigma_n\}$  contains a subsequence that converges in  $C^\infty(M, \mathbb{R})$  to  $\sigma \in C^\infty(M, \mathbb{R})$  and  $S(\sigma) = 0$ . The corresponding metric  $e^\sigma h dz \otimes d\bar{z}$  is the unique metric on  $M$  of negative curvature  $K$ .*

### 2.2 Uniform bounds

Since  $\{\sigma_n\}$  is a minimizing sequence, we have the obvious inequality

$$S(\sigma_n) = \int_M (K_n - K)^2 e^{2\sigma_n} d\mu = \int_M \left( K_0 - \frac{1}{2} \Delta_h \sigma_n - K e^{\sigma_n} \right)^2 d\mu \leq m \quad (2.1)$$

for some  $m > 0$ , where we denoted by  $K_n$  the Gaussian curvature  $K(\sigma_n)$  of the metric  $e^{\sigma_n} h$  and by  $K_0$  that of the metric  $h$ , and used that

$$K_n = e^{-\sigma_n} \left( K_0 - \frac{1}{2} \Delta_h \sigma_n \right).$$

*Note.* Here  $\Delta_h = 4h^{-1}(\partial^2/\partial z \partial \bar{z})$  stands for the Laplacian defined by the metric  $h$  on  $M$ .

*Lemma 2.2.* *There exist constants  $C_1$  and  $C_2$  such that, uniformly in  $n$ ,*

$$(a) \quad \int_M (\Delta_h \sigma_n)^2 d\mu < C_1,$$

$$(b) \quad \int_M e^{2\sigma_n} d\mu < C_2.$$

*Proof.* By Minkowski inequality, and using (2.1), we get

$$\begin{aligned} \left[ \int_M \left( -\frac{1}{2} \Delta_h \sigma_n - K e^{\sigma_n} \right)^2 d\mu \right]^{1/2} &\leq \left[ \int_M \left( K_0 - \frac{1}{2} \Delta_h \sigma_n - K e^{\sigma_n} \right)^2 d\mu \right]^{1/2} \\ &+ \left[ \int_M (K_0)^2 d\mu \right]^{1/2} \leq m^{1/2} + c = C, \end{aligned}$$

so that

$$\frac{1}{4} \int_M (\Delta_h \sigma_n)^2 d\mu + \int_M K^2 e^{2\sigma_n} d\mu + \int_M \Delta_h \sigma_n e^{\sigma_n} K d\mu \leq C^2. \quad (2.2)$$

Let  $A_+^n = \{\Delta_h \sigma_n > 0\}$  and  $A_-^n = \{\Delta_h \sigma_n < 0\}$ .

$$\begin{aligned} \int_M \Delta_h \sigma_n K e^{\sigma_n} d\mu &= \int_{A_+^n} \Delta_h \sigma_n K e^{\sigma_n} d\mu + \int_{A_-^n} \Delta_h \sigma_n K e^{\sigma_n} d\mu \\ &\geq \min K \int_{A_+^n} (\Delta_h \sigma_n) e^{\sigma_n} d\mu + \max K \int_{A_-^n} (\Delta_h \sigma_n) e^{\sigma_n} d\mu \\ &= \tau_n \int_M (\Delta_h \sigma_n) e^{\sigma_n} d\mu \end{aligned}$$

where

$$\begin{aligned} \tau_n &= \frac{(\min K) \int_{A_+^n} (\Delta_h \sigma_n) e^{\sigma_n} d\mu + (\max K) \int_{A_-^n} (\Delta_h \sigma_n) e^{\sigma_n} d\mu}{\int_M (\Delta_h \sigma_n) e^{\sigma_n} d\mu} \\ &= -\tau_n \int_M |\partial_z \sigma_n|^2 e^{\sigma_n} d\mu, \quad \text{by Stoke's theorem.} \end{aligned}$$

Note that  $\tau_n \leq \max K < 0$ . Thus by (2.2), we get

$$\frac{1}{4} \int_M (\Delta_h \sigma_n)^2 d\mu + \int_M K^2 e^{2\sigma_n} d\mu - \tau_n \int_M |\partial_z \sigma_n|^2 e^{\sigma_n} d\mu \leq C^2. \quad (2.3)$$

Since each term is positive, the result follows.  $\square$

### 2.3. Pointwise convergence of zero mean-value part

Next, for  $\sigma \in C^\infty(M)$  denote by  $m(\sigma)$  its mean value,

$$m(\sigma) = \int_M \sigma d\mu,$$

and by  $\tilde{\sigma} = \sigma - m(\sigma)$  denote its zero-mean value part. For the minimizing sequence  $\{\sigma_n\}$  we denote the corresponding mean values by  $m_n$ . (Note: We had normalized the volume  $\int_M d\mu = 1$ .)

