Proc. Indian Acad. Sci. (Math. Sci.), Vol. 111, No. 2, May 2001, pp. 203–219.
Printed in India

Multipliers for the absolute Euler summability of Fourier series

PREM CHANDRA

School of Studies in Mathematics, Vikram University, Ujjain 456 010, India

MS received 30 December 1999; revised 30 October 2000

Abstract. In this paper, the author has investigated necessary and sufficient conditions for the absolute Euler summability of the Fourier series with miltipliers. These conditions are weaker than those obtained earlier by some workers. It is further shown that the multipliers are best possible in certain sense.

Keywords. Multipliers; absolute summability; summability of factored Fourier series; absolute Euler summability.

1. Definitions and notations

Let $\sum_{n=0}^{\infty} w_n$ be a given infinite series and let q be a real or complex number such that $q \neq -1$. Then we write

$$w_n^q = (1+q)^{-n-1} \sum_{m=0}^{n} \binom{n}{m} q^{n-m} w_m; \quad w_n^0 = w_n. \tag{1.1}$$

Following Chandra [2], $\sum w_n$ is said to be absolutely summable by (E, q) means (or Euler means) or simply $\sum_{n=0}^{\infty} a_n \in |E, q|$ if

$$\sum_{n=0}^{\infty} \quad |w_n^q| \quad < \quad \infty. \tag{1.2}$$

For $q > 0$, a reference may be made to Hardy ([9]; p. 237). It may be observed that the method $|E, q|$ $(q > 0)$ is absolutely regular.

Let $L_{2\pi}$ be the space of all 2π-periodic and Lebesgue-integrable functions over $[-\pi, \pi]$. Then the Fourier series of $f \in L_{2\pi}$ at x is given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x), \tag{1.3}$$

where a_n and b_n are the Fourier coefficients of f.

Throughout the paper, we assume that the constant term $a_0 = 0$. For real x, $q > 0$ and $\delta \geq 0$, we write

$$\phi(t) \quad = \quad \frac{1}{2}\{f(x+t) + f(x-t)\}, \tag{1.4}$$

$$\phi_1(t) \quad = \quad \frac{1}{t}\int_0^t \phi(u)\mathrm{d}u, \tag{1.5}$$

$$P(t) = \phi(t) - \phi_1(t), \tag{1.6}$$

$$y(t) = (1+q)^{-1}(1+q^2+2q\cos t)^{1/2}, \tag{1.7}$$

$$V_m^q(n) = (1+q)^{-n-1}\binom{n+1}{m+1}q^{n-m} \quad (m \le n), \tag{1.8}$$

$$s_m(t) = \sum_{r=1}^{m} V_r^q(n)\sin rt, \tag{1.9}$$

$$d_n = \log^{-\delta}(n+1), \tag{1.10}$$

$$\Delta d_n = d_n - d_{n+1}, \tag{1.11}$$

$$g(t) = P(t)\log^{-\delta}\frac{k}{t}, \tag{1.12}$$

$$b(t) = t\log^{\delta}\frac{k}{t}, \tag{1.13}$$

$$H_n(t) = b(t)\frac{\sin nt}{nt} + \int_t^c \frac{\sin nu}{nu}\,\mathrm{d}\,b(u), \tag{1.14}$$

where $0 < c \le \pi$ and k is a suitable positive constant taken for the convenience in the analysis and possibly depending upon δ.

2. Introduction

In 1968, Mohanty and Mohapatra [12] began the study of absolute Euler summability of Fourier series by proving the following:

Theorem A. *Let*

$$\phi(t)\log\frac{1}{t} \in BV(0,c), \quad 0 < c < 1. \tag{2.1}$$

Then

$$\sum A_n(x) \in |E,q| \quad (q > 0). \tag{2.2}$$

Among other results the above result was also proved by Kwee [10] independently. He also proved that the condition (2.1) cannot be replaced by the weaker condition

$$\phi(t)\log^{\eta}\frac{1}{t} \in BV(0,c), \quad 0 < \eta < 1, \tag{2.3}$$

in Theorem A. This result of Kwee [10] was further improved by the present author and Dikshit [7].

In 1978, the present author [4] proved the following:

Theorem B. *Let*

$$\phi(t) \in BV(0,\pi). \tag{2.4}$$

Then

$$\sum_{n=1}^{\infty}\frac{A_n(x)}{\log(n+1)} \in |E,q| \qquad (q > 0). \tag{2.5}$$

Recently, Ray and Sahoo [15] have not only bridged the gap in between Theorems A and B but they have also improved Theorem B by proving the following:

Theorem C. *Let* $0 \leq \delta \leq 1$ *and let*

$$\phi(t)\log^{1-\delta}\frac{k}{t} \in BV(0,c), \quad 0 < c < 1. \tag{2.6}$$

Then

$$\sum_{n=1}^{\infty}\frac{A_n(x)}{\log^{\delta}(n+1)} \in |E,q| \qquad (q > 0). \tag{2.7}$$

It may be remarked that in Theorem C, δ has been restricted to be in [0,1] since for $\delta > 1$, (2.6) implies the absolute convergence of

$$\sum_{n=1}^{\infty}\frac{A_n(x)}{\log^{\delta}(n+1)}. \tag{2.8}$$

A reference may be made to Chandra [1]; Theorem 2 on page 6, and hence (2.8) is necessarily summable $|E,q|$ $(q > 0)$.

In a different setting, very recently, Dikshit [8] has obtained a few more results concerning the absolute Euler summability factors for Fourier series.

