

On a Tauberian theorem of Hardy and Littlewood

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Abstract. In this paper, we give a simple alternative proof of a Tauberian theorem of Hardy and Littlewood (Theorem E stated below, [3]).

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1. Introduction

Let $\sum_{n=0}^{\infty} a_n$ be an infinite series of real terms. Let

$$0 \leq \lambda_0 < \lambda_1 < \dots, \lambda_n \rightarrow \infty$$

and let $\sum a_n e^{-\lambda_n x}$ be convergent for all $x > 0$. If

$$f(x) = \sum a_n e^{-\lambda_n x} \rightarrow s$$

as $x \rightarrow 0$, then we say that $\sum a_n$ is summable (A, λ_n) to s . When $\lambda_n = n$, the method (A, λ_n) reduces to the classical method summability (A) , named after Abel.

It is a famous result due to Abel that if $\sum a_n$ is convergent to s , then $\sum a_n$ is summable (A) to s . That the converse is not necessarily true is evident from the example of the series

$$1 - 1 + 1 - 1 \dots$$

which is summable (A) to $\frac{1}{2}$, but not convergent. The question naturally arises as to whether one can determine a suitable restriction or restrictions on the general term a_n so that $\sum a_n$ will be convergent to s whenever it is summable (A) . The first answer to this question was given by Tauber in 1897 in the form of the following theorem.

Theorem A [7]. *If $\sum a_n$ is summable (A) to s and $na_n = o(1)$, then $\sum a_n$ is convergent to s .*

A generalization of Theorem A to the set-up of summability (A, λ_n) was proved by Landau [4].

Another significant generalization of Theorem A was obtained by Littlewood in 1910 in the form of

Theorem B. *If $\sum a_n$ is summable (A) to s , and $na_n = O(1)$, then $\sum a_n$ is convergent to s .*

In fact Littlewood proved the following more general theorem.

Theorem C [5]. If $\sum \mu_n$ is a series of positive terms such that, as $n \rightarrow \infty$,

$$\lambda_n = \mu_1 + \mu_2 + \cdots + \mu_n \rightarrow \infty, \quad \mu_n/\lambda_n \rightarrow 0,$$

$\sum a_n e^{-\lambda_n x} \rightarrow s$ as $x \rightarrow 0$, and

$$a_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right),$$

then $\sum a_n$ is convergent to s .

Littlewood had stated that Theorem C is true even without the restriction: $\mu_n/\lambda_n \rightarrow 0$. This result is stated below as Theorem C*. It was proved in 1928 by Ananda-Rau [1]. A simple alternative proof was supplied by Bosanquet (see Hardy [2]).

Theorem C*. If $\sum \mu_n$ is a series of positive terms such that $\lambda_n = \mu_1 + \mu_2 + \cdots + \mu_n \rightarrow \infty$ as $n \rightarrow \infty$, $\sum a_n e^{-\lambda_n x} \rightarrow s$ as $x \rightarrow 0$ and

$$a_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right), \quad (1.1)$$

then $\sum a_n$ is convergent to s .

Littlewood also conjectured [5] that the following theorem is true.

Theorem D. If $\lambda_1 > 0$, $\lambda_{n+1}/\lambda_n \geq \theta > 1$ ($n = 1, 2, \dots$), and

$$f(x) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} \rightarrow s \text{ as } x \rightarrow 0,$$

then $\sum a_n$ converges to s .

The truth of this conjecture was proved by Hardy and Littlewood [3]¹. Theorems of this kind are called ‘high indices’ theorems, as distinguished from ‘Tauberian’ theorems, since in such theorems no restriction is needed to be imposed upon the general term a_n of the series in question, excepting, of course, that $\sum a_n e^{-\lambda_n x}$ is convergent for every $x > 0$. Such a theorem shows that the method (A, λ_n) with the type of λ_n involved does not sum any series which is not convergent, and therefore shows the ‘ineffectiveness’ of the method (A, λ_n) .