*Lemma 2.3.* *The mean-value-zero part  $\{\tilde{\sigma}_n\}_{n=1}^\infty$  of the minimizing sequence  $\{\sigma_n\}_{n=1}^\infty$  is uniformly bounded in the Sobolev space  $W^{2,2}(M)$ .*

*Proof.* By Hodge theory, there exists an operator  $G$  such that  $G\Delta_h = I - P$ , where  $I$  is the identity operator in  $L^2(M)$  and  $P$  is the orthogonal projection onto kernel of  $\Delta_h$ . We also know  $\Delta_h : W^{2,2} \rightarrow L^2$  boundedly and  $G : L^2 \rightarrow W^{2,2}$  is a bounded operator.

Now, by Lemma 2.2  $\{\Delta_h \sigma_n\}$  are bounded uniformly in  $L^2$ . Thus,  $\{G\Delta_h \sigma_n\}$  are bounded uniformly in  $W^{2,2}$ . But  $G\Delta_h \sigma_n = (I - P)\sigma_n = \tilde{\sigma}_n$ .

Now we can formulate the main result of this subsection.

#### PROPOSITION 2.4

*The sequence  $\{\tilde{\sigma}_n\}_{n=1}^\infty$  contains a subsequence  $\{\tilde{\sigma}_{l_n}\}_{n=1}^\infty$  with the following properties.*

- (a) The sequences  $\{\tilde{\sigma}_n\}_{n=1}^\infty$  and  $\{e^{\tilde{\sigma}_n+m_n}\}$  converge in  $C^0(M)$  topology to continuous functions  $\tilde{\sigma}$  and  $u$  respectively. Moreover,  $\tilde{\sigma} \in W^{2,2}(M)$ .
- (b) The subsequence  $\{\Delta_h \tilde{\sigma}_{l_n}\}$  converges weakly in  $L^2$  to  $f \doteq \Delta_h^{\text{distr}} \tilde{\sigma}$  — a distribution Laplacian of  $\tilde{\sigma}$ .
- (c) Passing to this subsequence  $\{\tilde{\sigma}_{l_n}\}$ , the following limits exist

$$\lim_{n \rightarrow \infty} \|\Delta_h \tilde{\sigma}_{l_n}\|_2 = \|\Delta_h^{\text{distr}} \tilde{\sigma}\|_2,$$

$$\lim_{n \rightarrow \infty} S(\sigma_{l_n}) = S_0 = \int_M (K_0 - \frac{1}{2} \Delta_h^{\text{distr}} \tilde{\sigma} - Ku)^2 d\mu,$$

where

$$\lim_{n \rightarrow \infty} e^{\tilde{\sigma}_{l_n}+m_n} = u.$$

In fact, the convergence in (b) is strong in  $L^2$ .

*Proof.* Part (a) follows from the Sobolev embedding theorem and Rellich lemma since, for  $\dim M = 2$ , the space  $W^{2,2}(M)$  is compactly embedded into  $C^0(M)$  (see, e.g. [1, 7]). Therefore the sequence  $\{\tilde{\sigma}_n\}$ , which, according to Lemma 2.3, is uniformly bounded in  $W^{2,2}(M)$ , contains a convergent subsequence in  $C^0(M)$ . Passing to this subsequence  $\{\tilde{\sigma}_{l_n}\}$  we can assume that there exists mean-value zero function  $\tilde{\sigma} \in C^0(M)$  such that

$$\lim_{n \rightarrow \infty} \tilde{\sigma}_{l_n} = \tilde{\sigma}.$$

Since  $\tilde{\sigma}_n$ 's are uniformly bounded in a Hilbert space  $W^{2,2}(M)$ , they weakly converge to  $s \in W^{2,2}(M)$  (after passing to a subsequence if necessary). The uniform limit coincides with  $s$  so that  $\tilde{\sigma} = s \in W^{2,2}(M)$ .

