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Proximinal subspaces of finite codimension in direct sum spaces

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Abstract. We give a necessary and sufficient condition for proximinality of a closed subspace of finite codimension in c_0-direct sum of Banach spaces.

Keywords. Proximinality and strong proximinality.

0. Notation and preliminaries

Let X be a normed linear space and A be a closed subset of X. We say A is proximinal in X if for each $x \in X$ there exists an element $a \in A$ such that $\|x - a\| = d(x, A)$.

We say A is strongly proximinal in X if A is proximinal in X and given $\epsilon > 0$, there exists $\delta > 0$ such that

$$a \in A, \|x - a\| < d(x, A) + \delta \Rightarrow d(a, P_A(x)) < \epsilon,$$

where $P_A(x) = \{a \in A : \|x - a\| = d(x, A)\}$.

Proximinal subspaces of finite codimension have been studied by various authors (see [1–4, 7–10]). In this paper we obtain a necessary and sufficient condition for proximinality of subspaces of finite codimension in c_0-direct sum of Banach spaces in terms of the proximinality of the corresponding subspaces of finite codimension of the coordinate spaces. We also give an example to show that similar result does not hold in l_1-direct sum of Banach spaces.

Let X be a real normed linear space and X^* its dual. The closed unit ball of X is denoted by B_X and the unit sphere by S_X. Let Y be a closed, linear subspace of codimension n in X. For a set $f_1, f_2, \ldots, f_n$ of linear functionals in the annihilator space $Y^{\perp}$, we give the following definitions from [4]. We have modified the notation used in [4].

$$J_X(f_1) = \{x \in B_X : f_1(x) = \|f_1\|\}$$
$$J_X(f_1, f_2, \ldots, f_i) = \{J_X(f_1, \ldots, f_{i-1}) : f_i(x) = \sup_{x \in J_X(f_1, \ldots, f_{i-1})} f_i(x)\}$$

for $i = 2, 3, \ldots, n$.

Similarly we set

$$J_{(Y^{\perp})^*}(f_1) = \{\Phi \in B_{(Y^{\perp})^*} : \Phi(f_1) = \|f_1\|\}$$
$$J_{(Y^{\perp})^*}(f_1, f_2, \ldots, f_i) = \{\Phi \in J_{(Y^{\perp})^*}(f_1, f_2, \ldots, f_{i-1}) : \Phi(f_i) = \max_{\psi \in J_{(Y^{\perp})^*}(f_1, f_2, \ldots, f_{i-1})} \psi(f_i)\}$$

for $i = 2, 3 \ldots n$. Since $Y^{\perp}$ is finite dimensional, the above n sets are nonempty. We also set

$$M(f_1) = \|f_1\| = N(f_1)$$
$$M(f_1, \ldots, f_i) = \sup\{f_i(x) : x \in J_X(f_1, \ldots, f_{i-1})\}$$

and

$$N(f_1, \ldots, f_i) = \max\{\Phi(f_i) : \Phi \in J_{(Y^{\perp})^*}(f_1, \ldots, f_{i-1})\}.$$

We also need the following Theorem from [4].

Theorem A. *Let X be a normed linear space and Y be a subspace of codimension n in X. Then Y is proximinal in X if and only if for every basis $f_1, \ldots, f_n$ of $Y^{\perp}$ we have*

1. $J_X(f_1, \ldots, f_i) \neq \emptyset$ *for* $1 \leq i \leq n$.
2. $M(f_1, \ldots, f_i) = N(f_1, \ldots, f_i)$ *for* $1 \leq i \leq n$.

We shall first show that condition 2 of the above theorem can be reformulated with conditions only involving the normed linear space X. This is easily done using the weak* density of B_X in $B_{X^{**}}$ and in a manner similar to that of Vlasov [10]. For this purpose we make the following definitions for $\epsilon > 0$ and any finite subset $f_1, \ldots, f_n$ of $Y^{\perp}$.

$$\tilde{N}(f_1) = \|f_1\|,$$
$$J_X(f_1, \epsilon) = \{x \in B_X : f_1(x) > \|f_1\| - \epsilon\},$$
$$\tilde{N}(f_1, \ldots, f_i, \epsilon) = \sup\{f_i(x) : x \in J_X(f_1, \ldots, f_{i-1}, \epsilon)\},$$
$$\tilde{N}(f_1, \ldots, f_i) = \inf_{\epsilon>0} \tilde{N}(f_1, \ldots, f_i, \epsilon)$$

and

$$J_X(f_1, \ldots, f_i, \epsilon) = \{x \in J_X(f_1, \ldots, f_{i-1}, \epsilon) : f_i(x) > \tilde{N}(f_1, \ldots, f_i) - \epsilon\}.$$

1. Proximinality of subspaces of finite codimension

We begin with the following proposition.

