Proc. Indian Acad. Sci. (Math. Sci.), Vol. 111, No. 3, August 2001, pp. 263–269.

Stability of Picard bundle over moduli space of stable vector bundles of rank two over a curve

INDRANIL BISWAS and TOMÁS L GÓMEZ

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India
E-mail: indranil@math.tifr.res.in; tomas@math.tifr.res.in

MS received 14 September 2000

Abstract. Answering a question of [BV] it is proved that the Picard bundle on the moduli space of stable vector bundles of rank two, on a Riemann surface of genus at least three, with fixed determinant of odd degree is stable.

Keywords. Picard bundle; Hecke lines.

0. Introduction

Let X be a compact connected Riemann surface of genus g, with $g \geq 3$. Let ξ be a holomorphic line bundle over X of odd degree d, with $d \geq 4g - 3$. Let M denote the moduli space of stable vector bundles E over X of rank two and $\bigwedge^2 E \cong \xi$. Take a universal vector bundle $\mathcal{E}$ on $X \times M$. Let $p : X \times M \longrightarrow M$ be the projection. The vector bundle $\mathcal{P} := p_* \mathcal{E}$ on M is called the *Picard bundle for M*. In [BV] it was proved that the Picard bundle $\mathcal{P}$ is simple, and a question was asked whether it is stable. In [BHM] a differential geometric criterion for the stability of $\mathcal{P}$ was given. But there is no evidence for this criterion to be valid.

In Theorem 3.1 we prove that the Picard bundle $\mathcal{P}$ over M is stable.

1. Preliminaries

In this section we prove some lemmas that will be needed.

A vector bundle E of rank two and degree d is called *superstable* if for every subline bundle L of E the inequality

$$\deg(L) < \frac{d}{2} - \frac{1}{2}$$

is valid. Clearly, a superstable bundle is stable. The first lemma ensures existence of superstable bundles.

Lemma 1.1. *There is a nonempty open subset U of M corresponding to superstable bundles.*

Proof. Here we need $g \geq 3$. Let T be the subset of M of vector bundles that are not superstable, i.e., $E \in T$ if and only if there exists a subline bundle L such that $\deg(L) \geq$

$(d-1)/2$. Since E is stable, $\deg(L) < d/2$, and since d is odd, $\deg(L) = (d-1)/2$. There is a short exact sequence

$$0 \longrightarrow L \longrightarrow E \longrightarrow \xi \otimes L^{-1} \longrightarrow 0\,.$$

Note that the quotient is torsion free (hence a line bundle) because E is stable and L has degree $(d-1)/2$.

Therefore, all vector bundles in T can be constructed by choosing a line bundle L of degree $(d-1)/2$ together with an extension class in $\mathrm{Ext}^1(\xi \otimes L^{-1}, L)$. It follows immediately that T is a closed subset of M with dimension

$$\begin{aligned}\dim(T) &\le g + h^1(\xi^{-1} \otimes L^2) - 1 = g - \chi(\xi^{-1} \otimes L^2) - 1\\ &= 2g - 1 < 3g - 3 = \dim(M),\end{aligned}$$

and hence the complement $U := M \backslash T$ is open and nonempty. □

Lemma 1.2. *Choose m distinct points $\{x_1, \ldots, x_m\} \subset X$, with $m > d/2$. Let $E \in M$ be a vector bundle and $0 \neq s \in H^0(E)$ a nontrivial section. Then s cannot simultaneously vanish at all the chosen points $\{x_1, \ldots, x_m\}$.*

Proof. If s vanishes at all chosen points $x_1, \ldots, x_m$, then $s : \mathcal{O} \longrightarrow E$ factors as

$$s : \mathcal{O} \longrightarrow E(-D) \hookrightarrow E,$$

where D is the divisor $D = x_1 + \cdots + x_m$. Since $\deg E(-D) = d - 2m < 0$, the stability condition of E forces s to be the zero section. □

