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# The multiplication map for global sections of line bundles and rank 1 torsion free sheaves on curves

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**Abstract.** Let $X$ be an integral projective curve and $L \in \mathrm{Pic}^a(X)$, $M \in \mathrm{Pic}^b(X)$ with $h^1(X, L) = h^1(X, M) = 0$ and $L, M$ general. Here we study the rank of the multiplication map $\mu_{L,M} : H^0(X, L) \otimes H^0(X, M) \to H^0(X, L \otimes M)$. We also study the same problem when $L$ and $M$ are rank 1 torsion free sheaves on $X$. Most of our results are for $X$ with only nodes as singularities.

**Keywords.** Singular projective curve; rank 1 torsion free sheaf; nodal curve; cuspidal curve; line bundle; special divisor.

## 1. Introduction

Let $X$ be a smooth projective curve of genus $g \geq 0$ and $L, M \in \mathrm{Pic}(X)$ with $L, M$ spanned. Call $h_L : X \to \mathbf{P}(H^0(X, L))$ and $h_M : X \to \mathbf{P}(H^0(X, M))$ the associated morphisms. Denote with $\mu_{L,M} : H^0(X, L) \otimes H^0(X, M) \to H^0(X, L \otimes M)$ the multiplication map and $\mathbf{i}_{L,M} : \mathbf{P}(H^0(X, L)) \times \mathbf{P}(H^0(X, M)) \to \mathbf{P}(H^0(X, L) \otimes H^0(X, M))$ the Segre embedding. Let $h_{L,M} : X \to \mathbf{P}(H^0(X, L)) \times \mathbf{P}(H^0(X, M))$ be the morphism induced by $h_L$ and $h_M$ on the two factors. Call $f_{L,M} : X \to \mathbf{P}(H^0(X, L \otimes M))$ the morphism obtained from $h_{L,M}$ and the multiplication map $\mu_{L,M}$. The surjectivity of $\mu_{L,M}$ means that $f_{L,M}(X)$ is linearly normal in its linear span and $\dim(\mathrm{Ker}(\mu_{L,M}))$ is the codimension of its linear span. For any $L, M$ the surjectivity of $\mu_{L,M}$ has several important geometric consequences (see e.g. [7]) and very good criteria for the surjectivity of $\mu_{L,M}$ are known (see [10], Th. 4.a.1, and [7], p. 514).

In §2 we will give a proof the following result, proved also in [3].

**Theorem 1.1.** *Fix integers m, n and g with $m \geq 1, n \geq 1$ and $g \geq 0$. Let X be a general smooth projective curve of genus g. Take a general pair $(L, M) \in \mathrm{Pic}^{g+m}(X) \times \mathrm{Pic}^{g+n}(X)$. Then the multiplication map $\mu_{L,M} : H^0(X, L) \otimes H^0(X, M) \to H^0(X, L \otimes M)$ has maximal rank, i.e. it is injective if $g \geq mn$ and it is surjective if $g \leq mn$.*

*Remark* 1.2. In the set-up of 1.1 since $\deg(L) \geq g, \deg(M) \geq g$ and both $L$ and $M$ are general, we have $h^1(X, L) = h^1(X, M) = 0$. Hence by Riemann–Roch we have $h^0(X, L) = m + 1$ and $h^0(X, M) = n + 1$. We explain the numerology in the statement of 1.1 with the following example. Fix positive integers $m$ and $n$. Let $C$ be a smooth projective curve of genus $mn$ and $A \in \mathrm{Pic}^{m+mn}(C)$, $B \in \mathrm{Pic}^{n+mn}(C)$ with $h^1(C, A) = h^1(C, B) = 0$. We have $h^0(C, A) = m+1$, $h^0(C, B) = n+1$, $\deg(A \otimes B) = n+m+2mn$, $h^1(C, A \otimes B) = 0$ and $h^0(C, A).h^0(C, B) = (m + 1)(n + 1) = n + m + 1 + mn = h^0(C, A \otimes B)$.

At the end of §2 we will prove the following result.