PROPOSITION 1.1

Let X be a normed linear space and Y be a closed subspace of codimension n. Then for every finite subset $f_1, \ldots, f_n$ of $Y^{\perp}$ we have

$$\tilde{N}(f_1, \ldots, f_i) = N(f_1, \ldots, f_i) \text{ for } 1 \leq i \leq n.$$

Proof. By induction. The case $i = 1$ is trivial. Assume

$$\tilde{N}(f_1, \ldots, f_k) = N(f_1, \ldots, f_k) \text{ for } 1 \leq k \leq i - 1.$$

Select any $\Phi \in J_{(Y^{\perp})^*}(f_1, f_2 \ldots f_i)$. Since B_X is weak* dense in $B_{X^{**}}$, there exists a net (x_α) in B_X that weak* converges to Φ. In particular,

$$\lim_{\alpha} f_k(x_\alpha) = \Phi(f_k) \text{ for } 1 \leq k \leq n.$$

Thus, given $\epsilon > 0$, $\exists\, \alpha_0$ such that

$$f_k(x_\alpha) > \tilde{N}(f_1, \ldots, f_k) - \epsilon \;\forall\, \alpha \geq \alpha_0 \text{ and } 1 \leq k \leq n.$$

This together with the induction hypothesis implies that

$$(x_\alpha) \in J_X(f_1, \ldots, f_{i-1}, \epsilon) \;\forall\; \alpha \geq \alpha_0.$$

and so

$$\tilde{N}(f_1, \ldots, f_i) \geq N(f_1, \ldots, f_i).$$

To prove the other inequality, for each positive integer n, select an element (x_n) in $J_X\,(f_1, \ldots, f_i, \frac{1}{n})$. Then

$$\tilde{N}(f_1, \ldots, f_k, \frac{1}{n}) > f_k(x_n) > \tilde{N}(f_1, \ldots, f_k) - \frac{1}{n} \text{ for } 1 \leq k \leq i.$$

Let $\psi_n = x_n|Y^\perp$ for each n. Then ψ_n is in $B_{(Y^\perp)^*}$. Since $Y^\perp$ is finite dimensional, w.l.o.g we assume (ψ_n) converges to ψ in $B_{(Y^\perp)^*}$. Note for $1 \leq k \leq i$,

$$\begin{aligned} f_k(\psi) &= \lim_{n\to\infty} f_k(\psi_n) \\ &= \lim_{n\to\infty} f_k(x_n) \\ &\geq \lim_{n\to\infty} \tilde{N}(f_1, \ldots, f_k) - \frac{1}{n} \\ &= \tilde{N}(f_1, \ldots, f_k) \end{aligned}$$

Again by induction hypothesis $\psi \in J_{(Y^\perp)^*}(f_1, f_2, \ldots, f_{i-1})$ and

$$N(f_1, \ldots, f_i) \geq f_i(\psi) \geq \tilde{N}(f_1, \ldots, f_i).$$

Hence

$$N(f_1, \ldots, f_i) = \tilde{N}(f_1, \ldots, f_i)$$

and this completes the induction and the proof.

The above Proposition, along with theorem A, implies as follows:

COROLLARY 1.2

Let Y be a closed subspace of finite codimension n in a normed linear space X. Then Y is proximinal in X if and only if for every basis $f_1, \ldots, f_n$ of $Y^\perp$ the sets

$$\bigcap_{i=1}^{j}\{x \in B_X : f_i(x) = \tilde{N}(f_1, \ldots, f_i)\} \neq \emptyset \; for \; 1 \leq j \leq n.$$

Remark 1.3. If $f_1, \ldots, f_n$ is a finite subset of X^* and $f_{n_1}, \ldots, f_{n_k}$ is a maximal linearly independent subset of $f_1, \ldots, f_n$ satisfying $n_1 < n_2 < \ldots < n_k$ then $\bigcap_{i=1}^{j}\{x \in B_X : f_i(x) = \tilde{N}(f_1, \ldots, f_i)\} \neq \emptyset$ for $1 \leq j \leq n$ if and only if $\bigcap_{i=1}^{m}\{x \in B_X : f_{n_i}(x) = \tilde{N}(f_{n_1}, \ldots, f_{n_i})\} \neq \emptyset$ for $1 \leq m \leq k$.

We now recall some known proximinality results that are needed in the sequel. For any normed linear space X, let $NA(X)$ denote the set of norm attaining elements of X^*. Garkavi [1] has characterized proximinal subspaces of finite codimension in general normed linear spaces and the following is an easy corollary of his result.

Lemma B [5]. *Let X be a normed linear space and Y be a closed subspace of finite codimension in X. Then Y is proximinal in X if and only if every closed subspace $Z \supseteq Y$ is proximinal in X.*

Now, if $f \in X^*$ and H is the kernel of f, it is well-known that the hyperplane H is proximinal in X if and only if $f \in NA(X)$. Thus from Lemma B we have the following.

Remark 1.4. If Y is a proximinal subspace of finite codimension in X, then $Y^{\perp} \subseteq NA(X)$.

However $Y^{\perp} \subseteq NA$ is only a necessary but not a sufficient condition for proximinality of a subspace Y of finite codimension. (See the example of Phelps in [7], p. 309.) But the behaviour of the space $c_0(\Gamma)$ in the above respect is rather special.The following fact is well known, see for instance [6].

