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On initial conditions for a boundary stabilized hybrid Euler–Bernoulli beam

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Abstract. We consider here small flexural vibrations of an Euler–Bernoulli beam with a lumped mass at one end subject to viscous damping force while the other end is free and the system is set to motion with initial displacement $y^0(x)$ and initial velocity $y^1(x)$. By investigating the evolution of the motion by Laplace transform, it is proved (in dimensionless units of length and time) that

$$\int_0^1 y_{xt}^2 \, \mathrm{d}x \leq \int_0^1 y_{xx}^2 \, \mathrm{d}x, \quad t > t_0,$$

where t_0 may be sufficiently large, provided that $\{y^0, y^1\}$ satisfy very general restrictions stated in the concluding theorem. This supplies the restrictions for uniform exponential energy decay for stabilization of the beam considered in a recent paper.

Keywords. Euler–Bernoulli beam equation; hybrid system; initial conditions; small deflection; exponential energy decay.

1. Introduction

In a recent paper, Gorain and Bose [2] investigated the possibility of stabilization of transverse vibrations of a hybrid system consisting of an Euler–Bernoulli beam held by a lumped mass movable hub attached to one of its ends. The beam is assumed to be initially set in vibration by a displacement y^0 and velocity y^1 in the transverse direction and stabilization is sought by applying viscous damping force to the moving lumped mass. The system equations for simplicity can be written in dimensionless form by suitably choosing the units of length and time. If $y(x, t)$ be the transverse displacement of a point of the beam distant x from the lumped mass at time t, the equations are [2]

$$y_{tt}(x, t) + y_{xxxx}(x, t) = 0, \quad 0 \leq x \leq 1, \; t \geq 0, \tag{1}$$

along the length of the beam, while at the lumped mass and free ends,

$$y_{xxx}(0, t) + \alpha y_{tt}(0, t) + \lambda y_t(0, t) = 0, \quad y_x(0, t) = 0, \quad t \geq 0, \tag{2}$$

$$y_{xx}(1, t) = 0, \quad y_{xxx}(1, t) = 0, \quad t \geq 0, \tag{3}$$

where α is the dimensionless mass of the lump and similarly λ the damping coefficient. The system is set to vibration with initial conditions

$$y(x, 0) = y^0(x), \quad y_t(x, 0) = y^1(x), \quad 0 \leq x \leq 1. \tag{4}$$

We note in (1)–(4) that without loss of generality we can assume

$$y^0(0) = 0. \tag{5}$$

Such hybrid systems for general $y^0(x)$ and $y^1(x)$ have been investigated in detail in search of uniform exponential decay of total energy (kinetic and potential) for proving stability of the process. However Littman and Marcus [5] and Chen and Zhou [1] have found by calculating the eigenvalues of their hybrid systems that uniform stabilization is not possible because infinitely large wave number k, during the passage of a wave along the beam are present in the general case. Rao [6] arrives at the same conclusion by applying semigroup theory to the evolving system.

In [2] it was noted that eq. (1) is arrived at by assuming that the beam remains approximately straight during vibration, precluding infinitely large wave numbers. From this observation, heuristically an additional condition was suggested, which in nondimensional form is

$$\int_0^1 y_{xt}^2 \, \mathrm{d}x \le \int_0^1 y_{xx}^2 \, \mathrm{d}x, \quad t > t_0, \tag{6}$$

where t_0 may be as large as we please. Subject to this condition, it was proved in [2], that uniform exponential decay of total energy indeed takes place.

The condition (6) places restrictions on the initial conditions $y^0(x)$, $y^1(x)$ from which the system evolves. It is the purpose of this paper to determine them by investigating the actual evolution of the system (1)–(5) by Laplace transformation in the complex frequency domain s and invoking the final value theorem for the system behaviour for t tending to infinity.