**Theorem 1.3.** *Fix integers m, n, g and q with $g \geq q \geq 0, m \geq 1, n \geq 1$ and $g \geq 3$. Let $\pi : Y \to C$ be a birational morphism with Y general curve of genus q and C general nodal curve with $g - q$ nodes and Y as normalization, i.e. assume that $\pi^1(\mathrm{Sing}(C))$ is formed by $2g - 2q$ general points of Y. Take a general pair $(L, M) \in \mathrm{Pic}^{g+m}(C) \times \mathrm{Pic}^{g+n}(C)$. Then the multiplication map $\mu_{L,M} : H^0(C, L) \otimes H^0(C, M) \to H^0(C, L \otimes M)$ has maximal rank, i.e. it is injective if $g \geq mn$ and it is surjective if $g \leq mn$.*

In §3 we will use the classical Brill–Noether theory of special divisors to study the multiplication map for line bundles on nodal or cuspidal curves. In §4 we will use 1.3 to study some problems related to the multiplication map for rank 1 torsion free sheaves on nodal curves.

## 2. Proofs of 1.1 and 1.3

We work over an algebraically closed field **K** with $\mathrm{char}(\mathbf{K}) = 0$; for the case $\mathrm{char}(\mathbf{K}) > 0$, see Remark 3.4. For all positive integers $m$ and $n$ set $\prod(m, n) := \mathbf{P}^m \times \mathbf{P}^n$. Call $\pi_1(m, n) : \prod(m, n) \to \mathbf{P}^m$ and $\pi_2(m, n) : \prod(m, n) \to \mathbf{P}^n$ (or just $\pi_1$ and $\pi_2$) the projections. We have $\mathrm{Pic}(\prod(m, n)) \cong \mathbf{Z}^{\oplus 2}$ and we will take $\pi_1{}^*(\boldsymbol{O}_{\mathbf{P}^m}(1))$ and $\pi_2{}^*(\boldsymbol{O}_{\mathbf{P}^n}(1))$ as generators of $\mathrm{Pic}(\prod(m, n))$. Sometimes we will write $\prod$ instead of $\prod(m, n)$. Set $\boldsymbol{O} := \boldsymbol{O}_{\prod}$ and call $\boldsymbol{O}(1, 0)$ and $\boldsymbol{O}(0, 1)$ the two choosen generators of $\mathrm{Pic}(\prod)$. Every one-dimensional cycle $T$ of $\prod$ has a bidegree $(a, b)$ with $a := T.\boldsymbol{O}(1, 0)$ and $b := T.\boldsymbol{O}(0, 1)$. If $T$ is effective and irreducible we have $a = \deg(\pi_1|T)\deg(\pi_1(T))$ and $b = \deg(\pi_2|T)\deg(\pi_2(T))$. The tangent bundle, $T\prod(m, n)$, of $\prod(m, n)$ is isomorphic to $\pi_1{}^*(T\mathbf{P}^m) \oplus \pi_2{}^*(T\mathbf{P}^n)$. Notice that $T\mathbf{P}^m(-1)$ and $T\mathbf{P}^n(-1)$ are spanned (e.g. by the Euler sequence of $T\mathbf{P}^s$, $s = m$ or $n$). Hence for every integral curve $X \subset \prod$ of type $(a, b)$, the vector bundle $T\prod(m, n)|X$ is the direct sum of a rank $m$ vector bundle which is the quotient of $m + 1$ copies of $\boldsymbol{O}_X(1, 0)$ (and hence the quotient of line bundles of degree $a$) and a rank $n$ vector bundle which is a quotient of $n + 1$ copies of $\boldsymbol{O}_X(0, 1)$ (and hence a quotient of $n + 1$ line bundles of degree $b$). For any locally complete intersection curve $X \subset \prod(m, n)$, let $N_{X]/\prod(m,n)}$ be its normal bundle. If $X$ is smooth, then the normal bundle $N_{X/\prod(m,n)}$ of $X$ in $\prod(m, n)$ is a quotient of $T\prod(m, n)|X$. To prove Theorem 1.1 we introduce the following statement:

$H(m, n), m \geq 1, n \geq 1$: There exists a smooth connected curve $X[m, n] \subset \prod(m, n)$ such that $p_a(X[m, n]) = mn$, $X[m, n]$ has bidegree $(mn + m, mn + n)$, the embedding of $X[m, n]$ in $\prod(m, n)$ is induced by a pair of line bundles $(L, M)$ with $h^1(X[m, n], L) = h^1(X[m, n], M) = 0$, $X[m, n]$ spans $\mathbf{P}^{mn+m+n}$ and $h^1(X[m, n], N_{X[m,n]/\prod(m,n)}) = 0$.