Lemma C. Let Y be a closed subspace of finite codimension in $c_0(\Gamma)$ and $f_1, \ldots, f_n$ be a basis of $Y^{\perp}$. Then Y is proximinal if and only if $f_i \in NA(c_0(\Gamma))$ for $1 \leq i \leq n$.

Finally we quote a characterization of strongly proximinal subspaces of finite codimension from [3], which is needed in the proof of our main result.

Theorem B. *Let X be a normed linear space and Y be a proximinal subspace of codimension n in X. Then Y is strongly proximinal in X if and only if the following hold for every basis $f_1, \ldots, f_n$ of $Y^{\perp}$.*

Given $\epsilon > 0$ there exists $\delta > 0$ such that for each i, $1 \leq i \leq n$ and for each $x \in J_X(f_1, \ldots, f_i, \delta)$, we have $d(x, J_X(f_1, \ldots, f_i)) < \epsilon$.

2. Direct sum spaces

We now consider proximinality in c_0-direct sum spaces. Let Λ be an index set and X_λ be a Banach space for each $\lambda \in \Lambda$. Let $X = \oplus_{c_0} X_\lambda$. Then $X^* = \oplus_{l_1} X^*_\lambda$. Further $F = (f_\lambda)_{\lambda \in \Lambda}$ is in $NA(X)$ if and only if $f_\lambda = 0$ but for finite number of indices and $f_\lambda \in NA(X_\lambda)$ whenever $f_\lambda \neq 0$. Also, in this case

$$J_X(F) = \{(x_\lambda) \in X : \|x_\lambda\| = 1 \text{ and } f_\lambda(x_\lambda) = \|f_\lambda\| \; \forall \, \lambda \in \Lambda\} \neq \emptyset. \quad (1)$$

For X defined as above, we have the following Proposition.

PROPOSITION 2.1

Let $F_i \in X^*$ and $F_i = (f_{i\lambda})$ for $1 \leq i \leq n$. Assume further that for each i, $1 \leq i \leq n$, $f_{i\lambda} = 0$ but for finite indices λ. Then

$$\tilde{N}(F_1, \ldots, F_i) = \sum_{\lambda \in \Lambda} \tilde{N}(f_{1\lambda}, \ldots, f_{i\lambda}) \text{ for } 1 \leq i \leq n.$$

Remark. Observe that the above sum has only finite number of nonzero entries. Also, the condition of the above proposition is satisfied if $F_i \in NA(X)$ for $1 \leq i \leq n$.

Proof of the Proposition. Let

$$A = \cup_{i=1}^{n} \{\lambda \in \Lambda : \ f_{i,\lambda} \neq 0\}. \tag{2}$$

Then card $A = l < \infty$. Set

$$\sigma(\epsilon) = \max_{\lambda \in \Lambda} \max_{1 \leq i \leq n} [\tilde{N}(f_{1\lambda}, \ldots, f_{i\lambda}, \epsilon) - \tilde{N}(f_{1\lambda}, \ldots, f_{i\lambda})].$$

For $\epsilon > 0$, let

$$\epsilon_1 = \epsilon, \ \epsilon_i = l\sigma(\epsilon_{i-1}) + \epsilon_{i-1} \text{ for } 2 \leq i \leq n. \tag{3}$$

Then $\epsilon_1 \leq \epsilon_2 \leq \ldots \leq \epsilon_n$. Clearly, $\sigma(\epsilon)$ and therefore ϵ_i, $1 \leq i \leq n$ tend to 0 as $\epsilon \to 0$. Further,

$$\tilde{N}(F_1) = \|F_1\| = \sum_{\lambda} \|f_{1\lambda}\| = \sum_{\lambda} \tilde{N}(f_{1\lambda}).$$

and

$$x \in J_X(F_1, \epsilon) \quad \Leftrightarrow \quad \{x = (x_\lambda) \in B_X : F_1(x) > \|F_1\| - \epsilon\} \tag{4}$$

$$= \quad \{x = (x_\lambda) \in B_X : \sum_{\lambda} f_{1\lambda}(x_\lambda) > \Big(\sum_{\lambda} \tilde{N}(f_{1\lambda})\Big) - \epsilon. \tag{5}$$

Using (4) and (5) and the fact that $f_{1\lambda}(x_\lambda) \leq \tilde{N}(f_{1\lambda})$ we get

$$\begin{aligned}
\textstyle\prod_{\lambda\in\Lambda} J_{X_\lambda}(f_{1\lambda}, \epsilon_1) &= \textstyle\prod_{\lambda\in\Lambda} J_{X_\lambda}(f_{1\lambda}, \epsilon) \\
&\supseteq J_X(F_1, \epsilon) \\
&\supseteq \{x = (x_\lambda) \in B_X : \textstyle\sum_\lambda f_{1\lambda}(x_\lambda) > \\
&\quad \Big(\textstyle\sum_\lambda \tilde{N}(f_{1\lambda})\Big) - \frac{\epsilon}{l} \ \forall \ \lambda \in \Lambda\} \\
&= \textstyle\prod_{\lambda\in\Lambda} J_{X_\lambda}(f_{1\lambda}, \frac{\epsilon}{l}).
\end{aligned}$$

