Proc. Indian Acad. Sci. (Math. Sci.), Vol. 111, No. 3, August 2001, pp. 293–318.

Uncertainty principles on two step nilpotent Lie groups

S K RAY

Stat-Math Unit, Indian Statistical Institute, 203 B.T. Road, Kolkata 700 035, India
Present Address: Department of Mathematics, Indian Institute of Technology, Kanpur 208 016, India
E-mail: res9601@isical.ac.in; skray@iitk.ac.in

Abstract. We extend an uncertainty principle due to Cowling and Price to two step nilpotent Lie groups, which generalizes a classical theorem of Hardy. We also prove an analogue of Heisenberg inequality on two step nilpotent Lie groups.

Keywords. Uncertainty principles; Hardy's theorem; two step nilpotent Lie groups; Heisenberg's inequality.

1. Introduction

As a meta-theorem in harmonic analysis, the uncertainty principles can be summarized as: *A nonzero function and its Fourier transform cannot both be sharply localized.* When *sharp localization* is interpreted as very rapid decay, this meta-theorem becomes the following theorem due to Hardy ([4]).

Theorem 1.1 (*Hardy*). *Let $f : \mathbb{R} \to \mathbb{C}$ be measurable and for all x, y*
(i) $|f(x)| \leq C\mathrm{e}^{-a\pi x^2}$,
(ii) $|\hat{f}(y)| \leq C\mathrm{e}^{-b\pi y^2}$,
where $C, a, b > 0$. If $ab > 1$ then $f = 0$ almost everywhere. If $ab = 1$ then $f(x) = C\mathrm{e}^{-a\pi x^2}$. If $ab < 1$ then there exist infinitely many linearly independent functions satisfying (i) *and* (ii).

Considerable attention has recently been paid to discover analogues of Hardy's theorem in the context of Lie groups ([28, 27, 5, 1, 16, 25, 24, 12, 8, 22]). Coming back to $\mathbb{R}$, we see that the decay conditions can be stated as $\|e_{a\pi} f\|_{L^\infty(\mathrm{R})} < \infty$ and $\|e_{b\pi} \hat{f}\|_{L^\infty(\mathrm{R})} < \infty$, where $e_k(x) = \mathrm{e}^{kx^2}$. So one reasonable question is to ask: what happens if $\|e_{a\pi} f\|_{L^p(\mathrm{R})} < \infty$ and $\|e_{b\pi} \hat{f}\|_{L^q(\mathrm{R})} < \infty$, where $1 \leq p, q < \infty$? The answer is given by the following theorem due to Cowling and Price ([6]).

Theorem 1.2 (*Cowling and Price*). *Let $f : \mathbb{R} \to \mathbb{C}$ be measurable and*
(i) $\|e_{a\pi} f\|_{L^p(\mathrm{R})} < \infty$,
(ii) $\|e_{b\pi} \hat{f}\|_{L^q(\mathrm{R})} < \infty$,
where $a, b > 0$ and $\min(p, q) < \infty$. If $ab \geq 1$ then $f = 0$ almost everywhere. If $ab < 1$ then there exist infinitely many linearly independent functions satisfying (i) *and* (ii).

The proof of the above theorem uses the following result (see [6]).

Lemma 1.1. *If $g : \mathbb{C} \to \mathbb{C}$ is an entire function and for $1 \leq p < \infty$*

(i) $|g(x+iy)| \leq A\mathrm{e}^{\pi x^2}$,
(ii) $\left(\int_{\mathbb{R}} |g(x)|^p \mathrm{d}x\right)^{1/p} < \infty$,
then $g = 0$.

The importance of Theorem 1.2 is that even if the *pointwise decay* is replaced by *average decay,* Hardy's uncertainty principle continues to be true. As expected, the case $ab > 1$ of Hardy's theorem follows trivially from that of Cowling and Price. Actually, if we drop the case $ab = 1$, then, on the real line (more generally on $\mathbb{R}^n$), the above theorems are equivalent (see [3]).

The following theorem, which follows as a corollary from a deep theorem of Beurling ([14]), also suggests another generalization of Hardy's theorem.

Theorem 1.3. *Let* $f : \mathbb{R} \to \mathbb{C}$ *be measurable and for all* x, y
(i) $|f(x)| \leq C\mathrm{e}^{-a\pi|x|^p}$,
(ii) $|\hat{f}(y)| \leq C\mathrm{e}^{-b\pi|y|^q}$,
where $C, a, b > 0$, $p^{-1} + q^{-1} = 1$, $1 < p, q < \infty$. *If* $(ap)^{1/p}(bq)^{1/q} > 2$, *then* $f = 0$ *almost everywhere.*

In this paper our aim is to get analogues of Theorems 1.2, 1.3 on connected, simply connected, two step nilpotent Lie groups (see [9, 3] for analogues of Theorem 1.2 on other groups). We also prove an analogue of Heisenberg's inequality on two step nilpotent Lie groups which was previously known only in the case of Heisenberg groups (see [29]).

This paper is organized as follows: in §2 we fix notation and describe some background material leading to a proof of the Plancherel theorem via the description of the Hilbert–Schmidt norm of the group Fourier transform, and in §3 we prove the proposed analogue of Theorem 1.2 and indicate a proof of Theorem 1.3. In §4 we prove an analogue of Heisenberg's inequality.

Finally we would like to point out that all the results except Theorem 3.2 are from the author's 1999 Ph.D. thesis of the Indian Statistical Institute.

2. Notation and background material

For a Lie algebra $\mathfrak{g}$ (we will always work with Lie algebras over $\mathbb{R}$), we define $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}^n = [\mathfrak{g}, \mathfrak{g}^{n-1}]$, $n \geq 1$.

DEFINITION 2.1

A Lie algebra $\mathfrak{g}$ is called *two step nilpotent* if $\mathfrak{g}^2 = 0$ and $\mathfrak{g}^1 \neq 0$. The connected simply connected Lie group G corresponding to such a $\mathfrak{g}$ is called a *two step nilpotent Lie group*.

We find it more convenient to look at a two step nilpotent Lie algebra in another way. Let $B : \mathbb{R}^{n-m} \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ be a nondegenerate, alternating, bilinear map. Let $\mathfrak{g} = \mathbb{R}^m \oplus \mathbb{R}^{n-m}$, we define

$$[(z, v), (z', v')] = (B(v, v'), 0), \tag{2.1}$$

where $z, z' \in \mathbb{R}^m$ and $v, v' \in \mathbb{R}^{n-m}$. Then $[.,.]$ is a Lie bracket and $\mathfrak{g}$ is a two step nilpotent Lie algebra with $\mathbb{R}^m$ as the center of $\mathfrak{g}$. If on $G = \mathbb{R}^m \oplus \mathbb{R}^{n-m}$ we define the product

$$(z, v).(z', v') = \left(z + z' + \frac{1}{2}B(v, v'), v + v'\right), \tag{2.2}$$

then G is a connected, simply connected, two step nilpotent Lie group with $\mathfrak{g}$ as its Lie algebra and $\exp : \mathfrak{g} \to G$ is the identity diffeomorphism. In this section we will first describe the effective unitary dual of a connected, simply connected, two step nilpotent Lie group $\mathfrak{g}$ following Kirillov theory. Our notations are standard and can be found in [7].

Let $\mathfrak{g}^*$ be the real dual of $\mathfrak{g}$. Then G acts on $\mathfrak{g}^*$ by the coadjoint action, that is $G \times \mathfrak{g}^* \to \mathfrak{g}^*$, $(g, l) \to g.l$ is given by

$$\begin{aligned}(g.l)(X) &= l(\mathrm{Ad}g^{-1}(X)), \qquad g \in G, l \in \mathfrak{g}^*, X \in \mathfrak{g}, \\ &= l(\mathrm{Ad}(\exp -Y)(X)), \qquad Y \in \mathfrak{g}, g = \exp Y, \\ &= l(\mathrm{e}^{\mathrm{ad}-y}(X)) \\ &= l(X) - l([Y, X]).\end{aligned}$$

Let $l \in \mathfrak{g}^*$. Then we denote $O_l =$ the coadjoint orbit of l. $B_l =$ the skew symmetric matrix corresponding to l, that is, given a basis $\{X_1, \ldots, X_m, X_{m+1}, \ldots, X_n\}$ of $\mathfrak{g}$ through the center (that is, $X_1, \ldots, X_m$ span the centre of $\mathfrak{g}$, we consider the matrix $B_l = (B_l(i, j)) = \big(l([X_i, X_j]\big)$. $r_l =$ The radical of the bilinear form B_l, that is,

$$r_l = \{X \in \mathfrak{g} : l([X, Y]) = 0 \quad \text{for all } Y \in \mathfrak{g}\}.$$

Clearly r_l is an ideal of $\mathfrak{g}$ and $\mathfrak{z}(= \mathbb{R}^m) \subset r_l$. $\tilde{r}_l = \mathrm{span}_{\mathrm{R}}\{X_{m+1}, \ldots, X_n\} \cap r_l$. $\tilde{B}_l = B_l \mid \mathbb{R}^{n-m} \times \mathbb{R}^{n-m}$ that is restriction of B_l on the complement of the center of $\mathfrak{g}$.

It follows trivially for two step nilpotent Lie groups that all the coadjoint orbits are hyperplanes ([17, 23]). In fact we have from the above, the following.

Theorem 2.1. *Let $l \in \mathfrak{g}^*$. Then $O_l = l + r_l^{\perp}$ where $r_l^{\perp} = \{h \in \mathfrak{g}^* : h \mid r_l = 0\}$.*

In particular, $l' \in O_l$ if and only if $r_l = r_{l'}$ and $l \mid r_l = l' \mid r_{l'}$.

Let $\mathfrak{g}$ be a two step nilpotent Lie algebra such that $\dim \mathfrak{g} = n$ with the basis $\mathcal{B} = \{X_1, \ldots, X_m, X_{m+1}, \ldots, X_n\}$ through the centre. Then B_l is the $n \times n$ matrix whose (i, j)th entry is $l([X_i, X_j])$, $1 \le i, j \le n$. Let $\mathcal{B}^* = \{X_1^*, \ldots, X_n^*\}$ be the dual basis of $\mathfrak{g}^*$. This is a Jordan–Hölder basis, that is, $\mathfrak{g}_j^* = \mathrm{span}_{\mathrm{R}}\{X_1^*, \ldots, X_j^*\}$ is $\mathrm{Ad}^*(G)$ stable for $1 \le j \le n$.

Let $l \in \mathfrak{g}^*$ and $X_i \in \mathcal{B}$.

DEFINITION 2.2

The term i is called a *jump index* for l if the rank of the $i \times n$ submatrix of B_l, consisting of the first i rows, is strictly greater than the rank of the $(i-1) \times n$ submatrix of B_l, consisting of the first $(i-1)$ rows.

Since an alternating bilinear form has even rank the number of jump indices must be even. The set of jump indices is denoted by $J = \{j_1, \ldots, j_{2k}\}$. Notice that $j_1 \ge m + 1$. The subset of $\mathcal{B}$ corresponding to J is then $\{X_{j_1}, \ldots, X_{j_{2k}}\}$. Notice that if i is a jump index then $\mathrm{rank} B_l^i = \mathrm{rank} B_l^{i-1} + 1$, where B_l^i is the submatrix of B_l consisting of first i rows.

Note 2.1. These jump indices depend on l and on the order of the basis as well. But ultimately we will restrict ourselves to 'generic linear functionals' and they will have the same jump indices.

Now we are going to spell out what we mean by *generic linear functionals*. This is also a basis dependent definition. We work with the basis $\mathcal{B}$ chosen above. Let $R_i(l) = \operatorname{rank} B_l^i$ and $R_i = \max\{R_i(l) : l \in \mathfrak{g}^*\}$.

DEFINITION 2.3

A linear functional $l \in \mathfrak{g}^*$ is called *generic* if $R_i(l) = R_i$ for all i, $1 \leq i \leq n$.

Let $\mathcal{U} = \{l \in \mathfrak{g}^* : l \text{ is generic}\}$. Since for any $l \in \mathfrak{g}^*$, we have $g.l \mid \mathfrak{z} = l \mid \mathfrak{z}$ where $g.l = l \circ \operatorname{Ad} g^{-1}$, we get $R_i(l) = R_i(g.l)$, $1 \leq i \leq n$ and hence,

(i) $\mathcal{U}$ is a G-invariant Zariski open subset of $\mathfrak{g}^*$. So $\mathcal{U}$ is union of orbits.

(ii) If j is a jump index for some $l \in \mathcal{U}$, then j is a jump index for all $l \in \mathcal{U}$.

(iii) Let $l \in \mathcal{U}$, then the number of jump indices for l is the same as the dimension of O_l (as a manifold). For, the rank of the matrix B_l is equal to the number of jump indices ($= 2k$, say) and the dimension of the radical r_l is the nullity of the matrix of B_l, which is $n - 2k$. Since $\mathfrak{g}/r_l$ is diffeomorphic to O_l (see [7]), we have $\dim O_l = 2k$.