Since $h^1(X[m, n], L) = h^1(X[m, n], M) = h^1(X[m, n], L \otimes M) = 0$, the condition that $X[m, n]$ spans $\mathbf{P}^{mn+m+n}$ in the statement of $H(m, n)$ is equivalent to the condition that the two maps $X[m, n] \to \mathbf{P}^m$ and $X[m, n] \to \mathbf{P}^n$ induced the inclusion of $X[m, n]$ into $\prod(m, n)$ are given by a complete linear system (i.e. by Riemann–Roch, that they are non-degenerate) and that the multiplication map $\mu_{L,M} : H^0(X[m, n], L) \otimes H^0(X[m, n], M) \to H^0(X[m, n], L \otimes M)$ is bijective.

*Remark* 2.2. $H(1, 1)$ is true because a smooth quadric surface $\prod(1, 1) \subset \mathbf{P}^3$ contains a smooth non-degenerate elliptic curve of bidegree $(2, 2)$ and such curve has as normal bundle a degree 4 line bundle.

PROPOSITION 2.2

*Fix an integer $m \geq 1$. If $H(m, m)$ is true, then $H(m + 1, m + 1)$ is true.*

*Proof.* See $\mathbf{P}^{m^2+2m}$ as a codimension $2m+3$ linear subspace, $A$, of $\mathbf{P}^{m^2+4m+3}$. Take a solution $X[m,m] \subset \prod(m,m)$ for $H(m,m)$ and see $\prod(m,m)$ as a linear section of $\prod(m+1,m+1) \subset \mathbf{P}^{m^2+4m+3}$. Fix $S \subset X[m,m]$ with $\mathrm{card}(S) = 2m+2$ and $S$ spanning a linear subspace $\langle S\rangle$ of $\mathbf{P}^{m^2+2m}$ with $\dim(\langle S\rangle) = 2m+1$. Let $C$ be a smooth rational curve and consider the pair $(R,R) \in \mathrm{Pic}(C)\times\mathrm{Pic}(C)$ with $\deg(R) = 2m+2$. The multiplication map $\mu_{R,R} : H^0(C,R)\otimes H^0(C,R) \to H^0(C,R^{\otimes 2})$ is surjective and $R^{\otimes 2}$ embeds $C$ into a $(4m+4)$-dimensional projective space $W$ as a rational normal curve; call $D \subset W$ its image. Hence $D$ may be seen both as a smooth rational curve of degree $4m+4$ in $W$ and a curve of bidegree $(2m+2, 2m+2)$ in $\prod(t,t)$ for any $t \geq 2m+1$. We may take $W \subset \mathbf{P}^{m^2+4m+3}$ in such a way that $W \cap A$ contains $S$; here we use that $A$ has codimension $2m+3 = \dim(W) - (2m+1)$ in $\mathbf{P}^{m^2+4m+3}$ and that $\mathrm{card}(S) \leq \dim(W)$. The group $\mathrm{Aut}(\langle S\rangle)$ acts transitively on the set of ordered $(2m+2)$-ples of points in linear general in $\langle S\rangle$. Any such $(2m+2)$-ple is contained in a codimension $2m+3$ linear section of a rational normal curve of $W$. Hence we may assume that $D \cap A = S$. Set $Y := X[m,m] \cup D$. $Y$ has bidegree $((m+1)(m+2), (m+1)(m+2))$, the same bidegree of $X[m+1,m+1]$.