Moreover, since

$$m_n = \int_M \sigma_n d\mu \leq \int_M e^{2\sigma_n} d\mu,$$

then by part (b) of Lemma 2.2 the sequence  $\{m_n\}$  is bounded above. Passing to a subsequence, if necessary, we obtain that there exists  $u \in C^0(M)$  such that in the  $C^0(M)$  topology  $\lim_{n \rightarrow \infty} e^{\tilde{\sigma}_{l_n}+m_n} = u$ . The function  $u$  may be identically zero if  $m_n \rightarrow -\infty$ .

In order to prove (b), set  $\psi_n = \Delta_h \tilde{\sigma}_{l_n}$  and observe that, according to part (a) of Lemma 2.2, the sequence  $\{\psi_n\}$  is bounded in  $L^2$ . Therefore, passing to a subsequence, if necessary, there exists  $f \in L^2(M)$  such that

$$\lim_{n \rightarrow \infty} \int_M \psi_n g = \int_M f g$$

for all  $g \in L^2(M)$ . In particular, considering  $g \in C^\infty(M)$ , this implies  $f = \Delta_h^{\text{distr}} \tilde{\sigma}$ .

In order to prove (c) we use the following lemma.

*Lemma 2.5.* If a sequence  $\{\psi_n\}$  converges to  $f \in L^2$  in the weak topology, then

$$\lim_{n \rightarrow \infty} \|\psi_n\| \geq \|f\|.$$

Further  $\lim_{n \rightarrow \infty} \|\psi_n\| = \|f\|$  iff there is strong convergence.

*Proof.* The lemma follows from considering the following inequality:

$$\lim_{n \rightarrow \infty} \int (\psi_n - f)^2 d\mu \geq 0.$$

□

To continue with the proof of the proposition, suppose  $\lim_{n \rightarrow \infty} \|\psi_n\| > \|f\|$ . Using the definition of the functional, we have

$$\begin{aligned} S(\sigma_n) &= \int_M \left( K_0 - \frac{1}{2} \Delta_h \tilde{\sigma}_n - K e^{\tilde{\sigma}_n + m_n} \right)^2 d\mu \\ &= \frac{1}{4} \|\psi_n\|^2 + \|K_0 - K e^{\tilde{\sigma}_n + m_n}\|^2 - \int_M \psi_n (K_0 - K e^{\tilde{\sigma}_n + m_n}) d\mu. \end{aligned}$$

From parts (a) and (b) it follows that the sequence  $S(\sigma_n)$  converges to  $S_0$  and

$$\begin{aligned} S_0 &= \lim_{n \rightarrow \infty} S(\sigma_n) \\ &= \lim_{n \rightarrow \infty} \frac{1}{4} \|\psi_n\|^2 + \|K_0 - K u\|^2 - \int_M f(K_0 - K u) d\mu \\ &> \frac{1}{4} \|f\|^2 + \|K_0 - K u\|^2 - \int_M f(K_0 - K u) d\mu \\ &= \left\| -\frac{1}{2} f + K_0 - K u \right\|^2. \end{aligned}$$

We will show that this inequality contradicts that  $\{\sigma_n\}$  was a minimizing sequence, i.e. we can construct a sequence  $\{\tau + m_{l_n}\} \in C^\infty(M)$  such that  $S(\tau + m_{l_n})$  gets as close to  $\|-\frac{1}{2} f + K_0 - K u\|^2$  as we like.

Namely, for any  $\epsilon > 0$  we can construct, by the density of  $C^\infty$  in  $W^{2,2}$ , a function  $\tau \in C^\infty(M)$  approximating  $\tilde{\sigma} \in W^{2,2}$  such that  $\|\Delta_h \tau - f\| < \epsilon$  and  $\|4(v - u)\| < \epsilon/2$  where  $v = \lim_{n \rightarrow \infty} e^{\tau + m_{l_n}}$ . Since

$$S_\tau = \lim_{n \rightarrow \infty} S(\tau + m_{l_n}) = \left\| -\frac{1}{2} \Delta_h \tau + K_0 - K v \right\|^2,$$

we have

$$\left| \sqrt{S_\tau} - \left\| -\frac{1}{2} f + K_0 - K u \right\| \right| \leq \left\| \frac{1}{2} (f - \Delta_h \tau) - K(v - u) \right\| \leq \epsilon.$$

Now setting  $\delta = \sqrt{S_0} - \left\| -\frac{1}{2} f + K_0 - K u \right\| > 0$  and choosing  $\epsilon < \delta/2$ , and using  $\sqrt{S_\tau} \leq \left\| -\frac{1}{2} f + K_0 - K u \right\| + \epsilon$  we get  $\sqrt{S_\tau} < \sqrt{S_0} - \frac{\delta}{2}$ —a contradiction, since  $S_0$  was the infimum of the functional.