(iv) Every orbit in $\mathcal{U}$ is of maximum dimension though not every maximum dimensional orbit may be in $\mathcal{U}$.

Note 2.2. If $l \in \mathfrak{g}^*$ is such that $\tilde{B}_l$ is an invertible matrix, then $r_l = \mathfrak{z}$ and then $m+1, \ldots, n$ are all jump indices and moreover

$$\mathcal{U} = \{l \in \mathfrak{g}^* : \; \tilde{B}_l \text{ is an invertible matrix}\}.$$

Clearly, if the codimension of $\mathfrak{z}$ in $\mathfrak{g}$ is odd then this cannot happen. Following [18] and [19], we call, the two step nilpotent Lie algebra a *MW algebra*, if there exists $l \in \mathfrak{g}^*$ such that $\tilde{B}_l$ is nondegenerate (or the corresponding matrix is invertible). For example, Heisenberg Lie algebras and $\mathfrak{f}_{2n,2}$, the free nilpotent Lie algebras of step two are MW algebras (see [2]).

Our aim is to parametrize the orbits in $\mathcal{U}$. We will see that they constitute a set of full Plancherel measure. We again describe some notation.

$$N = \{1, \ldots, m, n_1, \ldots, n_r\} \subset \{1, \ldots, n\}$$

is the complement of J in $\{1, \ldots, n\}$, $V_J = \operatorname{span}_{\mathbb{R}}\{X_{j_i} : 1 \leq i \leq 2k, j_i \in J\}$, $V_N = \operatorname{span}_{\mathbb{R}}\{X_1, \ldots, X_m, X_{n_i} : 1 \leq i \leq r, n_i \in N\}$, $V_J^* = \operatorname{span}_{\mathbb{R}}\{X_{j_1}^*, \ldots, X_{j_{2k}}^*\}$, $V_N^* = \operatorname{span}_{\mathbb{R}}\{X_1^*, \ldots, X_m^*, X_{n_i}^* : n_i \in N\}$, $\tilde{V}_N^* = \operatorname{span}_{\mathbb{R}}\{X_{n_i}^* : 1 \leq i \leq r\}$.

The following theorem shows that there exist a vector subspace of $\mathfrak{g}^*$ which intersects *almost every* orbit contained in $\mathcal{U}$ at exactly one point (see [7]). In the two step case one can easily prove it using Theorem 2.1 (see [23]).

Theorem 2.2. (i) *V_N^* intersects every orbit in $\mathcal{U}$ at a unique point.* (ii) *There exist a birational homeomorphism $\Psi : (V_N^* \cap \mathcal{U}) \times V_J^* \to \mathcal{U}$.*

Note 2.3. For each coadjoint orbit in $\mathcal{U}$, we choose their representatives from $V_N^* \cap \mathcal{U}$. Note that $V_N^* \cap \mathcal{U}$ can be identified with the cartesian product of $\tilde{V}_N^*$ and a Zariski open subset $\mathcal{U}'$ of $\mathfrak{z}^*$, where $\mathcal{U}' = \{l \in \mathfrak{z}^* : R_i(l) = R_i, 1 \leq i \leq m\}$.

We begin with a brief discussion of Kirillov theory, for details see [7]. Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. G acts on $\mathfrak{g}^*$ by the coadjoint action. Given any $l' \in \mathfrak{g}^*$ there exist a subalgebra $\mathfrak{h}_{l'}$ of $\mathfrak{g}$ which is maximal with respect to the property

$$l'([\mathfrak{h}_{l'}, \mathfrak{h}_{l'}]) = 0. \tag{2.3}$$

Thus we have a character $\chi_{l'} : \exp(\mathfrak{h}_{l'}) \to \mathbb{T}$ given by

$$\chi_{l'}(\exp X) = e^{2\pi i l'(X)}, X \in \mathfrak{h}_{l'}.$$

Let $\pi_{l'} = \mathrm{ind}_{\exp(\mathfrak{h}_{l'})}^{G} \chi_{l'}$. Then

(1) $\pi_{l'}$ is an irreducible unitary representation of G.
(2) If $\mathfrak{h}'$ is another subalgebra maximal with respect to the property $l'([\mathfrak{h}', \mathfrak{h}']) = 0$, then $\mathrm{ind}_{\exp(\mathfrak{h}'_l)}^{G} \chi_l \cong \mathrm{ind}_{\exp(\mathfrak{h}')}^{G} \chi_{l'}$.
(3) $\pi_{l_1} \cong \pi_{l_2}$ if and only if l_1 and l_2 belong to the same coadjoint orbit.
(4) Any irreducible unitary representation π of G is equivalent to π_l for some $l \in \mathfrak{g}^*$.

So we have a map $\kappa : \mathfrak{g}^*/\mathrm{Ad}^*(G) \to \hat{G}$, which is a bijection. A subalgebra corresponding to $l \in \mathfrak{g}^*$, maximal with respect to (2.3) is called a *polarization*. It is known that the maximality of $\mathfrak{h}$ with respect to (2.3) is equivalent to the following *dimension condition*

$$\dim \mathfrak{h} = \frac{1}{2}(\dim \mathfrak{g} + \dim r_l).$$

Now suppose $\mathfrak{g}$ is a two step nilpotent Lie algebra and $l \in \mathfrak{g}^*$. The following technique for construction of a polarization corresponding to l, seems to be standard: we consider the bilinear form $\tilde{B}_l$ on the complement of the center, we restrict $\tilde{B}_l$ on its nondegenerate subspace, then on that subspace we can choose a basis with respect to which $\tilde{B}_l$ is the canonical symplectic form. With a little modification the basis can be chosen to be orthonormal as well. This is essentially what was done to obtain a canonical polarization in [19, 2, 26, 21]. We will set down the basis change explicitly; our main ingredient for that is the following lemma.

Lemma 2.1. *Let $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a nondegenerate, alternating, bilinear form. Then there exists an orthonormal basis $\{X_i, Y_i : 1 \le i \le k\}$ of $\mathbb{R}^n$ such that $B(X_i, Y_j) = \delta_{i,j}\lambda_j(B)$, $B(X_i, X_j) = B(Y_i, Y_j) = 0$, $1 \le i, j \le k$, $n = 2k$ where $\pm i\lambda_j(B)$ are the eigenvalues of the matrix of B.*

As a consequence we have the following.

COROLLARY 2.2.1

Let $l \in \mathfrak{g}^*$. Then there exist an orthonormal basis

$$\{X_1, \ldots, X_m, Z_1(l), \ldots, Z_r(l), W_1(l), \ldots, W_k(l), Y_1(l), \ldots, Y_k(l)\} \tag{2.4}$$

of $\mathfrak{g}$ such that

(a) $r_l = \mathrm{span}_{\mathbb{R}}\{X_1, \ldots, X_m, Z_1(l), \ldots, Z_r(l)\}$.
(b) $l([W_i(l), Y_j(l)]) = \delta_{ij}\lambda_j(l)$, $1 \le i, j \le k$ *and*

$$l([W_i(l), W_j(l)]) = l([Y_i(l), Y_j(l)]) = 0, 1 \le i, j \le k.$$

(c) $\mathrm{span}_{\mathbb{R}}\{X_1, \ldots, X_m, Z_1(l), \ldots, Z_r(l), W_1(l), \ldots, W_k(l)\} = \mathfrak{h}$ *is a polarization for l.*

For a proof see [23]. We call the above basis an *almost symplectic basis*. Given $X \in \mathfrak{g}$ and a basis (2.4) we write

$$X = \sum_{j=1}^{m} x_j X_j(l) + \sum_{j=1}^{r} z_j Z_j(l) + \sum_{j=1}^{k} w_j W_j(l) + \sum_{j=1}^{k} y_j Y_j(l) \equiv (x, z, w, y).$$

Since we are going to use induced representations we need to describe nice sections of G/H and a G-invariant measure on G/H. In our situation H will always be a normal subgroup of G. We identify G and $\mathfrak{g}$ via the exponential map. Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$ containing $\mathfrak{z}$ and $H = \exp \mathfrak{h}$.

We take $\{X_1, \ldots, X_m, X_{m+1}, \ldots, X_{m+k}, \ldots, X_n\}$ a basis of $\mathfrak{g}$ such that

$$\mathfrak{z} = \mathrm{span}_{\mathbb{R}}\{X_1, \ldots, X_m\}, \quad \mathfrak{h} = \mathrm{span}_{\mathbb{R}}\{X_1, \ldots, X_m, X_{m+1}, \ldots, X_{m+k}\}.$$

If $L_g(x) = g^{-1}x$ and $R_g(x) = xg$, $x, g \in G$, then it is clear from the group multiplication that the Jacobian matrix for either of the transformations is upper triangular with diagonal entries 1. Thus we have the following lemma whose proof can be found in [7].

Lemma 2.2. *Let* $\mathfrak{g}$, $\mathfrak{h}$, $\{X_1, \ldots, X_m, X_{m+1}, \ldots, X_{m+k}, \ldots, X_n\}$ *be as before. Then*

(i) $dx_1 \ldots dx_n$ *is a left and right invariant measure on* G.

(ii) $\sigma : G/H \to G$ *given by*

$$\sigma\left(\exp\left(\sum_{i=1}^{n} t_i X_i\right) H\right) = \exp\left(\sum_{i=1}^{n-m-k} t_{m+k+i} X_{m+k+i}\right),$$

is a section for G/H.

(iii) $dx_{m+k+1} \ldots dx_n$ *is a* G*-invariant measure on* G/H.

Now we come to the construction of representations corresponding to $l \in V_N^* \cap \mathcal{U}$. Let $\dim r_l = m + r$ and $\dim O_l = 2k$ so $m + r + 2k = n$. We choose an almost symplectic basis (2.4) of $\mathfrak{g}$ corresponding to l and get hold of $\mathfrak{h}_l$ as in Corollary 2.2.1, c). On $H_l = \exp(\mathfrak{h}_l)$ we have the character $\chi_l : H_l \to \mathbb{T}$. Let $\pi_l = \mathrm{ind}_{H_l}^{G} \chi_l$. We do not use the standard model for the induced representation as given in chapter 2 of [7], rather using the continuous section σ given in Lemma 2.2.2 and computing the unique splitting of a typical group element

$$(x, z, w, y) = (0, 0, 0, y)\left(x - \frac{1}{2}[(0, 0, 0, y), (0, z, w, 0)], z, w, 0\right),$$

corresponding to σ, the representation π_l is realized on $L^2(\mathbb{R}^k)$ and is given by

$$\begin{aligned}&(\pi_l(x, z, w, y) f)(\bar{y}) \qquad f \in L^2(\mathbb{R}^k),\\ &= \mathrm{e}^{2\pi i (l(x) + l(z) + l(w) - \frac{1}{2}\sum_{i=1}^{k} y_i w_i \lambda_i(\ell) + \sum_{i=1}^{k} \bar{y}_i w_i \lambda_i(\ell))} f(\bar{y} - y),\end{aligned} \tag{2.5}$$

for $\bar{y} \in \mathbb{R}^k$.

DEFINITION 2.4

For $l \in \mathfrak{g}^*$ we define

$$Pf(l) = \sqrt{\det((B_l')_{js})}$$

called the *Pfaffian* of l, where $(B_l')_{is} = l([X_{j_i}, X_{j_s}])$, $X_{j_i}, X_{j_s} \in V_J$.

Note 2.4. If J is the set of jump indices for l, then B_l' is nondegenerate on V_J and then $Pf(l)$ is the Pfaffian of B_l' (see [15]). It is easy to show that

(a) $\det((B_l')_{is})$ is always a square of a polynomial and hence $Pf(l)$ is a homogeneous polynomial in $l \mid \mathfrak{z}$.

(b) $Pf(l) \neq 0$ if $l \in \mathcal{U}$ and is Ad^*G invariant.

We restrict our attention to the representations π_l for $l \in V_N^* \cap \mathcal{U}$ and, motivated by the example of the Heisenberg groups ask the following question: suppose $f \in L^1(G) \cap L^2(G)$. What is the relation between $\hat{f}(\pi_l)$ and $\mathcal{F}_1 f(l \mid \mathfrak{z}, v)$? Here $\hat{f}$ is the operator valued group Fourier transform, (z, v) are elements of the group with $z \in \mathfrak{z}$ and $v \in \mathrm{Span}_{\mathbb{R}}\{X_{m+1}, \ldots, X_n\}$ and $\mathcal{F}_1 f(l \mid \mathfrak{z}, v)$ means the partial (Euclidean) Fourier transform of f in the central variables at the point $l \mid \mathfrak{z}$.