*Claim.* We may find such $D$ with $D \subset \prod(m+1,m+1)$, i.e. with $Y \subset \prod(m+1,m+1)$ and $Y \cap A = X[m,m]$.

*Proof of the Claim.* First we will check that $\mathrm{Pic}(Y)$ is an extension of $\mathrm{Pic}(X[m,m]) \times \mathrm{Pic}(D) \cong \mathrm{Pic}(X[m,m]) \times \mathbf{Z}$ by a multiplicative group isomorphic to $(\mathbf{K}^*)^{\oplus(2m+1)}$. More precisely, every $E \in \mathrm{Pic}(Y)$ is uniquely determined by $E|X[m,m]$, $E|D$ and by the gluing data at each of the $2m+2$ points of $S$; since $D \cong \mathbf{P}^1$, $E|D$ is uniquely determined by the integer $\deg(E|D)$; each of these gluing data is uniquely determined by a non-zero scalar (and vice versa, each non-zero scalar induces a gluing datum at one point of $S$); however, since for any $E' \in \mathrm{Pic}(X[m,m])$ and $E'' \in \mathrm{Pic}(D)$ we have $\mathrm{Aut}(E') \cong \mathrm{Aut}(E'') \cong \mathrm{Aut}(E) \cong \mathbf{K}^*$, we may multiply all these gluing data by a common non-zero scalar and obtain an isomorphic line bundle on $Y$. Hence $\mathrm{Pic}(Y)$ is an extension of $\mathrm{Pic}(X[m,m]) \times \mathrm{Pic}(D)$ by $(\mathbf{K}^*)^{\oplus(2m+1)}$. Take any $L' \in \mathrm{Pic}(Y)$, $M' \in \mathrm{Pic}(Y)$ with $L'|X[m;m] \cong L$, $M'|X[m,m] \cong M$ and $\deg(L'|D) = \deg(M'|D) = 2m+1$. Consider the Mayer–Vietoris exact sequence for $L'$,

$$0 \to L' \to L'|X[m,m] \oplus L'|D \to L'|S \to 0 \tag{1}$$

and the corresponding Mayer–Vietoris exact sequence for $M'$. Since $\mathrm{card}(S) = 2m+2$ and $\deg(L'|D) = \deg(M'|D) = 2m+1$, the restriction maps $H^0(D, L'|D) \to H^0(S, L'|S)$ and $H^0(D, M'|D) \to H^0(S, M'|S)$ are surjective. Hence by the Mayer–Vietoris exact sequences we obtain $h^0(Y,L') = m+1$, $h^0(Y,M') = m+1$, $h^1(Y,L') = 0$ and $h^1(Y,M') = 0$. Similarly, we obtain that $L'$ and $M'$ are spanned and (for general gluing data) induce an embedding of $Y$ into $\prod(m+1,m+1)$, proving the Claim.

The variety $\prod(m,m)$ is the complete intersection of two Cartier divisors of $\prod(m+1,m+1)$, one of type $(1,0)$ and one of type $(0,1)$. Hence $N_{X[m,m]/\prod(m+1,m+1)} \cong N_{X[m,m]/\prod(m,m)} \oplus L \oplus M$. Thus $h^1(X[m,m], N_{X[m,m]/\prod(m+1,m+1)}) = 0$. By construction $D \cap X[m,m] = S$ and $D$ intersects quasi-transversally $X[m,m]$. Hence $Y$ is a connected nodal curve with $p_a(Y) = m^2+2m+1$. Since $A \cap W = \langle S\rangle$ and $\dim(\langle S\rangle) + \dim(W) = \mathrm{codim}(A)$, $Y$ spans $\mathbf{P}^{m^2+4m+3}$. Hence by semicontinuity it is sufficient to prove that $Y$ is smoothable and that $h^1(Y, N_{Y/\prod(m+1,m+1)}) = 0$. Since $D$ has bidegree $(2m+1, 2m+1)$ in $\prod(m+1,m+1)$, its normal bundle is a quotient of a direct sum of line bundles of degree $2m+1$. Since every vector bundle on $D \cong \mathbf{P}^1$ is a direct sum of line bundles, we obtain that every line bundle appearing in a decomposition of $N_{D/\prod(m+1,\ m+1)}$ has degree at least $2m+1$. By [11], Cor. 3.2 and Prop. 3.3, or [13], $N_{Y/\prod(m+1,\ m+1)}|X[m,m]$ (resp. $N_{Y/\prod(m+1,\ m+1)}|D$) is obtained from $N_{X[m,m]/\prod(m+1,\ m+1)}$ (resp. $N_{D/\prod(m+1,\ m+1)}$) making $2m+2$ positive elementary transformations. Hence $h^1(X[m,m], N_{Y/\prod(m+1,\ m+1)}|X[m,m]) = 0$ and every line bundle appearing in a decomposition of $N_{Y/\prod(m+1,\ m+1)}|D$ has degree at least $2m+1$. The last remark implies the surjectivity of the restriction map $\rho : H^0(D, N_{Y/\prod(m+1,\ m+1)}|D) \to H^0(S, N_{Y/\prod(m+1,\ m+1)}|S)$. By the Mayer–Vietoris exact sequence