Thus,  $\lim_{n \rightarrow \infty} \|\Delta_h \tilde{\sigma}_n\| = \|f\|$ , so that, in fact, by Lemma 2.5, the convergence is in the strong  $L^2$  topology. This proves part (c).

#### 2.4 Convergence and the non-degeneracy

##### PROPOSITION 2.6

*The minimizing sequence  $\{\sigma_n\}_{n=1}^\infty$  contains a subsequence that converges in  $C^0(M)$  to a function  $\sigma \in C^0(M)$ , so that the resulting metric  $e^\sigma h$  is non-degenerate.*

*Proof.* Since  $\sigma_n = \tilde{\sigma} + m_n$ , by Proposition 2.4 and Lemma 2.3, it is enough to show that the sequence  $\{m_n\}$  is bounded below. Supposing the contrary and passing, if necessary, to a subsequence, we can assume that

$$\lim_{n \rightarrow \infty} m_n = -\infty,$$

so that, in notations of Proposition 2.4,  $u = 0$ . By Proposition 2.4(c) we get

$$S_0 = \lim_{n \rightarrow \infty} S(\sigma_n) = \int_M \left( K_0 - \frac{1}{2} \Delta_h^{\text{distr}} \tilde{\sigma} \right)^2 d\mu.$$

We shall show that this contradicts the fact that  $S_0$  is the infimum of the functional  $S$  and that  $\{\sigma_n\}$  is a minimizing sequence. First we have the following lemma.

*Lemma 2.7.* Let  $b = K_0 - \frac{1}{2} \Delta_h^{\text{distr}} \tilde{\sigma} \in L^2(M)$ , where  $\tilde{\sigma}_n \rightarrow \tilde{\sigma}$  and  $m_n \rightarrow -\infty$  as  $n \rightarrow \infty$ . Then

$$\int_M b \Delta_h \beta d\mu = 0$$

for all  $\beta \in C^\infty(M)$  and  $b \equiv -L$ , where  $L$  is a positive constant.

*Proof.* Consider  $G_n(t) = S(\sigma_n + t\beta) - S_0$ —a smooth function of  $t$  for a fixed  $\beta$ . Then by Proposition 2.4(c) we have

$$\begin{aligned} G(t) &= \lim_{n \rightarrow \infty} G_n(t) \\ &= \int_M \left( K_0 - \frac{1}{2} \Delta_h^{\text{distr}} (\tilde{\sigma} + t\beta) \right)^2 d\mu - \int_M \left( K_0 - \frac{1}{2} \Delta_h^{\text{distr}} \tilde{\sigma} \right)^2 d\mu, \end{aligned}$$

and  $G(t)$  is a smooth function of  $t$  for fixed  $\beta$ . Since  $S_0$  is the infimum of  $S$ , we have that  $G(t) \geq 0$  for all  $t$  and  $G(0) = 0$ . Therefore it follows that

$$\left. \frac{dG}{dt} \right|_{t=0} = 0$$

for all  $\beta \in C^\infty(M)$ . Straightforward computation yields

$$\left. \frac{dG}{dt} \right|_{t=0} = - \int_M b \Delta_h \beta d\mu.$$

Therefore,  $b$  satisfies the Laplace equation  $\Delta_h b = 0$  in a distributional sense and from elliptic regularity it follows that  $b$  is smooth. Thus  $b$  is harmonic and therefore is a constant. Finally, by the Gauss–Bonnet theorem, we have  $\int_M b d\mu = 4\pi(1 - g)$  and recalling that  $g > 1$ , we conclude that  $b = 4\pi(1 - g) = -L < 0$ .  $\square$

To complete the proof of the proposition, we get a contradiction as follows. By Lemma 2.7, we have that  $S_0 = \int_M (-L)^2 d\mu = L^2$  is the infimum of the functional. Since  $L > 0$ , and  $\{m_n\} \rightarrow -\infty$ , we consider  $\tau = \tilde{\sigma} + m_n$  and choose  $n$  large enough so that  $-Ke^\tau < L/2$ . We have

$$S(u) = \int_M \left( K_0 - \frac{1}{2} \Delta_h \tilde{\sigma} - Ke^\tau \right)^2 d\mu = \int_M (-L - Ke^\tau)^2 d\mu.$$

Then, since  $-L + \alpha < -L - Ke^\tau < -L/2$ , where  $\alpha > 0$  is the infimum of  $-Ke^\tau$ , we have  $(-L - Ke^\tau)^2 < (L - \alpha)^2$  so that  $S(\tau) < L^2$ —a contradiction.  $\square$

### 3. Smoothness and uniqueness

Here we complete the proof of the main Theorem 3.1.