In the case of the Heisenberg groups H_n, with Lie algebra

$$\mathfrak{h}_n = \mathrm{Span}_{\mathbb{R}}\{Z, W_1, \ldots, W_n, Y_1, \ldots, Y_n\}$$

with the only nontrivial Lie brackets $[W_i, Y_i] = Z$, $1 \leq i \leq n$, we have $V_N = \mathrm{Span}_{\mathbb{R}}\{Z\}$, $V_J = \mathrm{Span}_{\mathbb{R}}\{W_1, \ldots, Y_n\}$ and $V_N^* \cap \mathcal{U} = \{l \in \mathfrak{h}_n^* : l(Z) = \lambda \neq 0\}$. Then it can be proved easily (see [10]), that for $f \in L^1(H_n) \cap L^2(H_n)$ and $l \in V_N^* \cap \mathcal{U}$,

$$\|\hat{f}(\pi_l)\|_{HS}^2 = |\lambda|^{-n} \int_{\mathbb{R}^{2n}} |\mathcal{F}_1 f(\lambda, w, y)|^2 \mathrm{d}w \mathrm{d}y. \tag{2.6}$$

To find an analogue of (2.6), the most important thing is to find the Jacobian of a transformation which we are now going to describe.

Let $l \in \tilde{V}_N^* = \mathrm{Span}_{\mathbb{R}}\{X_{n_1}^*, \ldots, X_{n_r}^*\}$. Notice that for H_n and $F_{n,2}$, where n is even, $\tilde{V}_N^* = \{0\}$, so the transformation we are going to describe, appears only for those two step nilpotent Lie groups whose Lie algebras are not MW. Suppose $l_{n_i} = l(X_{n_i})$, $1 \leq i \leq r$; we also have $l(X_{j_i}) = 0$, $1 \leq i \leq 2k$. From $\tilde{B}_l$ we have constructed an orthonormal basis $\{Z_1(l), \ldots, Z_r(l), W_1(l), \ldots, W_k(l), Y_1(l), \ldots, Y_k(l)\}$ with respect to which the matrix of $\tilde{B}_l$ is of the following form

$$\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}, \tag{2.7}$$

where the $2k \times 2k$ matrix S is given by

$$\begin{pmatrix} & & & \lambda_1(l) & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_k(l) \\ -\lambda_1(l) & & & & & \\ & \ddots & & & 0 & \\ & & -\lambda_k(l) & & & \end{pmatrix}, \tag{2.8}$$

where $\lambda_i(l) > 0, 1 \leq i \leq k'$. Let $l(Z_i(l)) = \bar{l}_i, 1 \leq i \leq r$. We consider the map

$$\phi : \tilde{V}_N^*(\cong \mathbb{R}^r) \to \mathbb{R}^r$$
$$\phi(l_{n_1}, \ldots, l_{n_r}) = (\bar{l}_1, \ldots, \bar{l}_r). \tag{2.9}$$

Lemma 2.3. *The modulus of the Jacobian determinant of* ϕ *is given by*

$$|\det J_\phi| = \frac{|Pf(l)|}{\lambda_1(l)\lambda_2(l)\ldots\lambda_k(l)},$$

where J_ϕ *is the Jacobian matrix of* ϕ.

Proof. First we systematically describe the transformations which gave the almost symplectic basis. We restrict ourselves only to the complement of the center, because it is there that the change of basis takes place.

$$\begin{aligned} A_1 &: \{X_{m+1}, X_{m+2}, \ldots, X_n\} &\to\ & \{X_{n_1}, \ldots, X_{n_r}, X_{j_1}, \ldots, X_{j_{2k}}\} \\ A_2 &: \{X_{n_1}, \ldots, X_{n_r}, X_{j_1}, \ldots, X_{2k}\} &\to\ & \{\tilde{X}_{n_1}, \ldots, \tilde{X}_{n_r}, X_{j_1}, \ldots, X_{j_{2k}}\} \\ A_3 &: \{\tilde{X}_{n_1}, \ldots, \tilde{X}_{n_r}, X_{j_k}, \ldots, X_{j_{2k}}\} &\to\ & \{Z_1(l), \ldots, Z_r(l), W_1(l), \ldots, \\ &&& \quad W_k(l), Y_1(l), \ldots, Y_k(l)\}, \end{aligned}$$

where $\tilde{X}_{n_i} = X_{n_i} - \sum_{s=1}^{2k} c_s^i(l) X_{j_s}, 1 \leq i \leq r$, so that each $\tilde{X}_{n_i} \in r_l$. A_1 is just a rearrangement of basis and hence is given by an orthogonal matrix. A_2 is clearly given by a lower triangular matrix with diagonal entries equal to one. The matrix of A_3 looks like

$$\begin{pmatrix} A' & C' \\ 0 & D' \end{pmatrix},$$

where A' is a $r \times r$ matrix, C' is a $r \times 2k$ matrix and D' is a $2k \times 2k$ matrix, because A_3 is obtained from the following operations:

(i) Gram–Schmidt orthogonalization of $\{\tilde{X}_{n_i} : 1 \leq i \leq r\}$.

(ii) Finding the orthogonal complement of the span of $\{\tilde{X}_{n_i} : 1 \leq i \leq r\}$.

(iii) Choosing an almost symplectic basis on the nondegenerate subspace of $\tilde{B}_l$.

Notice that for $l \in \tilde{V}_N$, $l(X_{j_i}) = 0, 1 \leq i \leq 2k$; thus $l(\tilde{X}_{n_i}) = l(X_{n_i}), 1 \leq i \leq r$. Hence

$$|\det J_\phi| = |\det A'|.$$

Since $|\det A_1 . \det A_2 . \det A_3| = 1$, we have $|\det A_3| = 1$. But

$$|\det A_3| = |\det A'||\det D'|.$$

So $|\det J_\phi| = |\det D'|^{-1}$. If we write $\tilde{B}_l$ in terms of the basis $\{\tilde{X}_{n_1}, \ldots, \tilde{X}_{n_r}, X_{j_1}, \ldots, X_{j_{2k}}\}$, then the matrix of $\tilde{B}_l$ looks like

$$\begin{pmatrix} 0 & 0 \\ 0 & B_l' \end{pmatrix},$$

where $(B_l')_{is} = l([X_{j_i}, X_{j_s}])$. Thus clearly $|\det B_l'| = |Pf(l)|^2$. Because of A_3 the above matrix changes to

$$\begin{pmatrix} 0 & 0 \\ 0 & D'B_l'(D')^t \end{pmatrix}$$

which is nothing but the matrix in (2.7). So

$$|\det D'|^2 = \frac{|\lambda_1(l)\dots\lambda_k(l)|^2}{|Pf(l)|^2} \quad\Longrightarrow\quad |\det D'| = \frac{|\lambda_1(l)\dots\lambda_k(l)|}{|Pf(l)|}.$$

Thus $|\det J_\phi| = \dfrac{|Pf(l)|}{|\lambda_1(l)\dots\lambda_k(l)|}$ as claimed.

Now we come to the analogue of (2.6). Given $f \in L^1(G)\cap L^2(G)$ and $\pi \in \hat{G}$ the so called group Fourier transform at π is the bounded linear transformation (realized on the Hilbert space $\mathcal{H}_\pi$) given by

$$\langle \hat{f}(\pi)\xi, \eta\rangle = \int_G f(g)\langle \pi(g^{-1})\xi, \eta\rangle \mathrm{d}\mu(g), \qquad \xi, \eta \in \mathcal{H}_\pi.$$

We recall, for $l \in V_N^*\cap\mathcal{U}$, the almost symplectic basis (2.4) and because of the orthonormal basis change, $\mathrm{d}x\mathrm{d}z\mathrm{d}w\mathrm{d}y$ is the normalized Haar measure on G we started with, where

$$(x, z, w, y) = \sum_{i=1}^m x_i X_i + \sum_{i=1}^r z_i Z_i(l) + \sum_{i=1}^k w_i W_i(l) + \sum_{i=1}^k y_i Y_i(l).$$

The representation π_l corresponding to l is now given by (2.5). Let $\mathrm{d}l_{n_1}\dots\mathrm{d}l_{n_r}$ denote the usual Lebesgue measure on $\tilde{V}_N^*$ (after we identify $\tilde{V}_N^*$ with $\mathbb{R}^r$ through the basis $\{X_{n_1}^*,\dots,X_{n_r}^*\}$).

Theorem 2.3. *Let $f \in L^1(G)\cap L^2(G)$. Then*

$$\begin{aligned}&|Pf(l)|\int_{\tilde{V}_N^*} \|\hat{f}(\pi_l)\|_{HS}^2 \mathrm{d}l_{n_1}\dots\mathrm{d}l_{n_r}\\ &= \int_{\mathbb{R}^{r+2k}} |\mathcal{F}_1 f(l_1,\dots,l_m,x_{n_1},\dots,x_{n_r},u,v)|^2 \mathrm{d}x_{n_1}\dots\mathrm{d}v \qquad (2.10)\end{aligned}$$

for almost every $l \in V_N^*\cap\mathcal{U}$, where

$$\begin{aligned}&\mathcal{F}_1 f(l_1,\dots,l_m,x_{n_1},\dots,x_{n_r},u,v)\\ &= \int_{\mathbb{R}^m} f(x_1,\dots,x_m,x_{n_1},\dots,x_{n_r},u,v)\mathrm{e}^{-2\pi i\Sigma_{j=1}^m l_j x_j}\mathrm{d}x_1\dots\mathrm{d}x_m\end{aligned}$$

and $l(X_j) = l_j$, $1 \le j \le m$.

Proof. Let $\phi \in L^2(\mathbb{R}^k)$. Then from (2.5),

$$\begin{aligned}&(\hat{f}(\pi_l)\phi)(\bar{y})\\ &= \int_{\mathbb{R}^{m+r+2k}} f(x,z,w,y)(\pi_l(-x,-z,-w,-y)\phi)(\bar{y})\mathrm{d}x\mathrm{d}z\mathrm{d}w\mathrm{d}y\\ &= \int_{\mathbb{R}^{m+r+2k}} f(x,z,w,y)\mathrm{e}^{2\pi i[-l(x)-l(z)-\Sigma_{j=1}^k w_j\bar{y}_j\lambda_j(l)-(1/2)\Sigma_{j=1}^k w_j y_j\lambda_j(l)]}\\ &\quad\times\phi(\bar{y}+y)\mathrm{d}x\mathrm{d}z\mathrm{d}w\mathrm{d}y\\ &= \int_{\mathbb{R}^{r+m+2k}} f(x,z,w,y-\bar{y})\mathrm{e}^{2\pi i[-l(x)-l(z)-\Sigma_{j=1}^k w_j\bar{y}_j\lambda_j(l)-(1/2)\Sigma_{j=1}^k w_j(y_j-\bar{y}_j)\lambda_j(l)]}\\ &\quad\times\phi(y)\mathrm{d}x\mathrm{d}z\mathrm{d}w\mathrm{d}y \qquad \text{(by the change of variable } y' = y+\bar{y})\end{aligned}$$

$$= \int_{\mathbb{R}^{r+m+2k}} f(x,z,w,y-\bar{y}) \mathrm{e}^{2\pi i[-l(x)-l(z)-(1/2)\Sigma_{j=1}^{k} w_j y_j \lambda_j(l) - (1/2)\Sigma_{j=1}^{k} w_j \bar{y}_j \lambda_j(l)]}$$
$$\times \phi(y)\mathrm{d}x\mathrm{d}z\mathrm{d}w\mathrm{d}y$$
$$= \int_{\mathbb{R}^{r+m+2k}} f(x,z,w,y-\bar{y}) \mathrm{e}^{-2\pi i l(x)} \mathrm{e}^{-2\pi i l(z)} \mathrm{e}^{-\pi i \Sigma_{j=1}^{k}(y_j+\bar{y}_j) w_j \lambda_j(l)}$$
$$\times \phi(y)\mathrm{d}x\mathrm{d}z\mathrm{d}w\mathrm{d}y.$$

Let

$$K_l^f(y,\bar{y}) = \int_{\mathbb{R}^{m+r+k}} f(x,z,w,y-\bar{y}) \mathrm{e}^{-2\pi i l(x)} \mathrm{e}^{-2\pi i l(z)} \mathrm{e}^{-\pi i \Sigma_{j=1}^{k}(y+\bar{y})\lambda_j(l) w_j} \mathrm{d}x\mathrm{d}z\mathrm{d}w.$$

Since $f \in L^1(G) \cap L^2(G)$, it follows that $K_l^f \in L^2(\mathbb{R}^k \times \mathbb{R}^k)$ for almost every $l \in \tilde{V}_N^* \cap \mathcal{U}$. Let $l \mid \mathfrak{z} = (l_1, \dots, l_m)$ and $l \mid \mathrm{span}_{\mathbb{R}}\{Z_1(l), \dots, Z_r(l)\} = (\bar{l}_1, \dots, \bar{l}_r)$. Then

$$K_l^f(y,\bar{y}) = \mathcal{F}_{123} f\left(l_1, \dots, l_m, \bar{l}_1, \dots, \bar{l}_r, \frac{y_1+\bar{y}_1}{2}\lambda_1(l), \dots, \frac{y_k+\bar{y}_k}{2}\lambda_k(l), y-\bar{y}\right),$$

where $\mathcal{F}_{123}$ stands for the partial Fourier (Euclidean) transform in the variables x, z, w. Thus $\hat{f}(\pi_l)$ is a Hilbert–Schmidt operator on $L^2(\mathbb{R}^k)$ with the kernel K_l^f. Hence

$$\begin{aligned} \|\hat{f}(\pi_l)\|_{HS}^2 &= \int_{\mathbb{R}^{2k}} |k_l^f(y,\bar{y})|^2 \mathrm{d}y\mathrm{d}\bar{y} \\ &= \int_{\mathbb{R}^{2k}} \left|\mathcal{F}_{123} f\left(l_1, \dots, \bar{l}_r, \frac{y_1+\bar{y}_1}{2}\lambda_1(l), \dots, \frac{y_k+\bar{y}_k}{2}\right.\right. \\ &\qquad \left.\left. \times \lambda_k(l), y-\bar{y}\right)\right|^2 \mathrm{d}y\mathrm{d}\bar{y}. \end{aligned}$$