One of the main objects of the present paper is to improve Theorem C on replacing (2.6) by the following weaker condition:

$$\left.\begin{array}{lrcl} \text{(i)} & P(t)\log^{1-\delta}\dfrac{k}{t} & \in & BV(0,c) \\ \text{(ii)} & t^{-1}P(t)\log^{-\delta}\dfrac{k}{t} & \in & L(0,c) \end{array}\right\}, \tag{2.9}$$

where $0 \leq \delta \leq 1$ and $0 < c < 1$. The above claim that (2.9) is weaker than (2.6) has been settled in Lemma 1 of the present paper.

Secondly, we investigate necessary and sufficient conditions, imposed upon the generating functions of the Fourier series of f at x, for the truth of (2.7). Before we give the statement of the theorem to be proved, we give the following equivalent form of (2.9), which follows from Lemma 2 of the present paper:

$$\left.\begin{array}{ll} \text{(i)} & \displaystyle\int_0^c \log\frac{k}{t} \quad |\mathrm{d}g(t)| < \infty, \quad 0 < c < 1 \\ \text{(ii)} & g(0+) = 0 \end{array}\right\}. \tag{2.10}$$

Precisely, we prove the following:

Theorem. *Let* $\delta \geq 0$ *and let* (2.10) (i) *hold. Then in order that* (2.7) *should hold, it is necessary and sufficient that* (2.10) (ii) *must hold. Further, the condition* (2.10) (i) *is best possible in the sense that it cannot be replaced by*

$$\int_0^{\pi}\log^{\eta}\frac{k}{t} \quad |\mathrm{d}g(t)| < \infty \quad (0 < \eta < 1). \tag{2.11}$$

3. Estimates

To prove the theorem, we shall require the following estimates for $\delta \geq 0$ but proved for real δ: uniformly in $0 < t < c$,

$$H_n(t) = O(1)\left(b(t) + b\left(\frac{1}{n+1}\right)\right), \tag{3.1}$$

$$H_n(t) = b(t)\frac{\sin nt}{nt} + O(1)n^{-2}t^{-2}b(t), \tag{3.2}$$

$$s_n(t) \leq y^n(t) + \left(\frac{q}{1+q}\right)^n. \tag{3.3}$$

Proof of (3.1). We have

$$H_n(t) = b(t)\frac{\sin nt}{nt} + \int_0^c \frac{\sin nu}{nu}\mathrm{d}b(u) - \int_0^t \frac{\sin nu}{nu}\mathrm{d}b(u).$$

Now, since $b(u)$ is monotonic increasing therefore

$$\left|\int_0^t \frac{\sin nu}{nu}\mathrm{d}b(u)\right| \leq \int_0^t \mathrm{d}b(u) = b(t)$$

and

$$\left|\int_0^c \frac{\sin nu}{nu}\mathrm{d}b(u)\right| \leq \int_0^{n^{-1}} \mathrm{d}b(u) + \left|\int_{n^{-1}}^c \frac{\sin nu}{nu}\mathrm{d}b(u)\right|.$$

Also $u^{-1}\dfrac{\mathrm{d}}{\mathrm{d}u}b(u)$ decreases therefore, we have, by the second mean value theorem

$$\begin{aligned}\int_{n^{-1}}^c \frac{\sin nu}{nu}\mathrm{d}\,b(u) &= \left[u^{-1}\frac{\mathrm{d}}{\mathrm{d}u}b(u)\right]_{u=n^{-1}} \int_{n^{-1}}^{\theta} \frac{\sin nu}{n}\mathrm{d}u \quad (n^{-1} < \theta < c)\\ &= O(1)b\left(\frac{1}{n+1}\right).\end{aligned}$$

Collecting the results, we get (3.1).

Proof of (3.2). Since $u^{-1}\dfrac{\mathrm{d}}{\mathrm{d}u}b(u)$ decreases, therefore, by the second mean value therorem

$$\begin{aligned}\int_t^c \frac{\sin nu}{nu}\mathrm{d}\,b(u) &= (nt)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}b(t)\int_t^{c'} \sin nu\,\mathrm{d}u \qquad (t < c' < c)\\ &= O(n^{-2}t^{-2}b(t)).\end{aligned}$$

Using this estimate in the definition $H_n(t)$, we get (3.2).

Proof of (3.3). We have

$$s_n(t) = \text{imaginary part of} \quad \sum_{k=0}^{n} \quad V_k^q(n) \ \exp(ikt),$$

where

$$\begin{aligned}
&\sum_{k=0}^{n} \quad V_k^q(n) \ \exp(ikt) \\
&= \quad e^{-it} \quad \sum_{k=0}^{n} \quad V_k^q(n) \ \exp(i(k+1)t) \\
&= \quad e^{-it} \quad \sum_{m=1}^{n+1} \quad V_{m-1}^q(n) \ \exp(imt) \\
&= \quad e^{-it} \quad \sum_{m=0}^{n+1} \quad V_{m-1}^q(n) \ \exp(imt) \ - \ e^{-it} V_{-1}^q(n) \\
&= \quad e^{-it} \quad \frac{(q+\exp(it))^{n+1}}{(1+q)^n} \ - \ e^{-it} \ \frac{q^{n+1}}{(1+q)^{n+1}} \\
&= \quad e^{-it} \ \frac{R^{n+1}}{(1+q)^{n+1}} \ (\cos\theta + i \ \sin\theta)^{n+1} \ - \ e^{-it} \left(\frac{q}{1+q}\right)^{n+1} \\
&= \quad \left(\frac{R}{1+q}\right)^{n+1} \quad e^{i(n+1)\theta - it} \ - \ e^{-it} \left(\frac{q}{1+q}\right)^{n+1},
\end{aligned}$$