2. Hecke lines

Let $U \subset M$ be the open subset of superstable vector bundles (Lemma 1.1). Take a point $x \in X$. Let $E \in U$ and $l \subset E_x$ a line in the fiber of E at x (equivalently, $l \in \mathbb{P}(E_x)$). Define the vector bundle W by

$$0 \longrightarrow W(-x) \longrightarrow E \longrightarrow E_x/l \longrightarrow 0\,.$$

The vector bundle $W(-x)$ is called the *Hecke transform* of E with respect to x and l. The exact sequence implies $\bigwedge^2 W \cong \xi \otimes \mathcal{O}(x)$. The vector bundle W is stable. Indeed, a line subbundle L of W is realized as a subline bundle of $E(x)$ using the homomorphism $W \longrightarrow E(x)$. Now the superstability condition of E says

$$\deg(L) < \frac{d+1}{2} = \frac{\deg(W)}{2}\,.$$

In other words, W is stable.

We can reconstruct back E from W by doing another Hecke transform, and E is given as the middle row of the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & E_x/l & \longrightarrow & W_x & \longrightarrow & \mathbb{C}_x & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & E & \longrightarrow & W & \xrightarrow{f_0} & \mathbb{C}_x & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & W(-x) & = & W(-x) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{1}
$$

Here $\mathbb{C}_x$ is the skyscraper sheaf at x with stalk $\mathbb{C}$. Instead of $f_0 : W \longrightarrow \mathbb{C}_x$ we may consider an arbitrary nontrivial homomorphism

$$f \in \mathrm{Hom}(W, \mathbb{C}_x) = \mathrm{Hom}(W_x, \mathbb{C}_x) = W_x^\vee$$

and define E_f as the kernel

$$0 \longrightarrow E_f \longrightarrow W \xrightarrow{f} \mathbb{C}_x \longrightarrow 0. \tag{2}$$

This way we obtain a family of vector bundles parametrized by the projective line $\mathbb{P}(W_x^\vee)$, with $E_{f_0} \cong E$. More precisely, there is a short exact sequence on $X \times \mathbb{P}(W_x^\vee)$,

$$0 \longrightarrow \widetilde{E} \longrightarrow \pi_X^* W \xrightarrow{\tilde{f}} \mathcal{O}_{x \times \mathbb{P}(W_x^\vee)}(1) \longrightarrow 0,$$

where $\pi_X : X \times \mathbb{P}(W_x^\vee) \longrightarrow X$ is the projection to X. It has the property that if $f \in W_x^\vee$ and we restrict the exact sequence to the subvariety $X \times [f] \cong X$ of $X \times \mathbb{P}(W_x^\vee)$, then a sequence isomorphic to (2) is obtained.

For every $f \in W_x^\vee$, the vector bundle E_f is stable. Indeed, if L is a subline bundle of E_f, then by composition with the homomorphism $E_f \longrightarrow W$ in (2) it is a subline bundle of W. The stability condition for W says that $\deg(L) < (d+1)/2$. Since d is odd this is equivalent to

$$\deg(L) \leq \frac{d-1}{2} < \frac{d}{2} = \frac{\deg(E_f)}{2}.$$

Note that if E is stable but not superstable, then W is semistable but not necessarily stable. The semistability condition is not enough to ensure the stability of E_f for each f.

The universal property of the moduli space M gives a morphism $\varphi : \mathbb{P}(W_x^\vee) \longrightarrow M$ for the family $\widetilde{E}$.

DEFINITION 2.1

The data consisting of the pair $(\mathbb{P}(W_x^\vee), \varphi)$ is called the Hecke line associated to the triple (E, x, l).

Since φ is determined by W and $\mathbb{P}(W_x^\vee)$, the projective line $\mathbb{P}(W_x^\vee)$ will also be called a Hecke line. The Hecke line $\mathbb{P}(W_x^\vee)$ will also be denoted by $P_{E,x,l}$ or simply by P if the rest of the data is clear from the context. Note that there is a distinguished point $[f_0] \in \mathbb{P}(W_x^\vee)$ that maps to $E \in M$.