$$0 \to N_{Y/\prod(m+1,\ m+1)} \to N_{Y/\prod(m+1,\ m+1)}|X[m,\ m] \oplus N_{Y/\prod(m+1,\ m+1)}|D \\ \to N_{Y/\prod(m+1,\ m+1)}|S \to 0, \tag{2}$$

we obtain $h^1(Y, N_{Y/\prod(m+1,\ m+1)}) = 0$. Furthermore, as in [11], Th. 4.1, or [13] we obtain also that $Y$ is smoothable. Notice that we may apply the semicontinuity theorem for the dimension of the kernel of the multiplication map for a flat family of pairs of non-special line bundles on a flat family of curves, because the non-speciality condition implies that the corresponding cohomology groups have constant dimension. By semicontinuity we obtain the result for a general triple $(Z, L'', M'')$ with $Z$ of genus $(m+1)^2$ and $(L'', M'')$ a general pair of line bundles on $Z$ with degree $(m+1)^2 + m + 1$.

PROPOSITION 2.3

*Fix integers $m, n$ with $n \geq m \geq 1$. Assume that $H(m,n)$ is true. Then $H(m, n+1)$ is true.*

*Proof.* We will show how to modify the proof of 2.2. Notice that $p_a(X[m, n+1]) = p_a(X[m,n]) + m$. We start with $(X[m,n], L, M)$ satisfying $H(m,n)$. Hence $L, M \in \mathrm{Pic}(X[m,n])$, $\deg(L) = p_a(X[m,n]) + m = mn + n$ and $\deg(M) = mn + n$. We take $S \subset X[m,n] \subset A := \langle \prod(m,n) \rangle$ with $\mathrm{card}(S) = m+1$ and $\dim(\langle S \rangle) = m$. Now $D$ is a smooth rational curve and it is embedded into $\prod(x,y)$, $x \geq m$, $y \geq m+1$, by a pair $(R_1, R_2)$ with $\deg(R_1) = m$ and $\deg(R_2) = m+1$, i.e. of bidegree $(m,\ m+1)$. Hence $\deg(R_1 \otimes R_2) = 2m+1$. Set $Y := X[m,n] \cup D$. Since $h^0(D, R_i) = \deg(R_i) + 1 \geq \mathrm{card}(S)$ for $i = 1, 2$, every part of the proof of 2.2 works in our new set-up, proving 2.3.

*Proof of* 1.1. (i) Here we will cover the case $0 \leq g \leq mn$, i.e. when we need to prove that for a general triple $(X, L, M)$ the multiplication map $\mu_{L,M}$ is surjective. Since the case $g \leq 1$ is well-known and trivial, we assume $g \geq 2$ and hence $n \geq 2$. Since $H(m,n)$ is true, we know the case $g = mn$. Hence we may assume $2 \leq g < mn$. We start with $X[1,1]$ satisfying $H(1,1)$ and then we follow the proofs of 2.2 and 2.3 made to obtain a proof of $H(m,n)$. However, at each step of the proof we take $D$ intersecting the other curve in a subset, $S'$, of $S$. For instance if $n > m$ and $mn - m - 1 \leq g < mn$, we take $\mathrm{card}(S') = g - mn + n$. Call $Y'$ the curve $X[m, n-1] \cup D$ with $D \cap X[m, n-1] = S'$. The proofs of 2.2 and 2.3 and semicontinuity proves 1.1 for this triple $(m, n, g)$.