Inductively assume that for some i, $2 \leq i \leq n$ we have

$$\tilde{N}(F_1, \ldots, F_k) = \sum_{\lambda\in\Lambda} \tilde{N}(f_{1\lambda}, \ldots, f_{k\lambda}) \text{ for } 1 \leq k \leq i-1$$

and

$$\prod_{\lambda\in\Lambda} J_{X_\lambda}(f_{1\lambda}, \ldots, f_{i-1\lambda}, \frac{\epsilon}{l}) \subseteq J_X(F_1, \ldots, F_{i-1}, \epsilon) \subseteq \prod_{\lambda\in\Lambda} J_{X_\lambda}(f_{1\lambda}, \ldots, f_{i-1\lambda}, \epsilon_{i-1}).$$

Now, observing that the summation below, over λ, involves only finite number of nonzero terms, we have

$$\begin{aligned}
\tilde{N}(F_1, \ldots, F_i) &= \inf_{\epsilon>0} \sup\{F_i(x) : x \in J_X(F_1, \ldots, F_{i-1}, \epsilon)\} \\
&\leq \inf_{\epsilon>0} \sup \left\{\textstyle\sum_\lambda f_{i\lambda}(x_\lambda) : J_{X_\lambda}(f_{1\lambda}, \ldots, f_{i-1\lambda}, \epsilon_{i-1}) \ \forall\lambda \in \Lambda\right\} \\
&= \inf_{\epsilon>0} \textstyle\sum_\lambda \sup\{f_{i\lambda}(x_\lambda) : J_{X_\lambda}(f_{1\lambda}, \ldots, f_{i-1\lambda}, \epsilon_{i-1}) \ \forall\lambda \in \Lambda\} \\
&= \inf_{\epsilon>0} \textstyle\sum_\lambda \ \tilde{N}(f_{1\lambda}, \ldots, f_{i-1\lambda}, \epsilon_{i-1}) \\
&= \textstyle\sum_\lambda \inf_{\epsilon>0} \ \tilde{N}(f_{1\lambda}, \ldots, f_{i-1\lambda}, \epsilon_{i-1}) \\
&= \textstyle\sum_\lambda \ \tilde{N}(f_{1\lambda}, \ldots, f_{i\lambda}) \text{ for } \epsilon_{i-1} \to 0 \text{ as } \epsilon \to 0.
\end{aligned}$$

Similarly using the other inclusion we conclude

$$\tilde{N}(F_1, \dots, F_i) = \sum_{\lambda \in \Lambda} \tilde{N}(f_{1\lambda}, \dots, f_{i\lambda}).$$

Hence

$$\begin{aligned} D_i &= \{x = (x_\lambda) \in \textstyle\prod_\lambda J_{X_\lambda}(f_{1\lambda}, \dots, f_{i-1\lambda}, \epsilon_{i-1}) : \\ &\quad \textstyle\sum_\lambda f_{i\lambda}(x_\lambda) > \sum_{\lambda \in \Lambda} \tilde{N}(f_{1\lambda}, \dots, f_{i\lambda}) - \epsilon\} \\ &\supseteq \{x = (x_\lambda) \in J_X(F_1, \dots, F_{i-1}, \epsilon) : \\ &\quad \textstyle\sum_\lambda f_{i\lambda}(x_\lambda) > \sum_{\lambda \in \Lambda} \tilde{N}(f_{1\lambda}, \dots, f_{i\lambda}) - \epsilon\} \\ &= \{x = (x_\lambda) \in J_X(F_1, \dots, F_{i-1}, \epsilon) : F_i(x) > \tilde{N}(F_1, \dots, F_i) - \epsilon\} \\ &= J_X(F_1, \dots, F_i, \epsilon). \end{aligned}$$

We have

$$\begin{aligned} x = (x_\lambda) \in D_i \Rightarrow f_{i\lambda}(x_\lambda) &\leq \tilde{N}(f_{1\lambda}, \dots, f_{i\lambda}, \epsilon_{i-1}) \\ &\leq \tilde{N}(f_{1\lambda}, \dots, f_{i\lambda}) + \sigma(\epsilon_{i-1}) \ \forall \lambda. \end{aligned} \tag{6}$$

Further

$$x = (x_\lambda) \in D_i \Rightarrow \sum_\lambda f_{i\lambda}(x_\lambda) > \textstyle\sum_\lambda \tilde{N}(f_{1\lambda}, \dots, f_{i\lambda}) - \epsilon. \tag{7}$$