Hardy and Littlewood first established Theorem D in the special case in which

$$a_n = O(1)$$

and then, by further analysis, derived Theorem D itself. This is an instance of a Tauberian theorem leading to a high indices theorem. Thus Hardy and Littlewood first established the following Tauberian theorem.

Theorem E. If $\lambda_1 > 0$, $\lambda_{n+1}/\lambda_n \geq \theta > 1$ ($n = 1, 2, \dots$),

$$f(x) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} \rightarrow s \text{ as } x \rightarrow 0,$$

¹ For a proof of Theorem D due to A E Ingham, see [2], proof of Theorem 114, where too, the result has been obtained via a Tauberian theorem.

and $a_n = O(1)$, then $\sum a_n$ is convergent to s .

It should be observed that Theorem E is included in the theorem of Ananda-Rau, in which no extra restriction is imposed on λ_n , in view of the fact that whenever $\lambda_{n+1}/\lambda_n \geq \theta > 1$, and $a_n = O(1)$, $a_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right)$. On the other hand, under the hypotheses of Theorem C*, (1.1) implies: $s_n = O(1)$, and hence $a_n = O(1)$ (see Lemma 2 in the sequel).

The object of the present paper is to give an alternative proof of Theorem E which is quite straightforward, not requiring Lemmas 1 and 2 of Hardy and Littlewood [3].

2. Lemmas

We shall need the following lemmas.

Lemma 1 [5]. If, as $y \rightarrow 0$, $\psi(y) \rightarrow s$, and for every positive integer r ,

$$y^r \psi^{(r)}(y) = O(1),$$

then for every positive integer r , $y^r \psi^{(r)}(y) = o(1)$.

Lemma 2 [4].² If $0 < \lambda_1 < \lambda_2 < \dots, \lambda_n \rightarrow \infty$, as $n \rightarrow \infty$, $f(x) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} = O(1)$ as $x \rightarrow 0$, and

$$a_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right),$$

then

$$s_n = a_1 + a_2 + \dots + a_n = O(1).$$

Lemma 3 [3]. If $\lambda_1 > 0$ and

$$\frac{\lambda_{n+1}}{\lambda_n} \geq \theta > 1 \quad (n = 1, 2, \dots)$$

then, for $r = 1, 2, \dots$,

$$\sum_{n=1}^{\infty} \lambda_n^r e^{-\lambda_n x} = O(x^{-r}).$$

3. Proof of Theorem E

We may assume, without loss of generality, that $s = 0$. Thus $f(x) = o(1)$ as $x \rightarrow 0$. Also, since $a_n = O(1)$, for $r = 1, 2, \dots$,

$$\begin{aligned} x^r f^{(r)}(x) &= (-1)^r x^r \sum_{n=1}^{\infty} a_n \lambda_n^r e^{-\lambda_n x} \\ &= O\left(x^r \sum_{n=1}^{\infty} \lambda_n^r e^{-\lambda_n x}\right) \\ &= O(1), \end{aligned}$$

² As remarked by Ananda-Rau in [1], the argument in Landau [4], pp. 13–14, has only to be slightly modified to yield the result of Lemma 2.

by Lemma 3. Hence, by Lemma 1,

$$x^r f^{(r)}(x) = o(1).$$

Since

$$f(x) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} = \sum_{n=1}^{\infty} s_n (e^{-\lambda_n x} - e^{-\lambda_{n+1} x}) = x \sum_{n=1}^{\infty} s_n \int_{\lambda_n}^{\lambda_{n+1}} e^{-xu} du,$$

we have

$$\begin{aligned} (-1)^r x^r f^{(r)}(x) &= x^{r+1} \sum_{n=1}^{\infty} s_n \int_{\lambda_n}^{\lambda_{n+1}} u^r e^{-xu} du - r x^r \sum_{n=1}^{\infty} s_n \int_{\lambda_n}^{\lambda_{n+1}} u^{r-1} e^{-xu} du \\ &= V_r - r V_{r-1}, \text{ say.} \end{aligned}$$