If we do the change of variables

$$\begin{aligned} u_j &= \frac{y_j+\bar{y}_j}{2}\lambda_j(l), \quad 1 \le j \le k, \\ v_j &= y_j - \bar{y}_j, \quad 1 \le j \le k, \end{aligned}$$

then the modulus of the Jacobian determinant is $|\lambda_1(l)\dots\lambda_k(l)|$ and the above integral reduces to

$$|\lambda_1(l)\dots\lambda_k(l)|^{-1} \left(\int_{\mathbb{R}^{2k}} |\mathcal{F}_{123} f(l_1, \dots, l_m, \bar{l}_1, \dots \bar{l}_r, u, v)|^2 \mathrm{d}u\mathrm{d}v\right),$$

where $u = (u_1, \dots, u_k)$ and $v = (v_1, \dots, v_k)$. By applying the Euclidean Plancherel theorem in the variable u we get

$$\|\hat{f}(\pi_l)\|^2 = |\lambda_1(l)\dots\lambda_k(l)|^{-1} \int_{\mathbb{R}^{2k}} |\mathcal{F}_{12} f(l_1, \dots, l_m, \bar{l}_1, \dots, \bar{l}_r, u, v)|^2 \mathrm{d}u\mathrm{d}v.$$

If we integrate both sides of the above equation on $\tilde{V}_N^*$ with respect to the usual Lebesgue measure and use change of variables by the map ϕ defined in (2.9), we get

$$\begin{aligned} &\int_{\tilde{V}_N^*} \|\hat{f}(\pi_l)\|_{HS}^2 \mathrm{d}l_{n_1} \dots \mathrm{d}l_{n_r} \\ &= |\lambda_1(l)\dots\lambda_k(l)|^{-1} \frac{|\lambda_1(l)\dots\lambda_k(l)}{|Pf(l)|} \\ &\quad \times \int_{\mathbb{R}^{r+2k}} |\mathcal{F}_{12} f(l_1, \dots, l_m, l_{n_1}, \dots, l_{n_r}, u, v)|^2 \mathrm{d}l_{n_1} \dots \mathrm{d}l_{n_r} \mathrm{d}u\mathrm{d}v. \end{aligned}$$

Then by applying the Euclidean Plancherel theorem on the variables $(l_{n_1}, \ldots, l_{n_r}) \in \mathbb{R}^r$ we get

$$|Pf(l)| \int_{\tilde{V}_N^*} \|\hat{f}(\pi_l)\|_{HS}^2 \mathrm{d}l_{n_1} \ldots \mathrm{d}l_{n_r}$$
$$= \int_{\mathbb{R}^{r+2k}} |\mathcal{F}_1 f(l_1, \ldots, l_m, x_{n_1}, \ldots, x_{n_r}, u, v)|^2 \mathrm{d}x_{n_1} \ldots \mathrm{d}x_{n_r} \mathrm{d}u \mathrm{d}v.$$

This completes the proof.

Theorem 2.4 (*Plancherel theorem*). *For* $f \in L^1(G) \cap L^2(G)$

$$\int_{V_N^* \cap \mathcal{U}} \|\hat{f}(\pi_l)\|_{HS}^2 |Pf(l)| \mathrm{d}l = \|f\|_{L^2(G)}^2,$$

where $\mathrm{d}l$ *is the standard Lebesgue measure on* $V_N^* (\cong \mathbb{R}^{m+r})$ *with respect to the basis* $\{X_1^*, \ldots, X_m^*, X_{n_1}^*, \ldots, X_{n_r}^*\}$.

Proof. Regarding $V_N^* \cap \mathcal{U}$ as the Cartesian product of $\mathcal{U}'$ and $\mathbb{R}^r$ as in Note 2.3, we integrate both sides of (2.10) with respect to the standard Lebesgue measure on $\mathfrak{z}^*$ (upon identification with $\mathbb{R}^m$ via the basis $\{X_1^*, \ldots, X_m^*\}$) to get

$$\int_{V_N^* \cap \mathcal{U}} \|\hat{f}(\pi_l)\|_{HS}^2 |Pf(l)| \mathrm{d}l$$
$$= \int_{\mathcal{U}'} \left(|Pf(l)| \int_{\tilde{V}_N^*} \|\hat{f}(\pi_l)\|_{HS}^2 \mathrm{d}l_{n_1} \ldots \mathrm{d}l_{n_r} \right) \mathrm{d}l_1 \ldots \mathrm{d}l_m$$
$$= \int_{\mathcal{U}'} \left(\int_{\mathbb{R}^{r+2k}} |\mathcal{F}_c f(l_1, \ldots, l_m, x_{n_1}, \ldots, x_{n_r}, u, v)|^2 \mathrm{d}x_{n_1} \ldots \mathrm{d}v \right) \mathrm{d}l_1 \ldots \mathrm{d}l_m \quad \text{(by (2.10))}$$
$$= \int_{\mathbb{R}^m} \int_{\mathbb{R}^{r+2k}} |f(x_1, \ldots, x_m, x_{n_1}, \ldots, x_{n_r}, u, v)|^2 \mathrm{d}x_1 \ldots \mathrm{d}x_m \mathrm{d}x_{n_1} \ldots \mathrm{d}x_{n_r} \mathrm{d}u \mathrm{d}v,$$

by using the Euclidean Plancherel theorem in the outer integral ($\mathcal{U}'$ is a set of full Lebesgue measure in $\mathfrak{z}^*$). The last integral is, of course, $\|f\|_{L^2(G)}^2$ and the proof is complete.

Note 2.5. The situation is simpler if we consider the case of MW groups. In this case $V_N^* \cap \mathcal{U} \subseteq \mathfrak{z}^*$ is Zariski open and for $l \in \mathcal{U} \subseteq \mathfrak{z}^*$, the representation π_l is given by

$$(\pi_l(x, z, y) f)(\bar{y}) = \mathrm{e}^{2\pi i [l(x) + \Sigma_{j=1}^k \bar{y}_j \bar{w}_j \lambda_j(l) - (1/2) \Sigma_{j=1}^k y_j w_j \lambda_j(l)]} f(\bar{y} - y),$$

where $\bar{y} \in \mathbb{R}^k$, $f \in L^2(\mathbb{R}^k)$ and $\dim \mathfrak{g}/\mathfrak{z} = 2k$. Then it follows from the calculations done in theorem (2.3) that

$$\|\hat{f}(\pi_l)\|_{HS}^2 = \frac{1}{|\lambda_1(l) \ldots \lambda_k(l)|} \int_{\mathbb{R}^{2k}} |\mathcal{F}_1 f(l_1, \ldots, l_m, u, v)|^2 \mathrm{d}u \mathrm{d}v.$$

Clearly $|\lambda_1(l) \ldots \lambda_k(l)| = |Pf(l)|$, since $\tilde{B}_l$ is nondegenerate. The Plancherel theorem again follows from integrating both sides on $U \subseteq \mathfrak{z}^*$. *So the change of variables through the map* ϕ *is not needed for MW groups.*

Let $\mathfrak{g}$ be a two step nilpotent Lie algebra with a basis $\mathcal{B}$ as before. Now we consider elements of $\mathfrak{g}$ as left invariant differential operators acting on $C^\infty(G)$ where the action is given by

$$X(f)(g) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f(g \exp tX).$$

We define

$$\mathcal{L} = -\sum_{i=1}^{n-m} X_{m+i}^2 \tag{2.11}$$

and as on the Heisenberg groups, call it the sub-Laplacian of G.

Given an irreducible, unitary representation π of G, we look at the matrix functions of π given by

$$\begin{aligned} &\phi_{u,v}^\pi : G \to \mathbb{C}, \qquad u, v \in \mathcal{H}_\pi \\ &\phi_{u,v}^\pi(g) = \langle \pi(g)u, v\rangle. \end{aligned} \tag{2.12}$$

Our aim is to find: which matrix functions of representations are joint eigenfunctions of $\mathcal{L}$ and $\{X_i : 1 \le i \le m\}$?

Given $\pi \in \hat{G}$ and $X \in \mathfrak{g}$, we have

$$\mathrm{d}\pi(X)(u) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \pi(\exp tX)u, \tag{2.13}$$

where u, v are C^∞ vectors for π. If $A \in \mathcal{U}(\mathfrak{g})$ then it follows that

$$A\langle \pi(g)u, v\rangle = \langle \pi(g)\mathrm{d}\pi(A)u, v\rangle$$

(see [7]). Thus if u is an eigenvector for $\mathrm{d}\pi(A)$ then $\phi_{u,v}^\pi$ is an eigenfunction for A. Since for $1 \le i \le m$, $X_i \in \mathcal{Z}(\mathcal{U}(\mathfrak{g}))$, the center of the universal enveloping algebra, then $\mathrm{d}\pi(X_i)$ acts as a scalar (see [7]) and hence $\phi_{u,v}^\pi$ is an eigenfunction for X_i for any u, v. Thus our job reduces to finding the eigenfunctions of $\mathrm{d}\pi(\mathcal{L})$ which are also matrix functions of π. Looking at the case of the Heisenberg groups and the group $F_{2n,2}$ (see [26]) it is reasonable to expect that $\mathrm{d}\pi(\mathcal{L})$ is closely related to the Hermite operator and, indeed, that is the case.

We use on G the exponential coordinates given by the above chosen basis. Given $x = \sum_{i=1}^n x_i X_i$ and $x' = \sum_{i=1}^n x_i' X_i$ denote

$$[x, x']_p = \langle [x, x'], X_p\rangle, \quad 1 \le p \le m,$$

where $\langle .,. \rangle$ is the Euclidean inner product on $\mathfrak{g}$ for which $\{X_i : 1 \le i \le n\}$ is an orthonormal basis. Then it follows that, for $1 \le i \le m$,

$$(X_i f)(x) = \frac{\partial f}{\partial x_i}(x), \tag{2.14}$$

and for $m + 1 \le i \le n$,

$$(X_i f)(x) = \left(\frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{j=1}^m [x, X_i]_j \frac{\partial}{\partial x_j} \right) f(x). \tag{2.15}$$

Now we start with a representation $\pi_l \in \hat{G}$ such that $l \mid \mathfrak{z} \neq 0$. We get hold of an almost symplectic basis (2.4) with $\dim r_l = m + r$ and $\dim O_l = 2k$, so $n = 2k + m + r$. The representations π_l are realized on $L^2(\mathbb{R}^k)$ and are given by (2.5). Using the explicit description (2.5), it is easy to see that $C^\infty(\pi_l) = \mathcal{S}(\mathbb{R}^k)$, the Schwartz class functions on $\mathbb{R}^k$. By direct calculation we find the effect of applying $\mathrm{d}\pi_l$ on the elements of the almost symplectic basis, which in turn describes $\mathrm{d}\pi_l(\mathcal{L})$.