where $R\cos\theta = q + \cos t$, $R\sin\theta = \sin t$ and

$$\theta = \tan^{-1}\left(\frac{\sin\theta}{q+\cos\theta}\right).$$

Hence imaginary part of

$$\sum_{k=0}^{n} V_k^q(n) \ \exp(ikt) = \left(\frac{R}{1+q}\right)^{n+1} \sin[(n+1)\theta - t] \ + \ \left(\frac{q}{1+q}\right)^{n+1} \sin t,$$

where

$$R \ = \ \sqrt{1+q^2+2q\cos t} \quad = \quad y(t)(1+q).$$

Hence

$$|s_n(t)| \quad \leq \quad y^n(t) \quad + \quad \left(\frac{q}{1+q}\right)^n.$$

This completes the proof.

4. Lemmas

We require the following lemmas for the proof of the theorem:

Lemma 1. *For* $0 \leq \delta \leq 1$,

$$(2.6) \Rightarrow (2.9) \tag{4.1}$$

but its converse is not true in general.

Proof. It has been observed (Chandra [5]; p. 19) that (2.6) with $\delta = 1$ holds if and only if

$$\text{(i)} \quad P(t) \in BV(0, c), \quad \text{(ii)} \quad t^{-t} Pt \in L(0, c), \tag{4.2}$$

which is stronger than (2.9) with $\delta = 1$.

We now consider the case $0 \leq \delta < 1$. In this case, we observe that

$$\begin{aligned}(2.6) \quad &\Rightarrow \quad \phi(t) \quad \in \quad BV(0, c)\\ &\Rightarrow \quad t^{-1} P(t) \quad \in \quad L(0, c) \quad \text{(see (4.2) (ii))}\\ &\Rightarrow \quad t^{-1} P(t) \log^{-\delta} \frac{k}{t} \quad \in \quad L(0, c).\end{aligned}$$

Hence (2.9) (ii) holds. Now for the truth of (2.9) (i), we write

$$h(t) = \phi(t) \log^{1-\delta} \frac{k}{t} \quad \text{and} \quad h_1(t) = \frac{1}{t} \int_0^t h(u)\, \mathrm{d}u.$$

Then $h_1(t) \in BV(0, c)$, where

$$\begin{aligned} t h_1(t) &= \int_0^t \phi(u) \quad \log^{1-\delta} \frac{k}{u} \mathrm{d}u \\ &= t\phi_1(t) \quad \log^{1-\delta} \frac{k}{t} + (1-\delta) \int_0^t \phi_1(u) \quad \log^{-\delta} \frac{k}{u} \mathrm{d}u. \end{aligned}$$

Hence

$$h_1(t) = \phi_1(t) \quad \log^{1-\delta} \frac{k}{t} + (1-\delta) \frac{1}{t} \int_0^t \phi_1(u) \log^{-\delta} \frac{k}{u} \mathrm{d}u$$

from which one gets

$$\begin{aligned} P(t) \quad \log^{1-\delta} \frac{k}{t} &= h(t) - h_1(t) \\ &\quad + \frac{1-\delta}{t} \int_0^t \phi_1(u) \quad \log^{-\delta} \frac{k}{u} \mathrm{d}u. \end{aligned} \tag{4.3}$$

Observe that

$$\begin{aligned} h(t) \in BV(0, c) \quad &\Rightarrow \quad \phi_1(t) \quad \log^{-\delta} \frac{k}{t} \in BV(0, c) \\ \Rightarrow \quad &\left\{ \frac{1}{t} \int_0^t \phi_1(u) \quad \log^{-\delta} \frac{k}{u} \mathrm{d}u \right\} \in BV(0, c). \end{aligned}$$

Hence using these results in (3.3), we get

$$P(t) \quad \log^{1-\delta}\frac{k}{t} \in BV(0,c).$$

To prove that converse is not true in general, let f be even function and $x = 0$. Then $\phi(t) = f(t)$ in $[0,\pi]$. We define

$$f(t) = \begin{cases} \left(\log\frac{k}{t}\right)^{-\frac{1}{2}(1-\delta)} & \text{in } (0,c) \\ 0 & \text{elsewhere.} \end{cases}$$

Then (2.6) does not hold.

On the other hand, since $\phi(t) \in BV(0,c)$, therefore

$$P(t) = \frac{1}{t}\int_0^t u\,\mathrm{d}\phi(u) = \frac{1}{2}(1-\delta)\frac{1}{t}\int_0^t \left(\log\frac{k}{u}\right)^{(\delta-3)/2}\mathrm{d}u \tag{4.4}$$

and hence

$$\begin{aligned}\int_0^c \left|\frac{P(t)}{t\log^{\delta}\frac{k}{t}}\right|\mathrm{d}t &< \frac{1}{2}\int_0^c \frac{\mathrm{d}t}{t^2\log^{\delta}\frac{k}{t}}\int_0^t\left(\log\frac{k}{u}\right)^{(\delta-3)/2}\mathrm{d}u \\ &= \frac{1}{2}\int_0^c\left(\log\frac{k}{u}\right)^{(\delta-3)/2}\mathrm{d}u\int_u^c\frac{t^{-2}}{\log^{\delta}\frac{k}{t}}\,\mathrm{d}t \\ &< \int_0^c\frac{\mathrm{d}u}{u\log^{(3+\delta)/2}\left(\frac{k}{u}\right)} < \infty,\end{aligned}$$

which proves (2.9) (ii). Also from (4.4)

$$P(t)\log^{1-\delta}\frac{k}{t} = \frac{1}{2}(1-\delta)t^{-1}\log^{1-\delta}\frac{k}{t}\int_0^t\log^{(\delta-3)/2}\left(\frac{k}{u}\right)\mathrm{d}u$$

$$= \frac{1}{2}(1-\delta)\log^{-\frac{1}{2}(1+\delta)}\left(\frac{k}{t}\right) + \frac{1}{2}(1-\delta)\frac{\delta-3}{2}t^{-1}\log^{1-\delta}\left(\frac{k}{t}\right)\int_0^t\log^{(\delta-3)/2}(ku)\mathrm{d}u.$$