For any $f \in \mathbb{P}(W_x^\vee)$, let l_f denote the kernel of the homomorphism $(E_f)_x \longrightarrow W_x$ of fibers in (2). Clearly, the images of the two Hecke lines $P_{E,x,l}$ and P_{E_f,x,l_f} in M coincide.

Therefore, for each $E \in M$, there is a three parameter family of Hecke lines whose image contains E. On the other hand, if we identify two Hecke lines if their images in M coincide, then through each point of M there is a two parameter family of rational curves defined by Hecke lines.

Since the morphism φ is given by the universal property of the moduli space, the pullback of the universal bundle $\mathcal{E}$ on $X \times M$ to $X \times P$ by the map $\mathrm{id}_X \times \varphi$ is isomorphic (up to a twist by a line bundle coming from P) to $\widetilde{E}$. In other words, there is an integer k such that

$$0 \longrightarrow (\mathrm{id}_X \times \varphi)^* \mathcal{E} \longrightarrow W \boxtimes \mathcal{O}_P(k) \longrightarrow \mathcal{O}_{x \times P}(k+1) \longrightarrow 0 \tag{3}$$

is an exact sequence of sheaves on $X \times P$; $\mathcal{O}_P(1)$ is the tautological line bundle on $P = \mathbb{P}(W_x^\vee)$. Applying $(\pi_P)_*$, where π_P is the projection of $X \times P$ to P, the following sequence

$$0 \longrightarrow \varphi^* \mathcal{P} \longrightarrow H^0(W) \otimes \mathcal{O}_P(k) \longrightarrow \mathcal{O}_P(k+1) \longrightarrow 0 \tag{4}$$

on P is obtained, where $\mathcal{P}$ is the Picard bundle. Since $d \geq 4g-3$, the stability condition ensures that $H^1(X, E')$ vanishes for every $E' \in M$.

Let N denote the rank of $\mathcal{P}$. The following proposition describes the pullback $\varphi^* \mathcal{P}$.

PROPOSITION 2.2

The pullback $\varphi^* \mathcal{P}$ of the Picard bundle $\mathcal{P}$ to the $P = P_{E,x,l}$ satisfies

$$\varphi^* \mathcal{P} \cong \mathcal{O}_P(k)^{\oplus N-1} \oplus \mathcal{O}_P(k-1). \tag{5}$$

Hence $\varphi^* \mathcal{P}$ has a canonical subbundle

$$\mathcal{O}_P(k)^{\oplus N-1} \cong \mathcal{V} \hookrightarrow \varphi^* \mathcal{P}.$$

Let $V \subset H^0(X, E)$ be the fiber of this subbundle over the distinguished point $[f_0] \in P$. Then $s \in V$ if and only if $s(x) \in l$.

Proof. Grothendieck's theorem [Gr] says that a vector bundle on $\mathbb{P}^1$ is holomorphically isomorphic to a direct sum of line bundles. Hence

$$\varphi^* \mathcal{P} \cong \mathcal{O}_P(a_1) \oplus \cdots \oplus \mathcal{O}_P(a_N).$$

The sequence (4) gives $h^0(W) = 1 + N$, $\sum a_i = Nk - 1$ and $a_i \leq k$ for all i. Combining these, (5) is obtained immediately.