Lemma 2.4. *For $\phi \in \mathcal{S}(\mathbb{R}^k)$ and $\xi \in \mathbb{R}^k$*

(i) $\mathrm{d}\pi_l(Z_j(l))\phi(\xi) = 2\pi i \bar{l}_j \phi(\xi), \qquad 1 \leq j \leq r.$
(ii) $\mathrm{d}\pi_l(W_j(l))\phi(\xi) = 2\pi i \xi_j \lambda_j(l)\phi(\xi), \qquad 1 \leq j \leq k.$
(iii) $\mathrm{d}\pi_l(Y_j(l))\phi(\xi) = -\frac{\partial\phi}{\partial\xi_j}(\xi), \qquad 1 \leq j \leq k.$
(iv) $\mathrm{d}\pi_l(\mathcal{L})\phi(\xi) = \{4\pi^2 \sum_{j=1}^r \bar{l}_j^2 + L_l\}\phi(\xi)$, *where*

$$L_l = \sum_{j=1}^k \left(-\frac{\partial^2}{\partial \xi_j^2} + 4\pi^2 \lambda_j(l)^2 \xi_j^2 \right).$$

Because of (iv) now it is easy to describe the eigenfunctions of $\mathrm{d}\pi_l(\mathcal{L})$. Let $\mu(l) = 4\pi^2 \sum_{j=1}^r \bar{l}_j^2$. Then $\mathrm{d}\pi_l(\mathcal{L}) = \mu(l) + L_l$, and $\mu(l) \geq 0$. If ϕ is an eigenfunction of L_l with eigenvalue $c(l)$, then ϕ is an eigenfunction of $\mathrm{d}\pi_l(\mathcal{L})$ with eigenvalue $c(l) + \mu(l)$. Again, if ϕ_j^l is an eigenfunction of $-\frac{\partial^2}{\partial x^2} + 4\pi^2\lambda_j(l)^2 x^2$ on $\mathbb{R}$, then clearly

$$\phi^l(\xi_1, \ldots, \xi_k) = \phi_i^l(\xi_1) \ldots \phi_k^l(\xi_k)$$

is an eigenfunction of L_l. Since for $s \in \mathbb{N}$, the sth normalized Hermite function h_s is an eigenfunction of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2$ with eigenvalue $2s + 1$, it is clear that

$$h_s^l(x) = (2\pi\lambda_j(l))^{\frac{1}{4}} h_s(\sqrt{2\pi}\lambda_j(l)^{\frac{1}{2}} x)$$

is an eigenfunction of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 4\pi^2\lambda_j(l)^2x^2$ with eigenvalue $2\pi\lambda_j(l)(2s+1)$ and also $\|h_s^l\|_2 = 1$. So for $(\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ we define

$$h_\alpha^l(\xi_1, \ldots, \xi_k) = \Pi_{j=1}^k h_{\alpha_j}^l(\xi_j), \tag{2.16}$$

where

$$h_{\alpha_j}^l(\xi_j) = (2\pi\lambda_j(l))^{\frac{1}{4}} h_{\alpha_j}(\sqrt{2\pi}\lambda_j(l)^{\frac{1}{2}}\xi_j).$$

Then

$$L_l(h_\alpha^l) = \left(\sum_{j=1}^k 2\pi\lambda_j(l)(2\alpha_j + 1) \right) h_\alpha^l. \tag{2.17}$$

Thus

$$\mathrm{d}\pi_l(\mathcal{L})(h_\alpha^l) = \left(\mu(l) + \sum_{j=1}^k 2\pi\lambda_j(l)(2\alpha_j + 1) \right) h_\alpha^l. \tag{2.18}$$

Now we state a mild generalization of Theorem 1.2, which follows from Lemma 2.3 of [20].

Theorem 2.5. *Suppose* $f : \mathbb{R}^n \to \mathbb{C}$ *be a measurable function such that*
(i) $\int_{\mathbb{R}^n} e^{pa\pi \|x\|^2} |f(x)|^p dx < \infty$,
(ii) $\int_{\mathbb{R}^n} e^{qb\pi \|y\|^2} |\hat{f}(y)|^q |Q(y)|^r dy < \infty$,
where $a, b > 0$, Q *is a polynomial and* $r > 0$ *is any real number. If* $ab \geq 1$ *then* $f = 0$.

3. Extensions of Hardy's theorem

The principal result in this section is the analogue of the Theorem 1.2 for two step nilpotent Lie groups. Along the way we also talk about the analogue of Theorem 1.3. Hardy's theorem for Heisenberg groups was proved in [28] and its L^p-analogue (Theorem 1.2) and the analogue of Theorem 1.3 was proved in [3]. An analogue of Hardy's theorem on two step nilpotent Lie groups was proved in [1].

Remark 3.1. Our treatment in this section tacitly assumes that G is not MW. For the case of MW groups the treatment needs only obvious modifications using the description of $\|\hat{f}(\pi_l)\|_{HS}$ given in Note 2.5.

In the case of Heisenberg groups, Hardy's theorem and Cowling–Price theorem actually reduce to the corresponding problems on the center of the group by an application of (2.6). Two step nilpotent Lie groups having reasonable analogue of (2.6) in Theorem 2.3, it is expected that the same technique may work here also; and it does, as we shall show presently. Since we are going to talk about exponential decay of the group Fourier transform, we need a growth parameter on the dual, where usual exponential makes sense, but that has been addressed in §1. In our parametrization the dual is essentially a vector subspace (actually a Zariski open subset of that subspace) of $\mathfrak{g}^*$, which is good enough for us.

Let $\mathfrak{g}$ be a two step nilpotent Lie algebra with basis $\mathcal{B}$ as before. G is the corresponding connected, simply connected, Lie group. We write elements of $\mathfrak{g}$ (as well as G) by $(x, v) \equiv \sum_{i=1}^{m} x_i X_i + \sum_{i=1}^{n-m} v_i X_{m+i}$. The set $V_N^* \cap \mathcal{U}$ serves as the effective dual (that is, it is a set of full Plancherel measure in $\hat{G}$) of G and we put Euclidean norm there such that $\{X_1^*, \ldots, X_m^*, X_{n_i}^* : 1 \leq i \leq r\}$ is an orthonormal basis. We write elements of V_N^* as

$$(\lambda, \gamma) \equiv \sum_{i=1}^{m} \lambda_i X_i^* + \sum_{i=1}^{r} \gamma_i X_{n_i}^*.$$

To prove an analogue of Theorem 1.2, we need the following trivial lemma.

Lemma 3.1. *Let G be a two step nilpotent Lie group. Then there exists a constant C such that*

$$\|(x, v).(x_1, v_1)^{-1}\| \geq \|(x, v)\| - \|(x_1, v_1)\| - C\|(x, v)\|\|(x_1, v_1)\|, \qquad (3.1)$$

for all $(x, v), (x_1, v_1) \in G$.

Now we come to the proposed analogue of Theorem 1.2.

Theorem 3.1. *Let* $f \in L^1(G) \cap L^2(G)$ *satisfy*
(i) $\int_G e^{pa\pi \|(x,v)\|^2} |f(x, v)|^p dx dv < \infty$,
(ii) $\int_{V_N^*} e^{qb\pi \|(\lambda,\gamma)\|^2} \|\hat{f}(\pi_{\lambda,\gamma})\|_{HS}^q |Pf(\lambda)| d\lambda d\gamma < \infty$,
where $1 \leq p \leq \infty$ *and* $2 \leq q < \infty$. *If* $ab > 1$, *then* $f = 0$ *almost everywhere.*

Proof. We first prove the case $p = \infty$ and later, use this result for the case $1 \leq p < \infty$.

Case 1. $p = \infty$. In this case we interpret (i) as

$$|f(x, v)| \leq A\mathrm{e}^{-a\pi \|(x,v)\|^2}. \tag{3.2}$$

We define

$$\tilde{f}(x, v) = \overline{f(-x, v)},$$

and

$$h(x) = \int_{\mathsf{R}^{n-m}} (f_v * \tilde{f}_v)(x)\mathrm{d}v, \tag{3.3}$$

where $f_v(x) = f(x, v)$ and $*$ is the convolution on $\mathbb{R}^m$. Since $f \in L^1(G)$, $h \in L^1(\mathbb{R}^m)$ and the Euclidean Fourier transform of h is given by

$$\begin{aligned} \hat{h}(\lambda) &= \int_{\mathsf{R}^m} h(x)\mathrm{e}^{-2\pi i\langle\lambda,x\rangle}\mathrm{d}x \\ &= \int_{\mathsf{R}^{n-m}} |\mathcal{F}_1 f(\lambda, v)|^2\mathrm{d}v \\ &= |Pf(\lambda)| \int_{\tilde{V}_N^*} \|\hat{f}(\pi_{\lambda,\gamma})\|_{HS}^2\mathrm{d}\gamma \qquad \text{(by (2.10))}. \end{aligned} \tag{3.4}$$

Now writing $\mathrm{e}^x = \exp x$,

$$\begin{aligned} |h(x)| &\leq \int_{\mathsf{R}^{n-m}} |(f_v * \tilde{f}_v)(x)|\mathrm{d}v \\ &\leq A^2 \int_{\mathsf{R}^n} \exp(-a\pi[2\|v\|^2 + \|x-y\|^2 + \|y\|^2])\mathrm{d}y\mathrm{d}v \\ &\leq A^2 \exp\left(-a\pi\frac{\|x\|^2}{2}\right) \int_{\mathsf{R}^n} \exp\left(-a\pi\left[2\|v\|^2 + 2\left(\|y\| - \frac{\|x\|}{2}\right)^2\right]\right)\mathrm{d}y\mathrm{d}v \\ &\leq A_1\mathrm{e}^{-(a'\pi/2)\|x\|^2}, \end{aligned} \tag{3.5}$$

where $a' < a$ with $a'b > 1$ (the integral in the last line but one being a polynomial in $\|x\|$). Choosing $b' < b$ such that $a'b' > 1$ we have, on the other hand,

$$\begin{aligned} &\int_{\mathsf{R}^m} \exp\left(\frac{q}{2}\pi 2b'\|\lambda\|^2\right)|\hat{h}(\lambda)|^{q/2}\mathrm{d}\lambda \\ &= \int_{\mathsf{R}^m}\left(\int_{\tilde{V}_N^*}\exp(2b'\pi\|\gamma\|^2)\|\hat{f}(\pi_{(\lambda,\gamma)})\|_{HS}^2\exp(-2b'\pi\|\gamma\|^2)\mathrm{d}\gamma\right)^{q/2} \\ &\qquad \times \exp(q\pi b'\|\lambda\|^2)|Pf(\lambda)|^{q/2}\mathrm{d}\lambda \\ &\leq \int_{\mathsf{R}^m}\exp(qb'\pi\|\lambda\|^2)\left\{\left(\int_{\tilde{V}_N^*}\exp\left(\frac{q}{2}2b'\pi\|\gamma\|^2\right)\|\hat{f}(\pi_{\lambda,\gamma})\|_{HS}^q\mathrm{d}\gamma\right)^{2/q}\right. \\ &\qquad \left.\times\left(\int_{\tilde{V}_N^*}\exp(-2b'\pi\alpha\|\gamma\|^2)\mathrm{d}\gamma\right)^{1/\alpha}\right\}^{q/2}|Pf(\lambda)|^{q/2}\mathrm{d}\lambda \\ &\qquad \text{(by Hölder's inequality, where } 2/q + 1/\alpha = 1) \end{aligned}$$

$$= B \int_{V_N^*} \exp(qb\pi \|(\lambda, \gamma)\|^2) \|\hat{f}(\pi_{(\lambda,\gamma)})\|_{HS}^q$$
$$\times \{\exp((b' - b)\pi \|(\lambda, \gamma)\|^2) |Pf(\lambda)|^{q/2}\} \mathrm{d}\lambda \mathrm{d}\gamma < \infty \quad \text{(by (ii))}. \tag{3.6}$$

Since $(a'/2)2b' = a'b' > 1$, by 3.5, 3.6 and Theorem 1.2 for the case $p = \infty$ and $q/2$ (which is ≥ 1 as $q \geq 2$) we get that $h = 0$ almost everywhere. Thus $\|\hat{f}(\lambda, \gamma)\|_{HS} = 0$ for almost every (λ, γ) and hence $f = 0$ almost everywhere by the Plancherel theorem.

Case 2. $p < \infty$. Let $e_k(x, v) = \mathrm{e}^{k\|(x,v)\|^2}$ for $k \in \mathbb{R}^+$. Suppose $g \in C_c(G)$ is such that supp $g \subset \{(x_1, v_1) : \|(x_1, v_1)\| \leq \frac{1}{m}\}$, where $m \in \mathbb{N}$. We choose $(x, v) \in G$ with $\|(x, v)\| > 1$. Thus, if $(x_1, v_1) \in$ supp g we have $\|(x_1, v_1)\| \leq \|(x, v)\|/m$ and hence by Lemma 3.1,

$$\|(x, v)(x_1, v_1)^{-1}\| \geq \|(x, v)\| \left(1 - \frac{d}{m}\right), \tag{3.7}$$

where $d = 1 + C$. Thus for $(x, v) \in G$ with $\|(x, v)\| > 1$ we have

$$\begin{aligned} &(e_{a\pi}|f| * |g|)(x, v) \\ &= \int_{\mathrm{supp} g} \mathrm{e}^{a\pi\|(x,v)(x_1,v_1)^{-1}\|} |f((x, v)(x_1, v_1)^{-1})||g(x_1, v_1)| \mathrm{d}x_1 \mathrm{d}v_1 \\ &\geq \mathrm{e}^{a\pi(1-(d/m))^2\|(x,v)\|^2} (|f| * |g|)(x, v) \quad \text{(by (3.7))}. \end{aligned} \tag{3.8}$$

By (i) we have that $e_{a\pi}|f|$ is a L^p function ($p < \infty$) on G and $g \in C_c(G)$, thus $e_{a\pi}|f| * |g|$ is a continuous function vanishing at infinity. Thus from (3.8) we have that

$$|(f * g)(x, v)| \leq \beta \mathrm{e}^{-a\pi(1-(d/m))^2\|(x,v)\|^2}$$

for all $(x, v) \in G$ with Euclidean norm greater than 1. By continuity of $f * g$ we have

$$|(f * g)(x, v)| \leq \beta \mathrm{e}^{-a\pi(1-(d/m))^2\|(x,v)\|^2}, \tag{3.9}$$

for all $(x, v) \in G$ (possibly with a different constant). Since

$$\begin{aligned} \|(\widehat{f * g})(\pi_{(\lambda,\gamma)})\|_{HS} &\leq \|\hat{g}(\pi_{(\lambda,\gamma)})\|_{Op} \|\hat{f}(\pi_{(\lambda,\gamma)})\|_{HS} \\ &\leq \|g\|_{L^1(G)} \|\hat{f}(\pi_{(\lambda,\gamma)})\|_{HS}, \end{aligned}$$

from (ii) we get that

$$\int_{V_N^*} \mathrm{e}^{qb\pi\|(\lambda,\gamma)\|^2} \|(\widehat{f * g})(\pi_{(\lambda,\gamma)})\|_{HS}^q |Pf(\lambda)| \mathrm{d}\lambda \mathrm{d}\gamma < \infty. \tag{3.10}$$

We choose m so large that $ab(1 - (d/m))^2 > 1$. Then by (3.9) and (3.10) we are reduced to case 1. Hence $f * g = 0$ almost everywhere. Now by choosing g from an approximate identity we get $f = 0$ almost everywhere. This completes the proof.