Hence, by Lemma 1,

$$\begin{aligned} V_r &= r V_{r-1} + o(1) \\ &= r(r-1)V_{r-2} + o(r) + o(1) \\ &= r(r-1)(r-2)V_{r-3} + o(r(r-1)) + o(r) + o(1) \\ &= \dots \\ &= r! f(x) + o(r(r-1)\dots 2) + \dots + o(1), \end{aligned}$$

so that³

$$\begin{aligned} \frac{V_r}{r!} &= f(x) + o\left(1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{r!}\right) \\ &= f(x) + o(1) \\ &= o(1). \end{aligned} \tag{3.1}$$

This can be explicitly written as

$$\frac{1}{r!} |F(x)| = \frac{1}{r!} \left| x^{r+1} \sum s_n \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt} dt \right| \rightarrow 0 \quad (r = 1, 2, \dots) \tag{3.2}$$

as $x \rightarrow 0$. If s_n does not converge to zero, there exists a positive constant h such that $|s_n| > h$ for an infinite number of values of n . Let m be any one of these values. We shall show that, when r exceeds a sufficiently large positive integer r_0 ,

$$\overline{\lim}_{x \rightarrow 0} \frac{1}{r!} |F(x)| \geq \delta > 0,$$

where δ is a positive constant. This will contradict (3.2), and hence we will conclude that $\sum a_n$ converges to zero, which is required to be proved.

³ In Hardy and Littlewood [3], (2.41) should be replaced by our (3.1) $\frac{V_r}{r!} = o(1)$; line 4 from the top on p. 225 should be replaced by: $r! \sum_{n=0}^{\infty} s_n w_n = V_r$ so that $\sum_{n=0}^{\infty} s_n w_n = \frac{V_r}{r!} = o(1)$. For similar alterations needed in the papers [1], [5] and [8], see Pati [6].

Now, by the hypotheses of Theorem E and Lemma 2, $s_n = O(1)$ and hence

$$\begin{aligned}
 |F(x)| &\geq |s_m|x^{r+1} \sum_{n=1}^{\infty} \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt} dt - \left| x^{r+1} \sum_{n=1}^{\infty} (s_n - s_m) \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt} dt \right| \\
 &> |s_m|x^{r+1} \sum_{n=1}^{\infty} \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt} dt - Kx^{r+1} \sum_{m+1}^{\infty} \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt} dt \\
 &\quad - Kx^{r+1} \sum_{n=1}^{m-1} \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt} dt, \tag{3.3}
 \end{aligned}$$

where K is a positive constant. We choose

$$x = \frac{2r}{\lambda_{m+1} + \lambda_m}.$$

Then, for fixed r , $x \rightarrow 0$ iff $m \rightarrow \infty$. Since ([5], p. 440)

$$\begin{aligned}
 \lim_{x \rightarrow 0} x^{r+1} \sum_{n=1}^{\infty} \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt} dt &= r!, \\
 \frac{1}{r!} \overline{\lim}_{x \rightarrow 0} |s_m|x^{r+1} \sum_{n=1}^{\infty} \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt} dt &\geq h. \tag{3.4}
 \end{aligned}$$

We now use the transformation $u = xt$, so that, for $t = \lambda_{m+1}$,

$$u = r \frac{2\lambda_{m+1}}{\lambda_{m+1} + \lambda_m} = r(1 + \eta),$$

where

$$\eta = \frac{\lambda_{m+1} - \lambda_m}{\lambda_{m+1} + \lambda_m}. \tag{3.5}$$

Thus the second term in (3.3) gives

$$\overline{\lim}_{x \rightarrow 0} x^{r+1} \sum_{m+1}^{\infty} \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt} dt \leq \int_{r(1+\eta)}^{\infty} u^r e^{-u} du. \tag{3.6}$$

The third term in (3.3) gives

$$\overline{\lim}_{x \rightarrow 0} x^{r+1} \sum_1^{m-1} \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt} dt \leq \int_0^{r(1-\eta)} u^r e^{-u} du, \tag{3.7}$$

where η is as defined in (3.5).