Now it may be observed that each of the term on the right above is of bounded variation on $(0,c)$ and hence

$$P(t)\log^{1-\delta}\left(\frac{k}{t}\right) \in BV(0,c),$$

which proves (2.9) (i).

This completes the proof of the lemma.

Lemma 2 [11]. *If* $\eta > 0$, *then necessary and sufficient conditions that* (i) $h(t)\log\frac{k}{t} \in BV(0,\eta)$ *and* (ii) $t^{-1}h(t) \in L(0,\eta)$ *are that*

$$h(0+) = 0 \quad and \quad \int_0^{\eta} \log\left(\frac{k}{t}\right) |\,\mathrm{d}h(t)| < \infty.$$

Lemma 3 [15]. *Let, for* $0 < c < \pi$,

$$\alpha_n = \frac{2}{\pi}\int_c^{\pi} \phi(t)\cos nt\ \mathrm{d}t.$$

Then $\sum_{n=1}^{\infty} \alpha_n d_n \in |E,q|\ (q > 0)$.

This is really proved for $0 \le \delta \le 1$ *but the same arguments hold for* $\delta \ge 0$.

Lemma 4. *Let* $0 < \beta \le \pi$ *and* $\delta \ge 0$. *Then uniformly in* $0 < t < \beta$

$$\sum_{m=1}^{n} V_m^q(n)\ d_m\ \exp(imt) = O\left\{n^{-1/2}t^{-1}\log^{-\delta}\left(\frac{k}{t}\right)\right\}.$$

The case $\delta = 1$ *is dealt with in Lemma 2 of Chandra* [4]. *The general case may be obtained similarly.*

Lemma 5. *For* $0 < c < \pi$ *and for all real* β

$$\frac{2}{\pi}\int_0^{c} \frac{\sin nu}{u}\log^{\beta}\frac{k}{u}\ \mathrm{d}u \sim \log^{\beta} n.$$

The case $\beta = 1$ *with* $c = \pi$ *was dealt with by Mohanty and Ray* [13] *and for all real* β *with* $c = \pi$, *references may be made to Ray* [14] *or Chandra* [6]. *Since the same arguments hold if we replace* π *by c in Ray* [14] *or Chandra* [6], *therefore one can get the above result from either Ray* [14] *or Chandra* [6].

Lemma 6. *Uniformly in* $0 < t < \pi$,

$$\sum_{m=1}^{n} V_m^q(n)\ d_m \sin mt$$
$$= O(t^{-1})\Delta d_n + O\{d_n y^n(t)\} + O\left\{(d_n)\left(\frac{q}{1+q}\right)^n\right\}.$$

Proof. Let N denote the integral part of $\frac{n+1-q}{1+q}$ for $n > 2q$. Then we first observe that $V_m^q(n)$ increases monotonically with $m \le N$ and decreases with $m > N$. And, by Abel's transformation

$$\begin{aligned}
\sum_{m=1}^{n} & V_m^q(n) d_m \sin mt \\
&= \sum_{m=1}^{n-1} s_m(t)\Delta d_m + s_n(t) d_n \\
&= \sum_{m=1}^{N} s_m(t)\Delta d_m + \sum_{m=N+1}^{n-1} s_m(t)\Delta d_m + s_n(t) d_n \\
&= \sum_{m=1}^{N} s_m(t)\Delta d_m + \sum_{m=N+1}^{n-1} \left[s_n(t) - \sum_{k=m+1}^{n} V_k^q(n) \sin kt \right] \Delta d_m + s_n(t) d_n \\
&= \sum_{m=1}^{N} s_m(t)\Delta d_m + d_{N+1} s_n(t) - \sum_{m=N+1}^{n-1} \Delta d_m \sum_{k=m+1}^{n} V_k^q(n) \sin kt \\
&= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3, \quad \text{say.} \qquad (4.5)
\end{aligned}$$

However, by Abel's lemma

$$\begin{aligned}
|s_m(t)| &\le V_m^q(n) \max_{1 \le m' < m'' \le m} \left| \sum_{k=m'}^{m''} \sin kt \right| \quad (\text{for } m < N) \\
&= O(t^{-1}) V_m^q(n).
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_1 &= O(t^{-1}) \sum_{m=1}^{N} \Delta d_m V_m^q(n) \\
&= O(t^{-1}) \sum_{m=1}^{n} \Delta d_m V_m^q(n) \\
&= O(t^{-1}) n^2 \Delta d_n \sum_{m=1}^{n} m^{-2} V_m^q(n) \\
&= O(t^{-1}) \Delta d_n, \qquad (4.6)
\end{aligned}$$

since $m^2 \Delta d_m$ is increasing and

$$\sum_{m=1}^{n} m^{-2} V_m^q(n) = O(n^{-2}).$$

And by (3.3)

$$\sum_2 = O\left\{ d_n y^n(t) \right\} + O\left\{ d_n \left(\frac{q}{1+q} \right)^n \right\}. \qquad (4.7)$$

Finally, once again by applying Abel's lemma in the inner sum of $\sum_3$, we get

$$\sum_3 = \sum_{m=N+1}^{n-1} \Delta d_m O(t^{-1}) V_m^q(n)$$

$$= O(t^{-1}) \sum_{m=1}^{n} V_m^q(n) \Delta d_m$$
$$= O(t^{-1}) \Delta d_n, \tag{4.8}$$

as in $\sum_1$.