#### PROPOSITION 3.1

The minimizing function  $\sigma \in C^0(M)$  is smooth and corresponds to the unique Kähler metric of negative curvature  $K$ .

*Proof.* Let  $b = (K_0 - \frac{1}{2}\Delta_h^{\text{distr}}\sigma - Ke^\sigma) \in L^2(M)$ ; according to Proposition 2.4(c) and Proposition 2.6,  $S_0 = \int_M b^2 d\mu$ . Set  $G(t) = S(\sigma + t\beta) - S_0$ , where  $\beta \in C^\infty(M)$ . Repeating arguments in the proof of Lemma 2.7, we conclude that  $G(t)$  for fixed  $\beta$  is smooth,  $G(0) = 0$  and  $G(t) \geq 0$  for all  $t$ . Therefore,

$$\left. \frac{dG}{dt} \right|_{t=0} = 0.$$

A simple calculation yields

$$\left. \frac{dG}{dt} \right|_{t=0} = \int_M (-b\Delta_h\beta - 2Ke^\sigma b\beta) d\mu.$$

Thus  $b \in L^2(M)$  satisfies, in a distributional sense, the following equation

$$-\Delta_h b - 2Ke^\sigma b = 0. \quad (3.1)$$

First, we will show that  $b = 0$  is the only weak  $L^2$  solution to eq. (3.1). Indeed, by elliptic regularity  $b$  is smooth, so that multiplying (3.1) by  $b$  and integrating over  $M$  using the Stokes formula, we get

$$\int_M db \wedge * db + \int_M b^2 e^\sigma d\mu = 0,$$

which implies that  $b = 0$ . Thus we have shown that  $S_0 = 0$ .

Second, equation  $b = 0$  for the minimizing function  $\sigma \in C^0(M)$  reads

$$\frac{1}{2}\Delta_h^{\text{distr}}\sigma = K_0 - Ke^\sigma \in C^0(M). \quad (3.2)$$

Therefore,  $\Delta_h^{\text{distr}}\sigma$  belongs to  $L^p(M)$  so that  $\sigma \in W^{2,p}$  for all  $p$ . By the Sobolev embedding theorem it follows that  $\sigma \in C^{1,\alpha}(M)$  for some  $0 < \alpha < 1$ . Therefore, the right hand side of eq. (3.2) actually belongs to the space  $C^{1,\alpha}(M)$ , and therefore  $\sigma \in C^{3,\alpha}(M)$  and so on. This kind of bootstrapping argument shows that  $\sigma$  is smooth [7].

The equation  $b \equiv 0$  satisfied by  $\sigma$  now translates to  $K(\sigma) \equiv K$ , where  $K(\sigma)$  is the Gaussian curvature of the metric  $e^\sigma h dz \otimes d\bar{z}$ ,  $\sigma \in C^\infty(M)$ .

The minimizing function  $\sigma$  is unique: here is the standard argument, which goes back to Poincaré. Let  $\eta$  be another minimizing function, which is smooth and also satisfies eq. (3.2)

$$\frac{1}{2}\Delta_h\eta = K_0 - Ke^\eta,$$

so that

$$\Delta_h(\sigma - \eta) = -2K(e^\sigma - e^\eta).$$

Multiplying this equation by  $\sigma - \eta$  and integrating over  $M$  with the help of Stokes formula, we get

$$-\int_M d\xi \wedge * d\xi = \int_M -2K(\sigma - \eta)(e^\sigma - e^\eta)d\mu,$$

where we set  $\xi = \sigma - \eta$ . Since  $-2K(\sigma - \eta)(e^\sigma - e^\eta) \geq 0$ , we conclude that  $d\xi = 0$  and, in fact,  $\xi = 0$ .

The proof of Theorem 3.1 is complete.

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