Let $x = (x_\lambda) \in D_i$. If for some λ_0, $f_{i\lambda_0}(x_{\lambda_0}) \leq \tilde{N}(f_{1\lambda_0}, \dots, f_{i\lambda_0}) - \epsilon_i$ then using (6),

$$\begin{aligned} \textstyle\sum_\lambda f_{i\lambda}(x_\lambda) &\leq \tilde{N}(f_{1\lambda_0}, \dots, f_{i\lambda_0}) - \epsilon_i \\ &\quad + \textstyle\sum_{\substack{\lambda \in A \\ \lambda \neq \lambda_0}} \tilde{N}(f_{1\lambda}, \dots, f_{i\lambda}) + (l-1)\sigma(\epsilon_{i-1}) \\ &\leq \textstyle\sum_\lambda \tilde{N}(f_{1\lambda}, \dots, f_{i\lambda}) + (l-1)\sigma(\epsilon_{i-1}) \\ &\quad - (l\sigma(\epsilon_{i-1}) + \epsilon_{i-1}) \\ &= \textstyle\sum_\lambda \tilde{N}(f_{1\lambda}, \dots, f_{i\lambda}) - \sigma(\epsilon_{i-1}) - \epsilon_{i-1} \end{aligned}$$

which contradicts (7) as $\sigma(\epsilon_{i-1}) + \epsilon_{i-1} > \epsilon$. Thus $x = (x_\lambda) \in D_i$ implies

$$f_{i\lambda}(x_\lambda) > \tilde{N}(f_{1\lambda}, \dots, f_{i\lambda}) - \epsilon_i \ \ \forall \lambda. \tag{8}$$

Now $\epsilon_{i-1} \leq \epsilon_i$ and so

$$J_{X_\lambda}(f_{1\lambda}, \dots, f_{i-1\lambda}, \epsilon_{i-1}) \subseteq J_{X_\lambda}(f_{1\lambda}, \dots, f_{i-1\lambda}, \epsilon_i) \ \forall \lambda. \tag{9}$$

Since $D_i \subseteq \{x = (x_\lambda) \in \prod_\lambda J_{X_\lambda}(f_{1\lambda}, \dots, f_{i-1\lambda}, \epsilon_{i-1})$, using (8) and (9) we conclude

$$D_i \subseteq \{x = (x_\lambda) \in \prod_{\lambda \in \Lambda} J_{X_\lambda}(f_{1\lambda}, \dots, f_{i-1\lambda}, \epsilon_i) \ \ \forall \lambda \}.$$

But $J_X(F_1, \ldots, F_i, \epsilon) \subseteq D_i$ and therefore

$$J_X(F_1, \ldots, F_i, \epsilon) \subseteq \{x = (x_\lambda) \in \prod_{\lambda \in \Lambda} J_{X_\lambda}(f_{1\lambda}, \ldots, f_{i\lambda}, \epsilon_i)\}. \tag{10}$$

On the other hand

$$\begin{aligned} J_X(F_1, \ldots, F_i, \epsilon) &= \{x = (x_\lambda) \in J_X(F_1, \ldots, F_{i-1}, \epsilon) : \\ &\quad F_i(x) > \tilde{N}(F_1, \ldots, F_i) - \epsilon\} \\ &= \{x = (x_\lambda) \in J_X(F_1, \ldots, F_{i-1}, \epsilon) : \\ &\quad \textstyle\sum_\lambda f_{i\lambda}(x_\lambda) > \sum_\lambda \tilde{N}(f_{1\lambda}, \ldots, f_{i\lambda}) - \epsilon\} \\ &\supseteq \{x = (x_\lambda) \in \textstyle\prod_{\lambda \in \Lambda} J_{X_\lambda}(f_{1\lambda}, \ldots, f_{i-1\lambda}, \frac{\epsilon}{l}) : \\ &\quad \textstyle\sum_\lambda f_{i\lambda}(x_\lambda) > \sum_\lambda \tilde{N}(f_{1\lambda}, \ldots, f_{i\lambda}) - \epsilon\} \\ &\supseteq \{x = (x_\lambda) \in \textstyle\prod_{\lambda \in \Lambda} J_{X_\lambda}(f_{1\lambda}, \ldots, f_{i-1\lambda}, \frac{\epsilon}{l}) : \\ &\quad f_{i\lambda}(x_\lambda) > \tilde{N}(f_{1\lambda}, \ldots, f_{i\lambda}) - \tfrac{\epsilon}{l} \;\; \forall \;\; \lambda\} \\ &= \textstyle\prod_{\lambda \in \Lambda} J_{X_\lambda}(f_{1\lambda}, \ldots, f_{i\lambda}, \frac{\epsilon}{l}). \end{aligned}$$

This completes the induction and we have

$$J_X(F_1, \ldots, F_i, \epsilon) \supseteq \prod_{\lambda \in \Lambda} J_{X_\lambda}(f_{1\lambda}, \ldots, f_{i\lambda}, \frac{\epsilon}{l})$$

for $1 \leq i \leq n$. This completes the proof of the proposition.We now prove our main result.