Combining (3.5) through (4.8), we get the required result.

Lemma 7. *There exists an* $f \in L_{2\pi}$ *for which* (2.10) (i) *and* (2.11) *hold but the series* (2.8) *at* $x = 0$ *diverges properly for every real* δ *and hence not summable by any regular summability method.*

Proof. Let f be even and let $x = 0$. Then $\phi(t) = f(t)$. Define f by periodicity. We first consider the case $\delta = 0$ for which we define

$$f(t) = \begin{cases} \log\log\left(\dfrac{k}{t}\right), & 0 < t \le \pi \\ 0, & t = 0 \end{cases} \tag{4.9}$$

where $k \ge \pi e^2$. Then

$$g(t) = \phi(t) - \phi_1(t)$$
$$= -\frac{1}{\log\left(\dfrac{k}{t}\right)} + \frac{1}{t}\int_0^t \log^{-2}\left(\frac{k}{u}\right)\, \mathrm{d}u,$$

which is of bounded variation and $g(0+) = 0$. Hence

$$\int_0^\pi \log^\eta \frac{k}{t} |\, \mathrm{d}g(t)| < \int_0^\pi t^{-1} \log^{\eta-2} \frac{k}{t}\, \mathrm{d}t,$$

which converges whenever $0 < \eta < 1$. This proves that (2.10) (i) and (2.11) hold. However

$$A_n(x) = \frac{2}{\pi}\int_0^\pi \log\, \log \frac{k}{t} \cos nt\, \mathrm{d}t$$
$$= \frac{2}{\pi}\int_0^\pi \frac{\sin nt}{nt} \log^{-1} \frac{k}{t}\, \mathrm{d}t$$
$$\sim \frac{1}{n\, \log\, n},$$

by using Lemma 5. Thus $\sum_{n=1}^{\infty} A_n(x)$ diverges properly and hence it cannot be summable by any absolutely regular summability method and, *a fortiori*, (2.8) with $\delta = 0$ is not $|E, q|$ $(q > 0)$ summable.

In the case when δ is non-zero real number, we define

$$f(t) = \begin{cases} \log^\delta \dfrac{k}{t}, & \left(0 < t \le \pi\right) \\ 0, & t = 0 \end{cases}. \tag{4.10}$$

Then since $\phi(t) = f(t)$, we have

$$P(t) = \delta \frac{1}{t} \int_0^t \log^{\delta-1}\left(\frac{k}{u}\right) \mathrm{d}u$$

$$= \delta \log^{\delta-1}\left(\frac{k}{t}\right) + \frac{\delta(\delta-1)}{t} \int_0^t \log^{\delta-2}\left(\frac{k}{u}\right) \mathrm{d}u$$

and hence

$$g(t) = P(t)\log^{-\delta}\left(\frac{k}{t}\right)$$

$$= \frac{\delta}{\log\left(\frac{k}{t}\right)} + \frac{\delta(\delta-1)}{t\log^{\delta}\left(\frac{k}{t}\right)} \int_0^t \log^{\delta-2}\left(\frac{k}{u}\right) \mathrm{d}u,$$

which shows that $g(0+) = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) = \frac{\delta^2}{t\log^2\left(\frac{k}{t}\right)} + \delta(\delta-1)\left\{\frac{\delta}{t^2\log^{1+\delta}\left(\frac{k}{t}\right)} - \frac{1}{t^2\log^{\delta}\left(\frac{k}{t}\right)}\right\} \times \int_0^t \log^{\delta-2}\left(\frac{k}{u}\right) \mathrm{d}u$$

and for all real $\delta \neq 0$

$$\int_0^t \log^{\delta-2}\left(\frac{k}{u}\right) \mathrm{d}u \leq Mt\log^{\delta-2}\left(\frac{k}{t}\right),$$

where M is a positive constant not necessarily the same at each occurrence and possibly depending upon δ. Therefore

$$\int_0^\pi \log^\eta \frac{k}{t}|\mathrm{d}g(t)| \leq M \int_0^\pi t^{-1}\log^{\eta-2}\frac{k}{t}\,\mathrm{d}t,$$

which converges for $0 < \eta < 1$. This proves that (2.10) (i) and (2.11) hold for all real $\delta \neq 0$. But for the function defined by (3.10)

$$A_n(x) = \frac{2}{\pi}\int_0^\pi \log^{\delta}\left(\frac{k}{t}\right)\cos nt\,\mathrm{d}t \quad (\delta \neq 0)$$

$$= \frac{2\delta}{n\pi}\int_0^\pi \frac{\sin nt}{t}\log^{\delta-1}\left(\frac{k}{t}\right) \mathrm{d}t$$

$$\sim \frac{\delta}{n}\log^{\delta-1} n,$$

by Lemma 5 and hence,

$$\frac{A_n(x)}{\log^{\delta}(n+1)} \sim \frac{\delta}{n\ \log(n+1)}.$$

This shows that for every real $\delta \neq 0$, (2.8) is not $|E, q|$ $(q > 0)$ summable since

$$\sum \frac{1}{n\ \log(n+1)} = \infty.$$

This completes the proof of the lemma.