Now we are going to identify the subbundle $\mathcal{V}$. From (3) the following commutative diagram is obtained

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathrm{id}_X \times \varphi)^* \mathcal{E} & \longrightarrow & W \boxtimes \mathcal{O}_P(k) & \longrightarrow & \mathcal{O}_{x \times P}(k+1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & W(-x) \boxtimes \mathcal{O}_P(k) & = & W(-x) \boxtimes \mathcal{O}_P(k) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

and applying $(\pi_P)_*$ we obtain the following commutative diagram on P:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varphi^*\mathcal{P} & \longrightarrow & H^0(W)\otimes \mathcal{O}_P(k) & \longrightarrow & \mathcal{O}_P(k+1) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & & & \\ & & H^0(W(-x))\otimes \mathcal{O}_P(k) & = & H^0(W(-x))\otimes \mathcal{O}_P(k) & & & & . \\ & & \uparrow & & \uparrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Since $\varphi^*\mathcal{P} \cong \mathcal{O}_P(k)^{\oplus N-1} \oplus \mathcal{O}_P(k-1)$, we deduce that

$$\mathcal{V} = H^0(W(-x)) \otimes \mathcal{O}_P(k) \subset \varphi^*\mathcal{P}\,.$$

Let V denote the fiber of $\mathcal{V}$ at the point $[f_0] \in P$. So, $V \subset H^0(E)$. Now, $s \in V$ if and only if $s \in H^0(W(-x)) \subset H^0(E)$. Finally, taking global sections for the diagram (2) it is easy to see that this is equivalent to the condition that $s(x) \in l$. This completes the proof of the proposition. □

Proposition 2.2 has the following corollary.

COROLLARY 2.3

The morphism φ is a nonconstant one.

Indeed, if φ were a constant map, then the vector bundle $\varphi^*\mathcal{P}$ would be trivial.

3. Main theorem

In this section we will prove the main theorem of this paper.

Theorem 3.1. *Let $\mathcal{P}$ be the Picard bundle on the moduli space M of stable bundles of rank two and fixed determinant of odd degree d with $d \geq 4g-3$. Then $\mathcal{P}$ is stable*.

Proof. Since $\mathcal{P}$ is a vector bundle, to check stability it is enough to consider reflexive subsheaves of $\mathcal{P}$. Let

$$\mathcal{F} \longrightarrow \mathcal{P}$$

be a reflexive subsheaf of rank $r < N = \text{rank}(\mathcal{P})$. Fix m distinct points $x_1, \ldots, x_m$ in X, with $m > d/2$.

We need the following lemma for the proof of the theorem.

Lemma 3.2. *There is a nonempty open set of M such that if E is a vector bundle corresponding to a point of that open set, then E has the following four properties*:

(i) *E is superstable*;

(ii) *$\mathcal{F}$ is locally free at E*;

(iii) *$\mathcal{F}_E \to \mathcal{P}_E$ is an injection*;

(iv) *Let x_i be one of the fixed points and l any line on E_{x_i}. Let $P = P_{E,x_i,l}$ be the associated Hecke line. Then $\mathcal{F}$ is locally free at all points of the image of $\varphi : P \longrightarrow M$.*

Proof. The subset U of M where property (i) is satisfied is open and nonempty by Lemma 1.1. Let $U' \subset U$ be the subset where also property (ii) is satisfied and $U'' \subset U'$ the subset where furthermore property (iii) is satisfied. Clearly, U'' is a nonempty open subset of M.

Let $S \subset M$ denote the subvariety where $\mathcal{F}$ is not locally free. Since $\mathcal{F}$ is reflexive, $\mathrm{codim}(S) \geq 3$. Let $\widetilde{S}_i$ be the union of the images of all Hecke lines $P_{E,x_i,l}$, when E runs through all points in S and l runs through all lines of E_x. Then

$$\mathrm{codim}\widetilde{S}_i \geq 3-1-1 = 1\,.$$

Finally consider the union

$$\widetilde{S} := \bigcup_{i=1}^{m} \widetilde{S}_i\,.$$

Since this is a union of a finite number of subvarieties, we still have $\mathrm{codim}\widetilde{S} \geq 1$. Consequently, $U''' := U'' \bigcap (M\backslash\widetilde{S})$ is nonempty and open. By construction, any vector bundle E corresponding to a point in U''' satisfies conditions (i) to (iv). This finishes the proof of the lemma. □

Continuing the proof of Theorem 3.1, fix a vector bundle E satisfying the four properties in the above lemma. Let $v \in \mathcal{F}_E$ be a nonzero vector in the fiber, and let s be its image in the fiber $\mathcal{P}_E \cong H^0(E)$. It is still nonzero because of property (iii).