2. System evolution

Let the Laplace transform of $y(x, t)$ be

$$Y(x, s) = \int_0^\infty y(x, t)\mathrm{e}^{-st}\mathrm{d}t, \tag{7}$$

then according to the final value theorem, if s be complex (with x fixed) and $Y(x, s)$ be analytic in $\mathrm{Re}\{s\} \ge c, \; c < 0$,

$$\lim_{t\to\infty} y(x, t) = \lim_{s\to 0} sY(x, s) \tag{8}$$

and so we would be interested in the transformed quantities as $s \to 0$. The transformation of equations (1)–(4) in the usual way yield

$$Y_{xxxx}(x, s) + s^2 Y(x, s) = sy^0(x) + y^1(x), \tag{9}$$

with boundary conditions, using (5):

$$Y_{xxx}(0, s) + \alpha s^2 Y(0, s) + \lambda s Y(0, s) = \alpha y^1(0), \quad Y_x(0, s) = 0, \tag{10}$$

$$Y_{xx}(1, s) = 0, \quad Y_{xxx}(1, s) = 0. \tag{11}$$

In order to solve (9)–(11), we introduce 'wave number' k by the relation

$$s = -ik^2 : s^2 = -k^4. \tag{12}$$

The general solution of (9) is then

$$Y(x,-ik^2) = C_0 \sin kx + C_1 \cos kx + C_2 \sinh kx + C_3 \cosh kx + \frac{1}{2k^3}\int_0^x [-ik^2 y^0(\xi) + y^1(\xi)][\sin k(\xi - x) - \sinh k(\xi - x)]\,\mathrm{d}\xi. \quad (13)$$

For the differentiability of the particular solution of (9) represented by the integral in (13) we require that $y^0(x)$ and $y^1(x)$ are C^1 smooth. The boundary conditions (10), (11) yield for the coefficients C_0, C_1, C_2, C_3 the four equations

$$C_0 = -C_2, \quad (14a)$$

$$-k^2(\alpha k^2 + i\lambda)(C_1 + C_3) + 2k^3 C_2 = \alpha y^1(0), \quad (14b)$$

$$-C_1 \cos k + C_2(\sin k + \sinh k) + C_3 \cosh k = \frac{1}{2k^3}\int_0^1 [-ik^2 y^0(\xi) + y^1(\xi)][\sin k(\xi - 1) + \sinh k(\xi - 1)]\mathrm{d}\xi, \quad (14c)$$

$$C_1 \sin k + C_2(\cos k + \cosh k) + C_3 \sinh k = -\frac{1}{2k^3}\int_0^1 [-ik^2 y^0(\xi) + y^1(\xi)][\cos k(\xi - 1) + \cosh k(\xi - 1)]\mathrm{d}\xi. \quad (14d)$$

The exact solution of (14) can be explicitly written down by Cramer's rule. But here we are interested in the solution for large t, that is to say, for small s or k and so we expand the determinants formally in powers of k and do the same for the trigonometric and hyperbolic functions appearing in (13). Thus, restoring s in place of k defined in eq. (12) we obtain,

$$\begin{aligned} Y_x(x,s) = \frac{1}{4[\lambda + (\alpha+1)s + O(s^2)]}\Bigg[& -2I_1(s)(\lambda + s)x^2 \\ & + 2\left\{ I_2(s)[\lambda + (\alpha+1)s] + I_1(s)\left[\lambda - i + \left(\alpha + \frac{1}{2}\right)s\right] \right. \\ & \left. + \alpha y^1(0)\left(1 - \frac{is}{2}\right)\right\} x\left(1 - \frac{isx^2}{6}\right) \\ & + 2\left\{ I_2(s)[\lambda + (\alpha+1)s] + I_1(s)\left[\lambda + i + \left(\alpha + \frac{1}{2}\right)s\right] \right. \\ & \left. - \alpha y^1(0)\left(1 + \frac{is}{2}\right)\right\} x\left(1 + \frac{isx^2}{6}\right) + O(s^2)\Bigg] \\ & - \frac{1}{2}\int_0^x [sy^0(\xi) + y^1(\xi)][(\xi - x)^2 + O(s^4)]\mathrm{d}\xi, \end{aligned} \quad (15)$$

where

$$I_1(s) = \int_0^1 [sy^0(\xi) + y^1(\xi)]\mathrm{d}\xi, \quad I_2(s) = \int_0^1 (\xi - 1)[sy^0(\xi) + y^1(\xi)]\mathrm{d}\xi. \quad (16)$$

In §4 we shall prove that poles of $Y_x(x,s)$ for each x lie in $\mathrm{Re}\{s\} < c,\ c < 0$ when $\lambda > 0$. Hence, by the final value theorem of Laplace transform, we find that since $\lambda \neq 0$,

$$\lim_{t\to\infty} y_x(x,t) = \lim_{s\to 0} sY_x(x,s) = 0. \quad (17)$$

The limiting operation in (15) is essentially justified by expansion in powers of s therein and the assumed C^1 continuity of $y^0(x)$ and $y^1(x)$. The limit (17) means that in the presence of the viscous damping, as t becomes large, the beam approaches its original straight shape.