5. Proof of the theorem

In view of the inclusion: $|E, q| \subset |E, q'|$ $(q' > q > -1)$ (see Chandra [2]; Corollary 2) we assume $0 < q < 1$ for the proof of the theorem, without any loss of generality.

Let (2.10) (i) hold. Then proceeding as in Chandra ([3], p. 388–9), we have for $n \geq 1$

$$\begin{aligned} A_n(x) = {} & \frac{2}{\pi}\int_c^{\pi} \phi(t) \quad \cos nt\ \mathrm{d}t + \frac{2}{\pi}\phi_1(c)\frac{\sin nc}{n} \\ & + \frac{2}{\pi}\int_0^c tP(t)\frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right)\ \mathrm{d}t \end{aligned} \tag{5.1}$$

and integrating by parts, we get

$$\begin{aligned} & \int_0^c tP(t)\frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right)\ \mathrm{d}t \\ & \quad = \int_0^c g(t)b(t)\frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right)\ \mathrm{d}t \\ & \quad = g(0+)\int_0^c b(u)\frac{\partial}{\partial u}\left(\frac{\sin nu}{nu}\right)\ \mathrm{d}u + \int_0^c \mathrm{d}g(t)\int_t^c b(u)\frac{\partial}{\partial u}\left(\frac{\sin nu}{nu}\right)\ \mathrm{d}u \end{aligned} \tag{5.2}$$

and for $0 \leq t < c$

$$\int_t^c b(u)\frac{\partial}{\partial u}\left(\frac{\sin nu}{nu}\right)\ \mathrm{d}u = b(c)\frac{\sin nc}{nc} - b(t)\frac{\sin nt}{nt} - \int_t^c \frac{\sin nu}{nu}\ \mathrm{d}b(u). \tag{5.3}$$

Using (5.3) in (5.2), we get

$$\begin{aligned} & \int_0^c tP(t)\frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right)\ \mathrm{d}t \\ & = \quad g(0+)\left[b(c)\frac{\sin nc}{nc} - \int_0^c \frac{\sin nu}{nu}\ \mathrm{d}b(u)\right] + \int_0^c b(c)\frac{\sin nc}{nc}\ \mathrm{d}g(t) - \int_0^c H_n(t)\ \mathrm{d}g(t) \\ & = g(c)b(c)\frac{\sin nc}{nc} - g(0+)\int_0^c \frac{\sin nu}{nu}\ \mathrm{d}b(u) - \int_0^c H_n(t)\ \mathrm{d}g(t). \end{aligned} \tag{5.4}$$

And using (5.4) in (5.1), we get

$$\begin{aligned} A_n(x) &= \frac{2}{\pi}\int_c^{\pi}\phi(t)\cos\, nt\,\mathrm{d}t + \frac{2}{\pi}\phi(c)\frac{\sin\, nc}{n} \\ &- \frac{2}{\pi}g(0+)\int_0^c \frac{\sin\, nu}{nu}\,\mathrm{d}b(u) - \frac{2}{\pi}\int_0^c H_n(t)\,\mathrm{d}g(t) \\ &= \alpha_n + \beta_n - \gamma_n - \delta_n, \text{ say.} \end{aligned} \tag{5.5}$$

Since $A_0 = \frac{1}{2}a_0 = 0$, therefore

$$\sum_{n=1}^{\infty} A_n(x)d_n \in |E, q| \quad (q > 0)$$

if and only if

$$\sum_{n=1}^{\infty}\frac{1}{(q+1)^{n+1}}\left|\sum_{m=1}^{n}\binom{n}{m}q^{n-m}d_m A_m(x)\right| < \infty. \tag{5.6}$$

However, it follows from Lemma 3 that

$$\sum_{n=1}^{\infty}\alpha_n d_n \in |E, q| \quad (q > 0) \tag{5.7}$$

and since

$$\beta_n = \frac{2}{\pi}\phi(c)\sin nc\left[\frac{1}{n+1} + \frac{1}{n(n+1)}\right]$$

and

$$\sum_{n=1}^{\infty}\left|\frac{2}{\pi}\phi(c)d_n\frac{\sin nc}{n(n+1)}\right| < \infty.$$

Therefore, in view of absolute regularity of $|E, q|$ $(q > 0)$ method,

$$\sum_{n=1}^{\infty}\beta_n d_n \in |E, q| \quad (q > 0) \tag{5.8}$$

if

$$\sum_{n=1}^{\infty}\frac{1}{(q+1)^{n+1}}\left|\sum_{m=1}^{n}\binom{n}{m}q^{n-m}d_m\frac{\sin nc}{m+1}\right| = \sum_{n=1}^{\infty}\frac{1}{n+1}\left|\sum_{m=1}^{n}V_n^q(m)d_m\sin nc\right| < \infty,$$

which holds by Lemma 4. Now

$$\sum_{n=1}^{\infty}\delta_n d_n \in |E, q| \quad (q > 0) \tag{5.9}$$

if and only if

$$Q = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(q+1)^{n+1}}\left|\sum_{m=1}^{n}\binom{n}{m}q^{n-m}d_m\int_0^c H_m(t)\mathrm{d}g(t)\right| < \infty.$$

Clearly

$$Q \leq \frac{2}{\pi}\int_0^c |\,\mathrm{d}g(t)|\sum_{n=1}^{\infty}\frac{1}{(q+1)^{n+1}}\left|\sum_{m=1}^{n}\binom{n}{m}q^{n-m}d_m H_m(t)\right|$$

and since by (2.10) (i),

$$\int_0^c \log\frac{k}{t}|\,\mathrm{d}g(t)| < \infty$$

therefore for the proof of (5.9) it is suffiecient to prove that

$$Z = \sum_{n=1}^{\infty}\frac{1}{(q+1)^{n+1}}\left|\sum_{m=1}^{n}\binom{n}{m}q^{n-m}d_m H_m(t)\right| = O\left(\log\frac{k}{t}\right), \tag{5.10}$$

uniformly in $0 < t < c$.