Theorem 2.2. *Let X_λ be a normed linear space for each λ in an index set Λ and X be the c_0-direct sum of the spaces X_λ for $\lambda \in \Lambda$. Let Y be a closed subspace of finite codimension n in X. Then Y is (strongly) proximinal in X if and only if the following two conditions hold for every basis $\{F_i : 1 \leq i \leq n\}$ of $Y^\perp$, where $F_i = (f_{i\lambda})_{\lambda \in \Lambda}$, for $1 \leq i \leq n$.*

1. *For each i, $1 \leq i \leq n$, $f_{i\lambda}$ is nonzero only for finite number of indices λ.*
2. *$Y_\lambda = \cap\{\mathrm{Ker}\; f_{i\lambda} : 1 \leq i \leq n\}$ is (strongly) proximinal in X_λ for each $\lambda \in \Lambda$.*

Proof. Necessity. First we observe that by remark 1.4, $F_i \in NA(X)$ for any basis $\{F_i : 1 \leq i \leq n\}$ of $Y^\perp$. This, in particular, implies condition 1 above. Hence Y_λ is a proper subspace of X_λ only for finite number of indices λ.

To prove 2, for each λ such that Y_λ is a proper subspace of X_λ, choose any basis $g_{1\lambda}, \ldots, g_{n\lambda}$ of $(Y_\lambda)^\perp$. If $G_i = (g_{i\lambda})$ for $1 \leq i \leq n$ then $G_1, \ldots, G_n$ is a basis of $Y^\perp$. Since Y is proximinal in X,we can, by Corollary 1, get an element $x = (x_\lambda) \in X$ satisfying $G_i(x) = \tilde{N}(G_1, \ldots, G_i)$ for $1 \leq i \leq n$. In particular,

$$G_1(x) = \sum_\lambda g_{1\lambda}(x_\lambda) = \|G_1\| = \sum_\lambda \|g_{1\lambda}\|.$$

We have $\|x\| = \|\sup_\lambda \|x_\lambda\| \leq 1$ and so the above inequality implies $g_{1\lambda}(x_\lambda) = \|g_{1\lambda}\|$ for all λ. Assume inductively,

$$g_{k\lambda}(x_\lambda) = \tilde{N}(g_{1\lambda}, \ldots, g_{k\lambda}) \text{ for } 1 \leq k \leq i-1 \text{ and } \forall \;\lambda.$$

Now again by Remark 1.4, $G_i \in NA(X)$ for $1 \leq i \leq n$. Hence by Proposition 2.1 we have,

$$G_i(x) = \sum_\lambda g_{i\lambda}(x_\lambda) = \tilde{N}(G_1, \ldots, G_i) = \sum_\lambda \tilde{N}(g_{1\lambda}, \ldots, g_{i\lambda}). \tag{11}$$

Also by induction hypothesis, $x_\lambda \in \tilde{N}(g_{1\lambda}, \ldots, g_{i-1\lambda}, \epsilon)$ for every $\epsilon > 0$ and so we have

$$g_{i\lambda}(x_\lambda) \ \leq \ \tilde{N}(g_{1\lambda}, \ldots, g_{i\lambda}) \ \forall \, \lambda.$$

This with (11) implies

$$g_{i\lambda}(x_\lambda) \ = \ \tilde{N}(g_{1\lambda}, \ldots, g_{i\lambda}) \ \forall \, \lambda.$$

and completes the process of induction. Hence for all λ, $x_\lambda \in X_\lambda$ satisfies

$$g_{i\lambda}(x_\lambda) \ = \ \tilde{N}(g_{1\lambda}, \ldots, g_{i\lambda}) \, \forall \ 1 \leq i \leq n. \tag{12}$$

By Corollary 1.2, Y_λ is proximinal in X_λ for each λ.

If Y is strongly proximinal in X, then given $\epsilon > 0$ there exists $\delta > 0$ such that for any G in $x = (x_\lambda)$ in $J_X(G_1, \ldots, G_i, \delta)$,

$$d(G, J_X(G_1, \ldots, G_i)) \ < \ \epsilon \ \forall \ 1 \ \leq i \ \leq n. \tag{13}$$

It is easy to verify using (11) and (12) that

$$J_X(G_1, \ldots, G_i) \ = \ \prod_{\lambda \in \Lambda} J_{X_\lambda}(g_{1\lambda}, \ldots, g_{i\lambda}) \tag{14}$$

and

$$J_X(G_1, \ldots, G_i, \delta) \ \supseteq \ \prod_{\lambda \in \Lambda} J_{X_\lambda}(g_{1\lambda}, \ldots, g_{i\lambda}, \frac{\delta}{l})$$

for $1 \leq i \leq n$. Now using (13) and (14) we conclude that for any λ and h_i in $J_{X_\lambda}(g_{1\lambda}, \ldots, g_{i\lambda}, \frac{\delta}{l})$ we have

$$d(h_i, J_{X_\lambda}(g_{1\lambda}, \ldots, g_{i\lambda})) \ < \ \epsilon$$

for $1 \leq i \leq n$. Hence Y_λ is strongly proximinal in X_λ for each λ.