Note 3.1. For general two step nilpotent Lie groups we are unable to answer the case $q < 2$. But if G is a MW group then we have a complete answer, as is shown in the following theorem.

Theorem 3.2. *Let G be a connected, simply connected, two step nilpotent Lie group which is MW. Let* $f \in L^1(G) \cap L^2(G)$. *Suppose that for* $a, b > 0$ *and* $\min(p, q) < \infty$

(i) $\int_G e^{pa\pi\|(z,v)\|^2}|f(z,v)|^p dz dv < \infty$,
(ii) $\int_{V_N^*} e^{qb\pi l^2}\|\hat{f}(\pi_l)\|_{HS}^q |Pf(l)| dl < \infty$.

Then
(a) *If* $q \geq 2$, *then* $f = 0$ *for* $ab > 1$.
(b) *If* $1 \leq q < 2$ *then* $f = 0$ *if* $ab \geq 1$.

Proof. Part (a) is essentially in Theorem 3.1. So we prove (b). In this case, for $f \in L^1(G) \cap L^2(G)$ we have

$$\|\hat{f}(\pi_l)\|_{HS} = |Pf(l)|^{-1}\int_{\mathsf{R}^{2n}} |\mathcal{F}_1 f(l,z)|^2 dz,$$

(see Note 2.5). Starting from (ii) we have

$$\begin{aligned}
&\int_{V_N^*} e^{qb\pi\|l\|^2}\|\hat{f}(\pi_l)\|_{HS}^q |Pf(l)| dl \\
&= \int_{V_N^*} e^{qb\pi\|l\|^2}\left(|Pf(l)|^{-1}\int_{\mathsf{R}^{2n}} |\mathcal{F}_1 f(l,v)|^2 dv\right)^{\frac{q}{2}} |Pf(l)| dl \\
&= \int_{V_N^*}\left(\int_{\mathsf{R}^{2n}} g(l,v)^{\frac{2}{q}} dv\right)^{\frac{q}{2}} d\mu(l) \\
&\qquad \text{(where } g(l,v) = |\mathcal{F}_1 f(l,v)|^q \text{ and } d\mu(l) = e^{qb\pi\|l\|^2}|Pf(l)|^{(1-\frac{q}{2})}dl) \\
&\geq \left(\int_{\mathsf{R}^{2n}}\left(\int_{V_N^*} g(l,v) d\mu(l)\right)^{\frac{2}{q}} dv\right)^{\frac{q}{2}} \qquad \text{(by Minkowski's inequality).}
\end{aligned}$$

Thus for almost every v, $\int_{V_N^*} g(l,v) d\mu(l) < \infty$, that is

$$\int_{V_N^*} e^{qb\pi\|l\|^2}|\mathcal{F}_1 f(l,v)|^q |Pf(l)|^{(1-\frac{q}{2})} dl < \infty. \tag{3.11}$$

But from (i) it follows that for almost every v,

$$\int_{\mathsf{Z}} e^{pa\pi\|z\|^2}|f(z,v)|^p dz < \infty. \tag{3.12}$$

Thus for almost every v, the function $f(.,v)$ satisfies the condition of Theorem 2.5 and hence for $ab \geq 1$, $f = 0$ after all.

Going back to connected, simply connected, two step nilpotent Lie groups G, we observe that the same technique using the functions $\tilde{f}$ and h, as in Theorem 3.1, yields the following theorem.

Theorem 3.3. *Let* $f : G \to \mathbb{C}$ *be a measurable function. Suppose*
(i) $|f(x,v)| \leq Cg(v)e^{-a\pi\|x\|^p}$,
(ii) $\|\hat{f}(\pi_{\lambda,\gamma})\|_{HS} \leq Ch(\gamma)e^{-b\pi\|\lambda\|^q}$,
where $C > 0$, $p \geq 2$, $1/p + 1/q = 1$ *and* g, h *are nonnegative functions with* $g \in L^1(\mathbb{R}^{n-m}) \cap L^2(\mathbb{R}^{n-m})$ *and* $h \in L^1(\mathbb{R}^r) \cap L^2(\mathbb{R}^r)$. *If* $(ap)^{1/p}(bq)^{1/q} > 2$, *then* $f = 0$ *almost everywhere.*

4. Heisenberg's inequality

The classical inequality of Heisenberg for L^2 functions on $\mathbb{R}$ says that

$$\left(\int_{\mathsf{R}} |x|^2 |f(x)|^2 \mathrm{d}x\right)^{1/2} \left(\int_{\mathsf{R}} |y|^2 |\hat{f}(y)|^2 \mathrm{d}y\right)^{1/2} \geq C\|f\|_2^2, \tag{4.1}$$

where $\hat{f}$ is defined by

$$\hat{f}(y) = \int_{\mathsf{R}} f(x)\mathrm{e}^{-2\pi i x}\mathrm{d}x$$

and C is a constant independent of f.

In this section our aim is to extend the version of Heisenberg's inequality proved in [29] for the Heisenberg groups to all connected, simply connected, step two nilpotent Lie groups. Two other variants of Heisenberg's inequality on Heisenberg groups are available in [13] and [28], but since these results use the existence of rotations on Heisenberg groups, it is not clear how, without the notion of rotation, one should proceed to extend them to a general two step nilpotent Lie group (see [2]).

We state (4.1) in a slightly different way. Let $\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ be the Laplacian on $\mathbb{R}^n$. Then $(\widehat{\Delta f})(y) = 4\pi^2\|y\|^2\hat{f}(y)$ for any Schwartz class function on $\mathbb{R}^n$. We may relate Δ to the character $\gamma_y(x) = \mathrm{e}^{2\pi i y.x}$ of $\mathbb{R}^n$ by $\mathrm{d}\gamma_y\left(\frac{\partial}{\partial x_j}\right) = 2\pi i y_j$, and hence $\mathrm{d}\gamma_y(\Delta) = 4\pi^2\|y\|^2$. Thus we have

$$(\widehat{\Delta f})(y) = \mathrm{d}\gamma_y(\Delta)\hat{f}(y).$$

Since $\mathrm{d}\gamma_y(\Delta)$ is a positive, self adjoint operator, it has a (visible) square root, which is *multiplication by* $2\pi\|y\|$. Thus we define

$$(\widehat{\Delta^{\frac{1}{2}} f})(y) = 2\pi\|y\|\hat{f}(y) = (\mathrm{d}\gamma_y(\Delta))^{\frac{1}{2}}\hat{f}(y),$$

for all Schwartz class functions on $\mathbb{R}^n$. Since the Fourier transform is an isomorphism on Schwartz class functions, the operator $(\Delta)^{\frac{1}{2}}$ is defined completely. Then we can restate (4.1) as

$$\left(\int_{\mathsf{R}^n} \|x\|^2 |f(x)|^2 \mathrm{d}x\right)^{1/2} \left(|(\widehat{\Delta^{\frac{1}{2}}(f)})(y)|^2 \mathrm{d}y\right)^{1/2} \geq C\|f\|_2^2, \tag{4.2}$$

for all f of Schwartz class on $\mathbb{R}^n$, where C is a constant independent of f. It is (4.2), whose analogue on connected, simply connected, two step nilpotent Lie groups we are looking for. As in the case of Heisenberg groups, here also the proof, in principle, is close to the proof on $\mathbb{R}^n$ (see [11]) having the same basic ingredients, namely, integration by parts, Cauchy–Schwartz inequality and the Plancherel theorem.

We call *a function* f *on* G *a Schwartz class function if* $f \circ \exp$ *is a Schwartz class function on* $\mathfrak{g}$. We denote the Schwartz class functions by $\mathcal{S}(G)$.

Replacing $\Delta^{\frac{1}{2}}$ by $\mathcal{L}^{\frac{1}{2}}$, the main result of this section is as follows.

Theorem 4.1. *Let* G *be a connected, simply connected, step two nilpotent Lie group and* $f \in \mathcal{S}(G)$. *Then*

$$\left(\int_G \|v\|^2 |f(x,v)|^2 \mathrm{d}x\mathrm{d}v\right)^{1/2} \left(\int_{V_N^* \cap \mathcal{U}} \|(\widehat{\mathcal{L}^{\frac{1}{2}} f})(\pi_l)\|_{HS}^2 |Pf(l)| \mathrm{d}l\right)^{1/2}$$
$$\geq C\|f\|_{L^2(G)}^2, \tag{4.3}$$

where C is a constant independent of f and $\mathcal{L} = -\Sigma_{i=1}^{n-m} X_{m+i}^2$ is the sub-Laplacian.

Let us explain the meaning of $(\widehat{\mathcal{L}^{\frac{1}{2}} f})(\pi_l)$. We view $X \in \mathfrak{g}$ as a left invariant differential operator on $C^\infty(G)$. Then in view of our definition of the group Fourier transform, we have for $f \in \mathcal{S}(G)$

$$(\widehat{Xf})(\pi_l) = \mathrm{d}\pi_l(X) \circ \hat{f}(\pi_l), \tag{4.4}$$

where $\mathrm{d}\pi_l(X)$ is given by (2.13). We view the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ as the algebra of all left invariant differential operators on $C^\infty(G)$. Since $\mathrm{d}\pi_l$ is a representation of $\mathfrak{g}$, it extends to a representation of $\mathcal{U}(\mathfrak{g})$ realized on $C^\infty(\pi_l)$. By (4.4) we have

$$(\widehat{\mathcal{L}f})(\pi_l) = \mathrm{d}\pi_l(\mathcal{L}) \circ \hat{f}(\pi_l),$$

as $\mathcal{L} \in \mathcal{U}(\mathfrak{g})$. In §2 we have seen that the eigenfunctions of $\mathrm{d}\pi_l(\mathcal{L})$ are parametrized by $\mathbb{N}^k$ and are given by (2.16). Let $\{t_i(l) > 0 : i = 0, \ldots\}$ be an enumeration of those real numbers such that there exist $\alpha \in \mathbb{N}^k$ with

$$t_i(l) = \mu(l) + \sum_{j=1}^{k} 2\pi\lambda_j(l)(2\alpha_j + 1), \tag{4.5}$$

as α varies over $\mathbb{N}^k$. Let $E_i(l) = \mathrm{span}_{\mathbb{C}}\{h_\alpha^l : \mathrm{d}\pi_l(\mathcal{L})(h_\alpha^l) = t_i(l)h_\alpha^l\}$, that is, $E_i(l)$ is the eigenspace corresponding to the eigenvalue $t_i(l)$, which is clearly finite dimensional. If $P_i(l) : L^2(\mathbb{R}^k) \to E_i(l)$ is the projection, we have

$$\mathrm{d}\pi_l(\mathcal{L}) = \sum_{j=0}^{\infty} t_j(l) P_j(l). \tag{4.6}$$

Thus we define

$$\mathrm{d}\pi_l(\mathcal{L})^{\frac{1}{2}} = \sum_{j=0}^{\infty} t_j(l)^{\frac{1}{2}} P_j(l) \tag{4.7}$$

and

$$\mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}} = \sum_{j=0}^{\infty} t_j(l)^{-\frac{1}{2}} P_j(l). \tag{4.8}$$

Analogous to the Euclidean spaces, we define

$$(\widehat{\mathcal{L}^{\frac{1}{2}} f})(\pi_l) = \mathrm{d}\pi_l(\mathcal{L})^{\frac{1}{2}} \circ \hat{f}(\pi_l), \tag{4.9}$$

for all $f \in \mathcal{S}(G)$ and $l \in V_N^* \cap \mathcal{U}$. Thus the statement in theorem 4.1 makes sense.

It follows from (4.5) that the eigenvalues of $\mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}}$ are bounded by $\lambda_0(l)^{-\frac{1}{2}}$ where $\lambda_0(l) = \min\{\lambda_j(l) : 1 \leq j \leq k\}$. As a consequence we get the following Lemma.

Lemma 4.1. *The operator* $\mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}}$ *is bounded on* $L^2(\mathbb{R}^k)$.