For $T = [k/t]$, the integral part of k/t, we write

$$Z = \sum_{n\leq T} + \sum_{n>T} \qquad \text{say.} \tag{5.11}$$

By (3.1), we get

$$\begin{aligned}
\sum_{n\leq T} &= b(t)\sum_{n=1}^{T}\frac{1}{(q+1)^{n+1}}\sum_{m=1}^{n}\binom{n}{m}q^{n-m}d_m \\
&\quad + O(1)\sum_{n=1}^{T}\frac{1}{(q+1)^{n+1}}\sum_{m=1}^{n}\binom{n}{m}q^{n-m}\frac{1}{m+1} \\
&= O(1)b(t)\sum_{n=1}^{T}d_n\sum_{m=1}^{n}V_m^q(n) + O(1)\sum_{n=1}^{T}\frac{1}{n+1}\sum_{m=1}^{n}V_m^q(n) \\
&= O(1)b(t)\sum_{n=1}^{T}d_n + O(1)\sum_{n=1}^{T}\frac{1}{n+1} \\
&= O\left(\log\frac{k}{t}\right),
\end{aligned} \tag{5.12}$$

uniformly in $0 < t < c$, since $\sum_{m=1}^{n} V_m^q(n) \leq 1$. And by (3.2)

$$\begin{aligned}
\sum_{n\geq T} &= \log^{\delta}\left(\frac{k}{t}\right)\sum_{n=T}^{\infty}\frac{1}{(q+1)^{n+1}}\left|\sum_{m=1}^{n}\binom{n}{m}q^{n-m}d_m\frac{\sin mt}{m}\right| \\
&\quad + O\left(t^{-1}\log^{\delta}\frac{k}{t}\right)\sum_{n=T}^{\infty}\frac{1}{(q+1)^{n+1}}\sum_{m=1}^{n}\binom{n}{m}q^{n-m}\frac{d_m}{m(m+1)}
\end{aligned}$$

$$\begin{aligned} &= \log^{\delta}\frac{k}{t}\sum_{n=T}^{\infty}\frac{1}{(q+1)^{n+1}}\left|\sum_{m=1}^{n}\binom{n}{m}q^{n-m}d_m\frac{\sin mt}{m+1}\right| \\ &\quad +O\left(t^{-1}\log^{\delta}\frac{k}{t}\right)\sum_{n=T}^{\infty}\frac{1}{(q+1)^{n+1}}\sum_{m=1}^{n}\binom{n}{m}q^{n-m}\frac{d_m}{m(m+1)} \\ &= R(t)\log^{\delta}\frac{k}{t}+O\left(t^{-1}\log^{\delta}\frac{k}{t}\right)W(t), \quad \text{say,} \end{aligned} \tag{5.13}$$

where

$$W(t)=\sum_{n=T}^{\infty}\frac{1}{(q+1)^{n+1}}\sum_{m=1}^{n}\binom{n}{m}q^{n-m}\frac{d_m}{m(m+1)}.$$

Now, by using repeatedly the relation:

$$\binom{r}{s}=\frac{s+1}{r+1}\binom{r+1}{s+1},$$

where r and s are integers such that $r \geq s \geq 0$, we get

$$\begin{aligned} &\sum_{m=1}^{n}\binom{n}{m}q^{n-m}\frac{d_m}{m(m+1)} \\ &\qquad = \frac{1}{n+1}\sum_{m=1}^{n}\binom{n+1}{m+1}q^{n-m}\frac{d_m}{m} \\ &\qquad = \frac{1}{n+1}\sum_{m=1}^{n}\binom{n+1}{m+1}q^{n-m}\left(\frac{2}{m}+1\right)\frac{d_m}{m+2} \\ &\qquad = \frac{1}{(n+1)(n+2)}\sum_{m=1}^{n}\binom{n+2}{m+2}q^{n-m}\left(\frac{2}{m}+1\right)d_m \\ &\qquad < \frac{3}{n^2}\sum_{m=1}^{n}\binom{n+2}{m+2}q^{n-m}d_m \\ &\qquad = \frac{3}{n^2(n+3)}\sum_{m=1}^{n}\binom{n+3}{m+3}q^{n-m}(m+3)d_m. \end{aligned}$$