Combining (3.4), (3.6) and (3.7) we have

$$\frac{1}{r!} \overline{\lim}_{x \rightarrow 0} |F(x)| \geq h - \frac{K}{r!} \left[\int_0^{r(1-\eta)} u^r e^{-u} du + \int_{r(1+\eta)}^{\infty} u^r e^{-u} du \right]. \tag{3.8}$$

We show below that

$$I_1 \equiv \int_0^{r(1-\eta)} u^r e^{-u} du < K_1 r^r e^{-r} \tag{3.9}$$

and

$$I_2 \equiv \int_{r(1+\eta)}^{\infty} u^r e^{-u} du < K_2 r^r e^{-r}, \quad (3.10)$$

where the K in each inequality denotes a positive constant, independent of r .

Proof of (3.9). We have

$$I_1 = -u^r e^{-u} \Big|_0^{r(1-\eta)} + r \int_0^{r(1-\eta)} u^{r-1} e^{-u} du.$$

Hence

$$\int_0^{r(1-\eta)} \left(\frac{r}{u} - 1\right) u^r e^{-u} du = r^r (1-\eta)^r e^{-r(1-\eta)}.$$

Now, since $0 < \eta < 1$, $u \leq r(1-\eta)$ implies:

$$\frac{r}{u} - 1 \geq \frac{\eta}{1-\eta},$$

so that

$$\frac{\eta}{1-\eta} I_1 \leq r^r e^{-r} [(1-\eta)e^\eta]^r < r^r e^{-r},$$

since

$$e^\eta < 1 + \eta + \eta^2 + \dots = \frac{1}{1-\eta}.$$

Thus

$$I_1 < K_1 r^r e^{-r},$$

where

$$K_1 = \frac{1-\eta}{\eta} = \frac{2\lambda_m}{\lambda_{m+1} - \lambda_m} \leq \frac{2}{\theta - 1} \quad (\theta > 1).$$

Proof of (3.10). We have

$$I_2 = -u^r e^{-u} \Big|_{r(1+\eta)}^{\infty} + r \int_{r(1+\eta)}^{\infty} u^{r-1} e^{-u} du.$$

Hence

$$\begin{aligned} \int_{r(1+\eta)}^{\infty} \left(1 - \frac{r}{u}\right) u^r e^{-u} du &= r^r e^{-r} \left(\frac{1+\eta}{e^\eta}\right)^r \\ &< r^r e^{-r}; \end{aligned}$$

since

$$u \geq r(1+\eta) \quad \text{implies:} \quad 1 - \frac{r}{u} \geq \frac{\eta}{1+\eta},$$

we have

$$\frac{\eta}{1+\eta} I_2 < r^r e^{-r},$$

so that

$$I_2 < K_2 r^r e^{-r},$$

where

$$K_2 = \frac{1 + \eta}{\eta} = \frac{2\lambda_{m+1}}{\lambda_{m+1} - \lambda_m} = \frac{2}{1 - \frac{\lambda_m}{\lambda_{m+1}}} \leq \frac{2\theta}{\theta - 1} \quad (\theta > 1).$$

Hence, from (3.8), (3.9) and (3.10), we have

$$\overline{\lim}_{x \rightarrow 0} \frac{1}{r!} |F(x)| \geq h - 2K \frac{\theta + 1}{\theta - 1} \frac{r^r e^{-r}}{r!}.$$

Since by Stirling's theorem,

$$\frac{r^r e^{-r}}{r!} \sim \frac{1}{\sqrt{2\pi}} r^{-\frac{1}{2}},$$

taking $r > r_0$, a sufficiently large positive integer, we have

$$\overline{\lim}_{x \rightarrow 0} \frac{1}{r!} |F(x)| \geq \delta > 0,$$

which contradicts (3.2). Hence our assumption that $\{s_n\}$ does not converge to 0 is false.

This completes the proof of Theorem E.

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