Let us consider the following elements of $\mathfrak{g}_{\mathbb{C}}$, the complexification of $\mathfrak{g}$,

$$D_j(l) = Y_j(l) - iW_j(l), \qquad 1 \le j \le k, \tag{4.10}$$

$$\bar{D}_j(l) = Y_j(l) + iW_j(l), \qquad 1 \le j \le k. \tag{4.11}$$

Because of Lemma 2.4 we have

$$\mathrm{d}\pi_l(D_j(l))\phi(\xi) = \left(-\frac{\partial}{\partial \xi_j} + 2\pi\lambda_j(l)\xi_j\right)\phi(\xi), \tag{4.12}$$

$$\mathrm{d}\pi_l(\bar{D}_j(l))\phi(\xi) = \left(-\frac{\partial}{\partial \xi_j} - 2\pi\lambda_j(l)\xi_j\right)\phi(\xi). \tag{4.13}$$

For h_s the sth normalized hermite function on $\mathbb{R}$, we define $h_s^c(x) = c^{1/4}h_s(c^{1/2}x)$, then

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x} + cx\right)h_s^c = c^{1/2}(2s+2)^{1/2}h_{s+1}^c,$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + cx\right)h_s^c = c^{1/2}(2s)^{1/2}h_{s-1}^c.$$

Using this with (4.12) and (4.13) we get for $\alpha \in \mathbb{N}^k$,

$$\mathrm{d}\pi_l(D_j(l))(h_\alpha^l) = (2\pi\lambda_j(l))^{1/2}(2\alpha_j+2)^{1/2}h_{\alpha+e_j}, \tag{4.14}$$

$$\mathrm{d}\pi_l(\bar{D}_j(l))(h_\alpha^l) = -(2\pi\lambda_j(l))^{1/2}(2\alpha_j)^{1/2}h_{\alpha-e_j}^l, \tag{4.15}$$

where

$$\alpha + e_j = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1}, \ldots, \alpha_k) \in \mathbb{N}^k,$$

$$\alpha - e_j = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_j - 1, \alpha_{j+1}, \ldots, \alpha_k) \in \mathbb{N}^k.$$

Lemma 4.2. *The operators* $\mathrm{d}\pi_l(D_j(l)) \circ \mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}}$ *and* $\mathrm{d}\pi_l(\bar{D}_j(l)) \circ \mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}}$ *are bounded operators on* $L^2(\mathbb{R}^k)$, $1 \le j \le k$.

Proof. We consider the orthonormal basis $\{h_\alpha^l : \alpha \in \mathbb{N}^k\}$ of $L^2(\mathbb{R}^k)$. By (4.8), (4.14) and (4.15) we have

$$\mathrm{d}\pi_l(D_j(l)) \circ \mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}}(h_\alpha^l) = \left(\frac{2\pi\lambda_j(l)(2\alpha_j+2)}{\mu(l) + \sum_{p=1}^k 2\pi\lambda_p(l)(2\alpha_p+1)}\right)^{\frac{1}{2}} h_{\alpha+e_j}^l,$$

and

$$\mathrm{d}\pi_l(\bar{D}_j(l)) \circ \mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}}(h_\alpha^l) = -\left(\frac{2\pi\lambda_j(l)2\alpha_j}{\mu(l) + \sum_{p=1}^k 2\pi\lambda_p(l)(2\alpha_p+1)}\right)^{\frac{1}{2}} h_{\alpha-e_j}^l.$$

Since

$$\left(\frac{2\pi\lambda_j(l)(2\alpha_j+2)}{\mu(l) + \sum_{p=1}^k 2\pi\lambda_p(l)(2\alpha_p+1)}\right)^{\frac{1}{2}} \le \sqrt{2}$$

and

$$\left(\frac{2\pi\lambda_j(l)2\alpha_j}{\mu(l)+\sum_{p=1}^{k}2\pi\lambda_p(l)(2\alpha_p+1)}\right)^{\frac{1}{2}}\leq 1,$$

the operators $\mathrm{d}\pi_l(D_j(l))\circ\mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}}$ and $\mathrm{d}\pi_l(\bar{D}_j(l))\circ\mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}}$ are bounded operators on $L^2(\mathbb{R}^k)$. This completes the proof.

Suppose $f\in\mathcal{S}(G)$ and let $l\in V_N^*\cap\mathcal{U}$ be arbitrary but fixed. So we have an almost symplectic basis (2.4) of $\mathfrak{g}$. Let $l\mid_{\mathfrak{z}}=\lambda$. We define

$$\mathcal{F}_c f(\lambda,v)=\int_{\mathrm{R}^m}f(x,v)\mathrm{e}^{-2\pi i\lambda(x)}\mathrm{d}x, \tag{4.16}$$

that is, the partial Fourier transform in the central component. So $v\to\mathcal{F}_c f(\lambda,v)$ is a Schwartz class function on $\mathbb{R}^{n-m}$. On Euclidean spaces, differentiation and multiplication are intertwined by the Fourier transform. On two step groups, as analogues of differentiation we consider the operators $D_l(l)$ and $\bar{D}_j(l)$ and as analogue of Fourier transform we consider the partial Fourier transform defined in (4.16). We want to find what plays the role of multiplication?

Let $f\in\mathcal{S}(G)$ and $X_j\in\mathcal{B}\subset\mathfrak{g}$, $m+1\leq j\leq n$. By (2.15) it is clear that $X_jf\in\mathcal{S}(G)$, and an easy calculation shows that

$$\mathcal{F}_c(X_jf)(\lambda,v)=\left(\frac{\partial}{\partial x_j}+\pi i B_\lambda(v,X_j)\right)(\mathcal{F}_cf)(\lambda,v).$$

Thus using the basis in (2.4) we have

$$\begin{aligned}&\mathcal{F}_c(W_j(l)f)(\lambda,z,w,y)\\ =\;&\left(\frac{\partial}{\partial w_j}-\pi i\lambda_j(l)y_j\right)(\mathcal{F}_cf)(\lambda,z,w,y), \qquad (4.17)\\ &\mathcal{F}_c(Y_j(l)f)(\lambda,z,w,y)\\ =\;&\left(\frac{\partial}{\partial y_j}+\pi i\lambda_j(l)w_j\right)(\mathcal{F}_cf)(\lambda,z,w,y) \qquad (4.18)\end{aligned}$$

for $1\leq j\leq k$. Thus writing

$$V_j(l)=\left(\frac{\partial}{\partial y_j}-i\frac{\partial}{\partial w_j}\right)-\pi\lambda_j(l)(y_j-iw_j), \tag{4.19}$$

$$\bar{V}_j(l)=\left(\frac{\partial}{\partial y_j}+i\frac{\partial}{\partial w_j}\right)+\pi\lambda_j(l)(y_j+iw_j), \tag{4.20}$$

we have from (4.17) and (4.18)

$$\mathcal{F}_c(D_j(l)f)(\lambda,z,w,y)=V_j(l)(\mathcal{F}_cf)(\lambda,z,w,y), \tag{4.21}$$

$$\mathcal{F}_c(\bar{D}_j(l)f)(\lambda,z,w,y)=\bar{V}_j(l)(\mathcal{F}_cf)(\lambda,z,w,y). \tag{4.22}$$

Thus $V_j(l)$ and $\bar{V}_j(l)$ play the role of multiplication.

Now we come to the proof of Theorem 4.1.

Proof of theorem 4.1. Let $f \in \mathcal{S}(G)$ and $l \mid \mathfrak{z} = \lambda$. Now

$$\begin{aligned}
&\int_{\mathsf{R}^{n-m}} |\mathcal{F}_c f(\lambda, z, w, y)|^2 \mathrm{d}z\mathrm{d}w\mathrm{d}y \\
&= \int_{\mathsf{R}^{n-m}} \mathcal{F}_c f(\lambda, z, w, y)\overline{\mathcal{F}_c f(\lambda, z, w, y)}\mathrm{d}z\mathrm{d}w\mathrm{d}y.
\end{aligned}$$

Since

$$\begin{aligned}
&\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)(x_j + iy_j)g(x, y) \\
&= 2g(x, y) + (x_j + iy_j)\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)g(x, y),
\end{aligned}$$

we have from the above equality

$$\begin{aligned}
&\int_{\mathsf{R}^{n-m}} |\mathcal{F}_c f(\lambda, z, w, y)|^2 \mathrm{d}z\mathrm{d}w\mathrm{d}y \\
&= \int_{\mathsf{R}^{n-m}} \frac{1}{2}\left\{\left(\frac{\partial}{\partial y_j} - i\frac{\partial}{\partial w_j}\right)(y_j + iw_j)\mathcal{F}_c f(\lambda, z, w, y)\right. \\
&\quad \left. -(y_j + iw_j)\left(\frac{\partial}{\partial y_j} - i\frac{\partial}{\partial w_j}\right)\mathcal{F}_c f(\lambda, z, w, y)\right\} \\
&\qquad \times\overline{\mathcal{F}_c f(\lambda, z, w, y)}\mathrm{d}z\mathrm{d}w\mathrm{d}y \\
&= -\frac{1}{2}\int_{\mathsf{R}^{n-m}} (y_j + iw_j)\mathcal{F}_c f(\lambda, z, w, y) \\
&\quad \times\overline{\left(\frac{\partial}{\partial y_j} + i\frac{\partial}{\partial w_j}\right)\mathcal{F}_c f(\lambda, z, w, y)}\mathrm{d}z\mathrm{d}w\mathrm{d}y \\
&\quad -\frac{1}{2}\int_{\mathsf{R}^{n-m}} (y_j + iw_j)\left(\frac{\partial}{\partial y_j} - i\frac{\partial}{\partial w_j}\right)\mathcal{F}_c f(\lambda, z, w, y) \\
&\qquad \times\overline{\mathcal{F}_c f(\lambda, z, w, y)}\mathrm{d}z\mathrm{d}w\mathrm{d}y \qquad \text{(by integration by parts)} \\
&= -\frac{1}{2}\int_{\mathsf{R}^{n-m}} (y_j + iw_j)\mathcal{F}_c f(\lambda, z, w, y) \\
&\quad \times\overline{(\bar{V}_j(l) - \pi\lambda_j(l)(y_j + iw_j))\mathcal{F}_c f(\lambda, z, w, y)}\mathrm{d}z\mathrm{d}w\mathrm{d}y \\
&\quad -\frac{1}{2}\int_{\mathsf{R}^{n-m}} (y_j + iw_j)(V_j(l) + \pi\lambda_j(l)(y_j - iw_j))\mathcal{F}_c f(\lambda, z, w, y) \\
&\qquad \times\overline{\mathcal{F}_c f(\lambda, z, w, y)}\mathrm{d}z\mathrm{d}w\mathrm{d}y \qquad \text{(by (4.19) and (4.20))} \\
&= -\frac{1}{2}\int_{\mathsf{R}^{n-m}} (y_j + iw_j)\mathcal{F}_c f(\lambda, z, w, y)\overline{\mathcal{F}_c(\bar{D}_j(l)f)(\lambda, z, w, y)}\mathrm{d}z\mathrm{d}w\mathrm{d}y \\
&\quad -\frac{1}{2}\int_{\mathsf{R}^{n-m}} (y_j + iw_j)\mathcal{F}_c(D_j(l)f)(\lambda, z, w, y)\overline{\mathcal{F}_c f(\lambda, z, w, y)}\mathrm{d}z\mathrm{d}w\mathrm{d}y \\
&\qquad \text{(by (4.21) and (4.22)).}
\end{aligned} \tag{4.23}$$

Let us recall, if l varies over $V_N^* \cap \mathcal{U}$ then $l \mid \mathfrak{z} = \lambda$ varies over the Zariski open subset $\mathcal{U}'$ of $\mathfrak{z}^*$ (see Note 2.3). Hence

$$\int_{\mathfrak{z}}\int_{\mathbb{R}^{n-m}}|f(x,v)|^2\mathrm{d}x\mathrm{d}v = \int_{\mathcal{U}'}\int_{\mathbb{R}^{n-m}}|\mathcal{F}_c f(\lambda,v)|^2\mathrm{d}\lambda\mathrm{d}v$$
$$= \int_{\mathcal{U}'}\left(\int_{\mathbb{R}^{n-m}}|\mathcal{F}_c f(\lambda,z,w,y)|^2\mathrm{d}z\mathrm{d}w\mathrm{d}y\right)\mathrm{d}\lambda$$

(by Fubini's theorem and the orthogonal basis change on $\mathbb{R}^{n-m}$ by T_l : $\mathrm{span}_{\mathbb{R}}\{X_{m+1}, \ldots, X_n\} \to \mathrm{span}_{\mathbb{R}}\{Z_1(l), \ldots, Y_k(l)\}$)

$$= \int_{\mathcal{U}'}\left(-\frac{1}{2}\int_{\mathbb{R}^{n-m}}(y_j+iw_j)\mathcal{F}_c f(\lambda,z,w,y)\overline{\mathcal{F}_c(\bar{D}_j(l)f)(\lambda,z,w,y)}\mathrm{d}z\mathrm{d}w\mathrm{d}y\right.$$
$$\left.-\frac{1}{2}\int_{\mathbb{R}^{n-m}}(y_j+iw_j)\mathcal{F}_c(D_j(l)f)(\lambda,z,w,y)\overline{\mathcal{F}_c f(\lambda,z,w,y)}\mathrm{d}z\mathrm{d}w\mathrm{d}y\right)d\lambda$$