3. Validity of condition (6)

In order to prove that condition (6) holds for the motion, consider the functions $ty_{xx}(x,t)$ and $t^2 y_{xt}(x,t)$. The Laplace transforms of the two functions are respectively

$$-\frac{\partial}{\partial s}[Y_{xx}(x,s)] \quad \text{and} \quad \frac{\partial^2}{\partial s^2}[sY_x(x,s) - y_x^0(x)] = \frac{\partial^2}{\partial s^2}[sY_x(x,s)].$$

Hence by the final value theorem,

$$\lim_{t\to\infty} \frac{t^2 \int_0^1 y_{xt}^2 \mathrm{d}x}{\int_0^1 y_{xx}^2 \mathrm{d}x} = \lim_{s\to 0} \frac{\int_0^1 \left\{\frac{\partial^2}{\partial s^2}[sY_x(x,s)]\right\}^2 \mathrm{d}x}{\int_0^1 \left\{\frac{\partial}{\partial s}[Y_{xx}(x,s)]\right\}^2 \mathrm{d}x}. \tag{18}$$

The limit of the numerator in (18), from equations (15), (16) turns out to be

$$\int_0^1 \Big[2x\int_0^1 \xi y^0(\xi)\mathrm{d}\xi - x^2\int_0^1 y^0(\xi)\mathrm{d}\xi - \frac{1}{\lambda}\left(x - x^2 + \frac{x^3}{3}\right) \times \int_0^1 y^1(\xi)\mathrm{d}\xi - \frac{\alpha}{\lambda}y^1(0)\Big(x + \frac{x^3}{3}\Big) + \int_0^x y^0(\xi)(\xi - x)^2\mathrm{d}\xi\Big]^2 \mathrm{d}x, \tag{19}$$

while that of the denominator turns out to be

$$\frac{1}{4}\int_0^1 \Big[2\int_0^1 \xi y^0(\xi)\mathrm{d}\xi - 2(x+1)\int_0^1 y^0(\xi)\mathrm{d}\xi + \frac{1}{\lambda}(2\lambda - 1 + 2x - x^2) \times \int_0^1 y^1(\xi)\mathrm{d}\xi - \frac{\alpha}{\lambda}y^1(0)(1+x^2) + \int_0^x y^0(\xi)(\xi - x)\mathrm{d}\xi\Big]^2 \mathrm{d}x. \tag{20}$$

If the latter limit vanishes, it follows by differentiating twice that

$$y^0(x) = \frac{2}{\lambda}\Big[\int_0^1 y^1(x)\mathrm{d}x + \alpha y^1(0)\Big] = 0, \quad 0 \le x \le 1, \tag{21}$$

since $y^0(0) = 0$. If this is the case, (19) and (20) respectively become

$$\frac{1}{5}\Big[\frac{\alpha}{\lambda}y^1(0)\Big]^2 \quad \text{and} \quad \Big[\frac{\alpha}{\lambda}y^1(0)\Big]^2\Big[\Big(\lambda + \frac{1}{2}\Big)^2 + \frac{1}{12}\Big]. \tag{22}$$

Hence the limit in (18) exists finitely even in the case when the initial values $y^0(x)$ and $y^1(x)$ satisfy (21) together with the provision that $y^1(0) \neq 0$. This last condition means that the velocity at the end where viscous damping is applied should not vanish when the initial displacement is zero. Let the limit in (18) be $l \ge 0$. It then follows that given $\epsilon > 0$ however small, there exists t_0 such that

$$\frac{\int_0^1 y_{xt}^2 \mathrm{d}x}{\int_0^1 y_{xx}^2 \mathrm{d}x} < \frac{l+\epsilon}{t^2} < \frac{l+\epsilon}{t_0^2}, \quad \text{for } t > t_0.$$

Hence for $t > t_0 > \sqrt{l + \epsilon}$, the condition (6) must hold. Thus we have proved the following theorem.