From the fixed set of chosen points $\{x_1,\ldots,x_m\}$, pick one of them x_i such that the section s does not vanish at x_i. The existence of such a point is ensured by Lemma 1.2. Let $l \subset E_{x_i}$ be a line such that $s(x_i) \notin l$. Consider the Hecke line $P = P_{E,x_i,l}$ defined with this data.

Note that $\varphi^*\mathcal{F}$ is a vector bundle because $\mathcal{F}$ is locally free on all points of the image of P in M (property (iv)), and $\varphi^*\mathcal{F} \longrightarrow \varphi^*\mathcal{P}$ is injective as a sheaf homomorphism because both $\varphi^*\mathcal{F}$ and $\varphi^*\mathcal{P}$ are vector bundles and property (iii).

The Proposition 2.2 says that $\varphi^*\mathcal{P}$ has a canonical subbundle $\mathcal{V}$ with

$$\mathcal{O}_P(k)^{\oplus N-1} \cong \mathcal{V} \subset \varphi^*\mathcal{P} \cong \mathcal{O}_P(k)^{\oplus N-1} \oplus \mathcal{O}_P(k-1). \tag{6}$$

We can think of v and s as vectors in the fibers of $\varphi^*\mathcal{F}$ and $\varphi^*\mathcal{P}$ at $[f_0]$. Since $s(x_i) \notin l$, Proposition 2.2 also gives that $s \notin V = \mathcal{V}_E$. Consequently,

$$\varphi^*\mathcal{F} \not\subseteq \mathcal{V}\,. \tag{7}$$

By Grothendieck's theorem

$$\varphi^*\mathcal{F} \cong \mathcal{O}_P(b_1) \oplus \cdots \oplus \mathcal{O}_P(b_r)\,.$$

Since $\varphi^*\mathcal{F} \longrightarrow \varphi^*\mathcal{P}$ is injective, (6) implies that $b_i \leq k$ for all i, and (7) implies that for some i (say $i=1$), $b_1 \leq k-1$.

Fix a polarization L on M. Let δ be the degree of φ^*L. The Corollary 2.3 says that $\delta > 0$. Now,

$$\frac{1}{\delta}\frac{\deg(\mathcal{F})}{\mathrm{rank}(\mathcal{F})} = \frac{\deg(\varphi^*\mathcal{F})}{\mathrm{rank}(\varphi^*\mathcal{F})} \leq k - \frac{1}{r} < k - \frac{1}{N} = \frac{\deg(\varphi^*\mathcal{P})}{\mathrm{rank}(\varphi^*\mathcal{P})} = \frac{1}{\delta}\frac{\deg(\mathcal{P})}{\mathrm{rank}(\mathcal{P})}$$

and hence the Picard bundle $\mathcal{P}$ is stable. This completes the proof of the theorem. □

References

[BV] Balaji V and Vishwanath P R, Deformations of Picard sheaves and moduli of pairs, *Duke Math. J.* **76** (1994) 773–792

[BHM] Brambila-Paz L, Hidalgo-Solís L and Muciño-Raymondo J, On restrictions of the Picard bundle. Complex geometry of groups (Olmué, 1998) 49–56; *Contemp. Math.* 240; *Am. Math. Soc.* (Providence, RI) (1999)

[Gr] Grothendieck A, Sur la classification des fibrés holomorphes sur la sphère de Riemann. *Am. J. Math.* **79** (1957) 121–138

[NR] Narasimhan M S and Ramanan S, Moduli of vector bundles on a compact Riemann surface. *Ann. Math.* **89** (1969) 19–51