Sufficiency. If $G_1, \ldots, G_n$ is any basis of $Y^\perp$ and $G_i = (g_{i\lambda})_{\lambda \in \Lambda}$, then by condition 1, for each i, $1 \leq i \leq n$, $g_{i\lambda} = 0$ except for finite number of indices λ. So, Proposition 2.1 can be applied to the basis $\{G_i : 1 \leq i \leq n\}$ of $Y^\perp$.

Since Y_λ is proximinal for each λ, by Remark 1.3 and Corollary 1.2, there exists $x_\lambda \in B_{(X_\lambda)}$ satisfying for each λ,

$$g_{i\lambda}(x_\lambda) \ = \ \tilde{N}(g_{1\lambda}, \ldots, g_{i\lambda}) \ \forall \, 1 \leq i \leq n.$$

Now let $x = (x_\lambda)_{\lambda \in \Lambda}$. Clearly $x \in B_X$ and Proposition 2.1 implies

$$G_i(x) = \sum_\lambda g_{i\lambda}(x_\lambda) = \sum_\lambda \tilde{N}(g_{1\lambda}, \ldots, g_{i\lambda}) = \tilde{N}(G_1, \ldots, G_i) \text{ for } 1 \leq i \leq n.$$

The conclusion now follows from Corollary 1.2.

Assume now Y_λ strongly proximinal in X_λ for each λ. Let $\epsilon > 0$ be given. Since the set A given by (2) is finite we can get $\delta > 0$ such that for each $\lambda \in A$ and h_i in $J_{X_\lambda}(g_{1\lambda}, \ldots, g_{i\lambda}, \delta))$ we have

$$d(h_i, J_{X_\lambda}(g_{1\lambda}, \ldots, g_{i\lambda})) < \epsilon \ \text{ for } \ 1 \le i \le n. \tag{15}$$

Now choose $\eta > 0$ small enough so that η_i is given by

$$\eta_1 = \eta, \ \eta_i = l\sigma(\eta_{i-1}) + \eta_{i-1} \text{ for } 2 \le i \le n$$

as in (3), is less than δ. We have from (10)

$$J_X(G_1, \ldots, G_i, \eta) \subseteq \prod_{\lambda \in \Lambda} J_{X_\lambda}(g_{1\lambda}, \ldots, g_{i\lambda}, \eta_i) \subseteq \prod_{\lambda \in \Lambda} J_{X_\lambda}(g_{1\lambda}, \ldots, g_{i\lambda}, \delta). \tag{16}$$

Clearly (14), (15) and (16) imply that if $x = (x_\lambda) \in J_X(G_1, \ldots, G_i, \eta)$ then

$$d(x, J_X(G_1, \ldots, G_i)) < \epsilon$$

and this completes the proof.

We now give an alternate shorter proof for the proximinality of Y when conditions (1) and (2) of Theorem 2.2 are satisfied. This proof avoids the use of Proposition 2.1 and uses Lemma B.

Let $\{X_j : 1 \le j \le l\} = \{X_\lambda : \lambda \in A\}$, where the set A is given by (2). We set

$$G = X_1 \oplus_\infty X_2 \oplus_\infty \ldots \oplus_\infty X_l$$

and

$$Z_j = \cap_{1 \le i \le n} \operatorname{Ker} f_{ij}$$

for $1 \le j \le l$. We have Z_j to be proximinal subspace of finite codimension in X_j for $1 \le j \le l$, by (2) of Theorem 2.2. Further if

$$Z = Z_1 \oplus_\infty Z_2 \oplus_\infty \ldots \oplus_\infty Z_l,$$

then Z is a proximinal subspace of finite codimension in G. Set

$$Y_1 = \cap_{i=1}^n \{(x_1, \ldots, x_l) : x_j \in X_j \ \forall\ 1 \le j \le l \text{ and } \sum_{1 \le j \le l} f_{ij}(x_j) = 0\}.$$

Then Y_1 is a subspace of G, $Z \subseteq Y_1 \subseteq G$. Now we use Lemma B to conclude Y_1 is proximinal in G. It is easily verified that this, in turn, implies proximinality of Y in X.

Remark 2.1. It is easy to see that the above proof goes through when X is taken as a finite l_1-direct sum of normed linear spaces and condition (2) of Theorem 2.2 is satisfied. The example below shows that this is no longer the case when X is an infinite l_1-direct sum.

Remark 2.2. We observe here that the necessity of Theorem 2.2 does not hold even for finite l_1-direct sums. For instance, let X be a non-reflexive Banach space and pick f and g in the unit sphere of X^* such that there exists $x \in X$ with $\|x\| = 1 = f(x)$ and g does not attain its norm on X. Now $1 = \max\{\|f\|, \|g\|\} = \|(f, g)\| = 1 = f(x)$. Hence (f, g) attains its norm at $(x, 0)$ and $Z = \{(x, y) : f(x) + g(y) = 0\}$ is a proximinal subspace but $\operatorname{Ker}(g)$ is not proximinal in X.