However, the function $(x+2)\log^{-\delta}x$ increases with $x > \exp(3\delta)$, therefore

$$\begin{aligned} W(t) &= O(1)\sum_{n=T}^{\infty}(q+1)^{-n-1}n^{-2}d_n\sum_{m=1}^{n}\binom{n+3}{m+3}q^{n-m} \\ &= O(1)\sum_{n=T}^{\infty}(q+1)^{-n-1}n^{-2}d_n\sum_{m=0}^{m+3}\binom{n+3}{m}q^{n-m} \\ &= O(1)\sum_{n=T}^{\infty}(q+1)^{2}\frac{d_n}{n^2} \end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{n=T}^{\infty}\frac{d_n}{n^2}\\
&= O\left(t\log^{-\delta}\left(\frac{k}{t}\right)\right), \qquad (5.14)
\end{aligned}
$$

uniformly in $0 < t < c$. And, by Lemma 6,

$$
\begin{aligned}
R(t) &= \sum_{n=T}^{\infty}\frac{1}{n+1}\left|\sum_{m=1}^{n}V_m^q(n)d_m\sin mt\right|\\
&= O(t^{-1})\sum_{n=T}^{\infty}\frac{\Delta d_n}{n+1} + O(1)\sum_{n=T}^{\infty}\frac{d_n}{n+1}y^n(t)\\
&\quad +O(1)\sum_{n=T}^{\infty}\frac{d_n}{n+1}\left(\frac{q}{1+q}\right)^n\\
&= O(1)\log^{-\delta}\frac{k}{t} + O(1)\log^{-\delta}\left(\frac{k}{t}\right)\sum_{n=1}^{\infty}\frac{y^n(t)}{n}\\
&\quad +O(1)t\log^{-\delta}\left(\frac{k}{t}\right)\sum_{n=0}^{\infty}\left(\frac{q}{1+q}\right)^n\\
&= O(1)\log^{1-\delta}\left(\frac{k}{t}\right), \qquad (5.15)
\end{aligned}
$$

uniformly in $0 < t < c$, since

$$
\sum_{n=1}^{\infty}\frac{y^n(t)}{n} = \log\frac{1}{1-y(t)}
$$

and

$$
\frac{1}{1-y(t)} = O\left(\frac{k}{t}\right)^2 \ (t \to 0+).
$$

Combining (5.11) through (5.15) we get (5.10). Also in view of (5.5) through (5.9), $\sum_{n=1}^{\infty} A_n(x)d_n \in |E,q|(q > 0)$ if and only if

$$
\sum_{n=1}^{\infty}\gamma_n d_n \in |E,q| \quad (q > 0), \qquad (5.16)
$$

where

$$
\gamma_n d_n = \frac{2}{\pi}g(0+)d_n\int_0^c\frac{\sin nu}{nu}\,\mathrm{d}b(u)
$$

and, by Lemma 5,

$$
\begin{aligned}
\frac{2}{\pi}\int_0^c\frac{\sin nu}{nu}\mathrm{d}b(u) &\sim \frac{1}{n}[\log^{\delta}(n+1) - \delta\log^{\delta-1}(n+1)]\\
&\sim \frac{1}{n}\log^{\delta}(n+1)
\end{aligned}
$$

and hence

$$\frac{2}{\pi} d_n \int_0^c \frac{\sin nu}{nu}\, \mathrm{d}b(u) \sim \frac{1}{n}.$$

Thus in order that (5.16) should hold it is necessary and sufficient that

$$\sum_{n=1}^{\infty} \frac{g(0+)}{n} \in |E, q| \; (q > 0)$$

for which it is necessary and sufficient that (2.10)(ii) must hold, since $\sum_{n=1}^{\infty} 1/n$ diverges strictly.

The fact that the condition (2.10)(i) cannot be replaced by (2.11) follows by Lemma 7.
This proves the theorem completely.

Acknowledgement

The author is thankful to the referee for reading the manuscript carefully and giving his valuable suggestions to improve the presentation of the paper.

References

[1] Chandra P, Absolute summability factors for Fourier series, *Rend. Accad. Nazionale dei XL* **24-25** (1974) 3–23
[2] Chandra P, On some summability methods, *Boll Un. Mat. Ital. (4)* **12(3)** (1975) 211–224
[3] Chandra P, Absolute summability by (E, q)-means, *Riv. Mat. Univ. Parma (4)* **4** (1978) 385–393
[4] Chandra P, On the absolute Euler summability factors for Fourier series and its conjugate series, *Indian J. Pure Appl. Math.* **9** (1978) 1004–1018
[5] Chandra P, On a class of functions of bounded variation, *Jñānābha* **8** (1978) 17–24
[6] Chandra P, Absolute Euler summability of allied series of the Fourier series, *Indian J. Pure Appl. Math.* **11** (1980) 215–229
[7] Chandra P and Dikshit G D, On the $|B|$ and $|E, q|$ summability of a Fourier series, its conjugate series and their derived series, *Indian J. Pure Appl. Math.* **12** (1981) 1350–1360
[8] Dikshit G D, Absolute Euler summability of Fourier series *J. Math. Anal. Appl.* **220** (1998) 268–282
[9] Hardy G H, *Divergent Series* (Oxford) (1963)
[10] Kwee B, The absolute Euler summability of Fourier series, *J. Austra. Math. Soc.* **13** (1972) 129–140
[11] Mohanty R, On the absolute Riesz summability of Fourier series and allied series, *Proc. London Math. Soc.* **52** (1951) 295–320
[12] Mohanty R and Mohapatra S, On the $|E, q|$ summability of Fourier series and allied series, *J. Indian Math. Soc.* **32** (1968) 131–139
[13] Mohanty R and Ray B K, On the convergence factors of a Fourier series and a differentiated Fourier series, *Proc. Cambridge Philos. Soc.* **65** (1969) 75–85
[14] Ray B K, On the absolute summability factors of some series related to a Fourier series, *Proc. Cambridge Philos. Soc.* **67** (1970) 29–45
[15] Ray B K and Sahoo A K, Application of the absolute Euler method to some series related to Fourier series and its conjugate series, *Proc. Indian Acad. Sci. (Math. Sci.)* **106** (1996) 13–38