Theorem. *Let $y(x, t)$ be the solution of the system* (1)–(5) *corresponding to the initial conditions $\{y^0(x), y^1(x)\}$ which are $C^1[0, 1]$ continuous. Then condition* (6) *holds, provided that if $y^0(x) = 0$ on $[0, 1]$ then, either $\int_0^1 y^1(x)\mathrm{d}x \neq -\alpha y^1(0)$ or $\int_0^1 y^1(x)\mathrm{d}x = -\alpha y^1(0) \neq 0$.*

4. Poles of $Y_x(x, s)$

When s is considered complex, $Y(x, s)$ given by (13) together with (12) has poles at those of the coefficients C_0, C_1, C_2, C_3. These are at zeroes of the determinant of the coefficients on the right hand side of the equations (14b)–(14d), satisfying the equation (in terms of k),

$$k^2\Big[k(\sin k \cosh k + \cos k \sinh k) + (\alpha k^2 + i\lambda)(1 + \cos k \cosh k)\Big] = 0. \quad (23)$$

When a differentiation of (13) is performed, $k = 0$ no longer remains a pole of $Y_x(x, s)$ as is reflected in (15). The poles of $Y_x(x, s)$ are thus the nonzero zeroes of (23). We investigate their domain by a method similar to that of Krall [4] as given in Gorain [3].

The zeroes of (23) result from (14b)–(14d) when the right hand sides are taken zero. In other words, they crop up from the boundary value problem (9)–(11) with the right hand sides set to zero:

$$Y_{xxxx}(0, s) + s^2 Y(x, s) = 0, \quad s = u + iv \neq 0, \quad (24)$$

$$Y_{xxx}(0, s) = -(\alpha s^2 + \lambda s)Y(0, s), \quad Y_x(0, s) = 0, \quad (25a)$$

$$Y_{xx}(1, s) = 0, \quad Y_{xxx}(1, s) = 0. \quad (25b)$$

If we multiply (24) by the complex conjugate Y^* and then take its conjugate, we obtain

$$Y^* Y_{xxxx} + s^2|Y|^2 = 0 \quad \text{and} \quad Y Y^*_{xxxx} + s^{*2}|Y|^2 = 0.$$

Subtracting one from the other and integrating from 0 to 1, we have

$$\left(s^2 - s^{*2}\right)\int_0^1 |Y|^2 \mathrm{d}x = \int_0^1 \left(Y Y^*_{xxxx} - Y^* Y_{xxxx}\right)\mathrm{d}x.$$

Integrating by parts and applying boundary conditions (25), we obtain from the above after simplification,

$$\left(s^2 - s^{*2}\right)\int_0^1 |Y|^2 \mathrm{d}x = -\left(s - s^*\right)\left[\alpha\left(s + s^*\right) + \lambda\right]|Y(0, s)|^2.$$

If now $s - s^* = 2iv \neq 0$, it follows that

$$u = -\frac{1}{2}\frac{\lambda|Y(0, s)|^2}{\int_0^1 |Y|^2\mathrm{d}x + \alpha|Y(0, s)|^2} < 0. \quad (26)$$

In (26) $u \neq 0$, since otherwise $Y(0, s) = 0$ and then (24), (25) yield $Y(x, s)$ identical to zero.

If $s - s^* = 2iv = 0$, we have $s = u$ and the boundary value problem (24), (25) becomes one of real value. Equation (24) then yields

$$YY_{xxxx} + u^2Y^2 = 0.$$

Integrating by parts from 0 to 1 and applying the boundary conditions (25) with u in place of s, we obtain since $Y(0, s) \neq 0$ as before,

$$u = -\frac{u^2\left[\int_0^1 Y^2\mathrm{d}x + \alpha\{Y(0,s)\}^2\right] + \int_0^1 Y_{xx}^2\mathrm{d}x}{\lambda\{Y(0.s)\}^2} < 0. \tag{27}$$

In (27) $u \neq 0$, since otherwise $\int_0^1 Y_{xx}^2\mathrm{d}x = 0$, which implies that $Y_{xx} = 0$, that is to say, $y_{xx} = 0$ on $0 \leq x \leq 1,\ t \geq 0$, meaning that the beam is not bent.

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