3. Example

Theorem 2.2 is not true if we replace the c_0-direct sum by, for instance, the l_1 direct sum as the following example shows.

Example. Let $X = \oplus_{l_1} X_n$ where $X_n = c_0$ for $n = 1, 2, \ldots$. Then $X^* = \oplus_{l_\infty} X_n^*$. Select for each positive integer n, $f_{in} \in l_1$ with $\|f_{in}\| \le 1$ and $f_{in} \in NA(c_0)$. Further set

$$\begin{aligned} f_{12} &= f_{21} = 0, \\ \|f_{11}\| &= \|f_{22}\| = 1, \\ f_{1n} &= f_{2n} \text{ for } n \ge 3, \\ \|f_{in}\| &< 1 \text{ for } n \ge 3, i = 1, 2 \end{aligned}$$

and

$$\lim_{n\to\infty} \|f_{in}\| = 1.$$

Define $F_i \in X^*$, $i = 1, 2$, as

$$F_i = (f_{i1}, f_{i2} \ldots f_{in} \ldots), \quad i = 1, 2.$$

Let $Y = \cap\{\text{Ker } F_i : i = 1, 2\}$ and $Y_n = \cap\{\text{Ker } f_{in} : i = 1, 2\}$ for $n = 1, 2, \ldots$. Since $f_{in} \in NA(C_0)$ for $i = 1, 2$ and for all n, Y_n is proximinal in $X_n = C_0$ for all n.We will now show that Y is not proximinal in X.

Choose $x_i \in C_0$, $i = 1, 2$ such that $\|x_i\| = 1$ for $i = 1, 2$ and $f_{11}(x_1) = f_{22}(x_2) = 1$. Consider $x = (x_1, x_2, 0, 0, \ldots)$ in X. Then $\|x\| = 2$. Further $F_i \in NA(X)$ as

$$F_1(x) = f_{11}(x) = 1 = F_2(x) = f_{22}(x_2).$$

So, $d(x, Y) = \|x|Y^\perp\| \ge 1$.We now show that $d(x, Y) = 1$.

To see this, select for $n \ge 3$, $x_n \in C_0$ satisfying $f_{in}(x_n) = -1$ for $i = 1, 2$ and $\lim_{n\to\infty} \|x_n\| = 1$. Define a sequence $(y_k)_{k\ge3} \in X$ by

$$y_k(n) = \begin{cases} x_n & \text{if } n \in \{1, 2, k\} \\ 0 & \text{otherwise.} \end{cases}$$

Then $F_i(y_k) = f_{11}(x_1) + f_{ik}(x_k) = 0$ for $i = 1, 2$ and so $y_k \in Y$ for all k. Further

$$\|x - y_k\| = \|x_k\| \to 1 \text{ as } k \to \infty.$$

Hence $d(x, Y) = 1$.

We recall that a nearest element to x from Y exists if and only if there exists y in X satisfying

$$F_i(y) = F_i(x) = 1 \text{ for } i = 1, 2 \quad \|y\| = d(x, Y) = 1.$$

However $\|y\| = 1 = F_1(y)$ implies $y = (y_1, 0, 0 \ldots)$ where $f_{11}(y_1) = 1$, $\|y_1\| = 1$. But, in this case, $F_2(y) = 0 \ne F_2(x)$ and the above equality can not hold. Therefore, Y is not proximinal in X.

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References

[1] Garkavi A L, On the best approximation by the elements of infinite dimensional subspaces of a certain class, *Mat. Sb.* **62** (1963) 104–120

[2] Garkavi A L, Helley's problem and best approximation in spaces of continuous functions, *Izv. Akad. Nauk SSSR Ser. Mat.* **31** (1967) 641–656

[3] Godefroy Gilles and Indumathi V, Strong proximinality and polyhedral spaces, *Revista Matemática Complutense* (to appear)

[4] Indumathi V, Proximinal subspaces of finite codimension in general normed linear spaces, *Proc. London Math. Soc.* **45(3)** (1982) 435–455

[5] Indumathi V, On transitivity of proximinality, *J. Approx. Theory* **49(2)** (1987) 130–143

[6] Pollul W, Reflexvitat und Existenz-Teilraume in der linearen Approximations theorie, *Dissertation Bonn* (1971); Schriften der Ges fur Math und Datenverarbeitung, *Bonn* **53** (1972) 1–21

[7] Singer Ivan, Best approximation in normed linear spaces by elements of linear subspaces, Die Grundlehren der mathematischen Wissenschaften, *Band* 171 (Springer Verlag) (1970)

[8] Singer Ivan, On best approximation in normed linear spaces by elements of subspaces of finite codimension, *Rev. Roumaine Math. Pure. Appl.* **17** (1972) 1245–1256

[9] Vlasov L P, Elements of best approximation relative to subspaces of finite codimension, *Mat. Zametki* **32(3)** (1982) 325–341

[10] Vlasov L P, Subspaces of finite codimension: Existence of elements of best approximation, *Mat. Zametki* **37(1)** (1985) 78–85