(by (4.23))

$$= \int_{\mathcal{U}'}\left(-\frac{1}{2}\int_{\mathbb{R}^{n-m}}T_l^{-1}(y_j+iw_j)\mathcal{F}_c f(\lambda,v)\overline{\mathcal{F}_c(\bar{D}_j(l)f)(\lambda,v)}\mathrm{d}v\right.$$
$$\left.-\frac{1}{2}\int_{\mathbb{R}^{n-m}}T_l^{-1}(y_j+iw_j)\mathcal{F}_c(D_j(l)f)(\lambda,v)\overline{\mathcal{F}_c f(\lambda,v)}\mathrm{d}v\right)\mathrm{d}\lambda$$

(by change of variables)

$$\le \frac{1}{2}\left(\int_{\mathcal{U}'}\int_{\mathbb{R}^{n-m}}|T_l^{-1}(y_j+iw_j)|^2|\mathcal{F}_c f(\lambda,v)|^2\mathrm{d}v\mathrm{d}\lambda\right)^{\frac{1}{2}}$$
$$\times\left\{\left(\int_{\mathcal{U}'}\int_{\mathbb{R}^{n-m}}|\mathcal{F}_c(\bar{D}_j(l)f)(\lambda,v)|^2\mathrm{d}v\mathrm{d}\lambda\right)^{\frac{1}{2}}\right.$$
$$\left.+\left(\int_{\mathcal{U}'}\int_{\mathbb{R}^{n-m}}|\mathcal{F}_c(D_j(l)f)(\lambda,v)|^2\mathrm{d}v\mathrm{d}\lambda\right)^{\frac{1}{2}}\right\}$$

(by Cauchy–Schwartz inequality and nonnegativity of the integral)

$$\le \frac{1}{2}\left(\int_{\mathcal{U}'}\int_{\mathbb{R}^{n-m}}\|v\|^2|\mathcal{F}_c f(\lambda,v)|^2\mathrm{d}v\mathrm{d}\lambda\right)^{\frac{1}{2}}$$
$$\times\left\{\left(\int_{\mathcal{U}'}\int_{\mathbb{R}^{n-m}}|\mathcal{F}_c(\bar{D}_j(l)f)(\lambda,v)|^2\mathrm{d}v\mathrm{d}\lambda\right)^{\frac{1}{2}}\right.$$
$$\left.+\left(\int_{\mathcal{U}'}\int_{\mathbb{R}^{n-m}}|\mathcal{F}_c(D_j(l)f)(\lambda,v)|^2\mathrm{d}v\mathrm{d}\lambda\right)^{\frac{1}{2}}\right\}$$

$$= \frac{1}{2}\left(\int_{\mathfrak{z}}\int_{\mathbb{R}^{n-m}}\|v\|^2|f(x,v)|^2\mathrm{d}x\mathrm{d}v\right)^{\frac{1}{2}}$$
$$\times\left\{\left(\int_{\mathcal{U}'}\int_{\mathbb{R}^{n-m}}|\mathcal{F}_c(\bar{D}_j(l)f)(\lambda,v)|^2\mathrm{d}v\mathrm{d}\lambda\right)^{\frac{1}{2}}\right.$$
$$\left.+\left(\int_{\mathcal{U}'}\int_{\mathbb{R}^{n-m}}|\mathcal{F}_c(D_j(l)f)(\lambda,v)|^2\mathrm{d}v\mathrm{d}\lambda\right)^{\frac{1}{2}}\right\}, \tag{4.24}$$

by the Euclidean Plancherel theorem on $\mathfrak{z}$.

By Theorem (2.3) we have

$$\int_{\tilde{V}_N^*} \|\hat{f}(\pi_{(\lambda,\gamma)})\|_{HS}^2 \mathrm{d}l_{n_1} \ldots \mathrm{d}l_{n_r} = |Pf(l)|^{-1} \int_{\mathbb{R}^{n-m}} |\mathcal{F}_c f(\lambda, v)|^2 \mathrm{d}v,$$

where $l \mid \mathfrak{z} = \lambda$ and $l \mid \tilde{V}_N = \gamma = (l_{n_1}, \ldots, l_{n_r})$. Thus

$$\begin{aligned}
&\left(\int_{\mathcal{U}'} \int_{\mathbb{R}^{n-m}} |\mathcal{F}_c(\bar{D}(l)f)(\lambda, v)|^2 \mathrm{d}v \mathrm{d}\lambda\right)^{\frac{1}{2}} \\
&= \left(\int_{\mathcal{U}'} \int_{\tilde{V}_N^*} \|\widehat{(\bar{D}_j(l)} f)(\pi_l)\|_{HS}^2 |Pf(l)| \mathrm{d}l_{n_1} \ldots \mathrm{d}l_{n_r} \mathrm{d}\lambda\right)^{\frac{1}{2}} \\
&= \left(\int_{\mathcal{U}'} \int_{\tilde{V}_N^*} \|\mathrm{d}\pi_l(\bar{D}_j(l)) \circ \hat{f}(\pi_l)\|_{HS}^2 |Pf(l)| \mathrm{d}l_{n_1} \ldots \mathrm{d}l_{n_r} \mathrm{d}\lambda\right)^{\frac{1}{2}} \\
&= \left(\int_{\mathcal{U}'} \int_{\tilde{V}_N^*} \|\mathrm{d}\pi_l(\bar{D}_j(l)) \circ \mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}} \circ \mathrm{d}\pi_l(\mathcal{L})^{\frac{1}{2}} \circ \hat{f}(\pi_l)\|_{HS}^2 \right. \\
&\qquad \left. \times |Pf(l)| \mathrm{d}l_{n_1} \ldots \mathrm{d}l_{n_r} \mathrm{d}\lambda\right)^{\frac{1}{2}} \\
&\leq \left(\int_{\mathcal{U}'} \int_{\tilde{V}_N^*} \|\mathrm{d}\pi_l(\bar{D}_j(l) \circ \mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}}\|_{Op}^2 \|\widehat{(\mathcal{L}^{\frac{1}{2}} f)}(\pi_l)\|_{HS}^2 \right. \\
&\qquad \left. \times |Pf(l)| \mathrm{d}l_{n_1} \ldots \mathrm{d}l_{n_r} \mathrm{d}\lambda\right)^{\frac{1}{2}} \\
&\leq 2\left(\int_{\mathcal{U}'} \int_{\tilde{V}_N^*} \|\widehat{(\mathcal{L}^{\frac{1}{2}} f)}(\pi_l)\|_{HS}^2 |Pf(l)| \mathrm{d}l_{n_1} \ldots \mathrm{d}l_{n_r} \mathrm{d}l_1 \ldots \mathrm{d}l_m\right)^{\frac{1}{2}} \\
&\qquad \text{(by Lemma 4.2)}.
\end{aligned}$$

Similarly as above we can show that

$$\begin{aligned}
&\left(\int_{\mathcal{U}'} \int_{\mathbb{R}^{n-m}} |\mathcal{F}_c(D_j(l)f)(\lambda, v)|^2 \mathrm{d}v \mathrm{d}\lambda\right)^{\frac{1}{2}} \\
&\leq \left(\int_{\mathcal{U}'} \int_{\tilde{V}_N^*} \|\widehat{(\mathcal{L}^{\frac{1}{2}} f)}(\pi_l)\|_{HS}^2 |Pf(l)| \mathrm{d}l_{n_1} \ldots \mathrm{d}l_{n_r} \mathrm{d}l_1 \ldots \mathrm{d}l_m\right)^{\frac{1}{2}} \\
&= \left(\int_{V_N^* \cap \mathcal{U}} \|\widehat{(\mathcal{L}^{\frac{1}{2}} f)}(\pi_l)\|_{HS}^2 |Pf(l)| \mathrm{d}l\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus from (4.24) we have

$$\begin{aligned}
&\int_G |f(x, v)|^2 \mathrm{d}x \mathrm{d}v \\
&\leq C\left(\int_G \|v\|^2 |f(x, v)|^2 \mathrm{d}x \mathrm{d}v\right)^{\frac{1}{2}} \left(\int_{V_N^* \cap \mathcal{U}} \|\widehat{(\mathcal{L}^{\frac{1}{2}} f)}(\pi_l)\|_{HS}^2 |Pf(l)| \mathrm{d}l\right)^{\frac{1}{2}},
\end{aligned}$$

where C is a constant independent of f. This completes the proof.

Acknowledgement

I am grateful to my teacher Prof. S C Bagchi and my friend E K Narayanan for several useful discussions. Most of this work is contained in the author's Ph.D. thesis submitted to the Indian Statistical Institute (1999).

References

[1] Astengo F, Cowling M G, Di Blasio B and Sundari M, Hardy's uncertainty principle on some Lie groups, *J. London Math. Soc.* (to appear)
[2] Benson C, Jenkins J and Ratcliff G, On Gelfand pairs associated to solvable Lie groups, *Trans. Am. Math. Soc.* **321** (1990) 85–116
[3] Bagchi S C and Ray S, Uncertainty principles like Hardy's theorem on some Lie groups, *J. Austral. Math. Soc.* **A65** (1998) 289–302
[4] Chandrasekharan K, Classical Fourier transforms (Springer Verlag) (1989)
[5] Cowling M G, Sitaram A and Sundari M, Hardy's uncertainty principle on semisimple Lie groups, *Pacific J. Math.* **192** (2000) 293–296
[6] Cowling M G and Price J F, Generalizations of Heisenberg's inequality, in: Harmonic analysis (eds) G Mauceri, F Ricci and G Weiss (1983) LNM, no. 992 (Berlin: Springer) 443–449
[7] Corwin L J and Greenleaf F P, Representations of nilpotent Lie groups and their applications, Part 1 – Basic theory and examples (NY: Cambridge Univ. Press, Cambridge) (1990)
[8] Ebata M, Eguchi M, Koizumi S and Kumahara K, A generalisation of the Hardy theorem to semisimple Lie groups, *Proc. Japan Acad. Math. Sci.* **A75** (1999) 113–114
[9] Ebata M, Eguchi M, Koizumi S and Kumahara K, L^p version of the Hardy theorem for motion groups, *J. Austral. Math. Soc.* **A68** (2000) 55–67
[10] Folland G B, A course in abstract harmonic analysis, (London: CRC Press) (1995)
[11] Folland G B, Lectures on partial differential equations (New Delhi: Narosa Pub. House) (1983)
[12] Folland G B and Sitaram A, The uncertainty principles: A mathematical survey, *J. Fourier Anal. Appl.* **3** (1997) 207–238
[13] Garofalo N and Lanconeli E, Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation, *Ann. Inst. Fourier* **40** (1990) 313–356
[14] Hörmander L, A uniqueness theorem of Beurling for Fourier transform pairs, *Ark. Math.* **29** (1991) 237–240
[15] Jacobson N, Basic Algebra, (New Delhi: Hindustan Publishing Co.) (1993) vol. 1
[16] Kaniuth E and Kumar A, Hardy's theorem for simply connected nilpotent Lie groups, *Proc. Cambridge Philos. Soc.* (to appear)
[17] Lipsman R L and Rosenberg J, The behavior of Fourier transform for nilpotent Lie groups, *Trans. Am. Math. Soc.* **348** (1996) 1031–1050
[18] Moore C C and Wolf J A, Square integrable representations of nilpotent Lie groups, *Trans. Am. Math. Soc.* **185** (1973) 445–462
[19] Muller D and Ricci F, Solvability for a class of doubly characteristic differential operators on two step nilpotent Lie groups, *Ann. Math.* **143** (1996) 1–49
[20] Narayanan E K and Ray S K, L^p version of Hardy's theorem on semisimple Lie groups, *Proc. Am. Math. Soc.* (to appear)
[21] Park R, A Paley–Wiener theorem for all two- and three-step nilpotent Lie groups, *J. Funct. Anal.* **133** (1995) 277–300
[22] Pati V, Sitaram A, Sundari M and Thangavelu S, An uncertainty principle for eigenfunction expansions, *J. Fourier Anal. Appl.* **5** (1996) 427–433
[23] Ray S K, Uncertainty principles on some Lie groups, Ph.D. Thesis (Indian Statistical Institute) (1999)
[24] Sundari M, Hardy's theorem for the n-dimensional Euclidean motion group, *Proc. Am. Math. Soc.* **126** (1998) 1199–1204
[25] Sengupta J, An analogue of Hardy's theorem for semi-simple Lie groups, *Proc. Am. Math. Soc.* **128** (2000) 2493–2499

[26] Strichartz R F, L^p harmonic analysis and radon transforms on the Heisenberg groups, *J Funct. Anal.* **96** (1991) 350–406
[27] Sitaram A and Sundari M, An analogue of Hardy's theorem for very rapidly decreasing functions on semisimple Lie groups, *Pacific J. Math.* **177** (1997) 187–200
[28] Sitaram A, Sundari M and Thangavelu S, Uncertainty principles on certain Lie groups, *Proc. Ind. Acad. Sci.* (*Math. Sci.*) **105** (1995) 135–151
[29] Thangavelu S, Some uncertainty inequalities, *Proc. Ind. Acad. Sci.* (*Math. Sci.*) **100** (1990) 137–145