(ii) Now we assume $g > mn$. By induction on $g$ for a fixed pair $(m,n)$ and the case $g' = mn$ (the bijective case) proved in part (i) we may assume the result for the triple $(m, n, g-1)$. Let $(C, A, B)$ a general triple satisfying the statement of 1.1 for the triple $(m, n, g-1)$. Fix two general points $\{P, Q\}$ of $C$ and let $Y$ be the nodal curve $C \cup D$ with $D \cong \mathbf{P}^1$ and $C \cap D = \{P, Q\}$. By semistable reduction $Y$ is the flat limit of a flat family of smooth connected curves of genus $g$. Take any $L, M \in \mathrm{Pic}(Y)$ with $L|C \cong A$, $M|C \cong B$ and $\deg(L|D) = \deg(M|D) = 1$. We saw in the proof of 2.2 that the set of all such $L$ (resp. $M$) is not empty and parametrized by an extension of $\mathrm{Pic}^0(C)$ by $\mathbf{K}^*$. Since the restriction maps $H^0(D, L|D) \to \boldsymbol{O}_{\{P,Q\}}$ and $H^0(D, M|D) \to \boldsymbol{O}_{\{P,Q\}}$

are surjective, as in the proof of 2.2 a Mayer–Vietoris exact sequence similar to (1) shows that $h^0(Y, L) = m + 1$, $h^0(Y, M) = m + 1$ and $h^1(Y, L) = h^1(Y, M) = 0$. Furthermore, the same exact sequence induces an isomorphism of $H^0(C, A)$ (resp. $H^0(C, B)$) with $H^0(Y, L)$ (resp. $H^0(Y, M)$) and a surjection of $H^0(C, A \otimes B)$ onto $H^0(Y, L \otimes M)$. Hence the injectivity of $\mu_{A,B}$ implies the injectivity of $\mu_{L,M}$. By semicontinuity we conclude as in the last part of the proof of 2.2.

*Proof of* 1.3. Look again to the proof of 1.1 and in particular to the proof of 2.2. Now we take as $X[1, 1]$ a rational curve with an ordinary node as only singularity. As in the proof of 2.2 we obtain the result in the case $m = n = 1$. Now we consider the inductive step in the proofs of 2.2 and 2.3. Just to fix the notation we assume the case $(m, m)$ and prove the case $(m+1, m+1)$. Now $X[m, m]$ is the general rational curve with $mn$ ordinary nodes as only singularities. Set $Y := X[m, m] \cup D$. We need to deform $Y$ inside $\prod(m + 1, m + 1)$ to an irreducible rational curve with only nodes as singularities. Hence it is sufficient to prove that we may smooth exactly one node (any node we chose) in $Y \cap D$ keeping singular the other singular points of $Y \cap D$ and without smoothing the other points and keeping singular the singular points of $X[m, n]$. If instead of $X[m, n]$ we would have a smooth curve, this would be the notion of strong smoothability considered in [11], §1. The part concerning the nodes in $X[m, m] \cap D$ is easy because $\mathrm{card}(X[m, m] \cap D) = 2m + 2$ and every line bundle appearing in a decomposition of $N_{Y/\prod(m+1,\ m+1)}|D$ has degree at least $2m + 1$. Hence $h^1(D, (N_{Y/\prod(m+1,\ m+1)}|D)(-S)) = 0$ and we may apply the proof of [11], Th. 4.1. We know that $h^1(Y, N_{Y/\prod(m+1,m+1)}) = 0$ and hence that $Y$ is a smooth point of $\mathrm{Hilb}(\prod(m + 1, m + 1))$. Furthermore, by induction on $m$ we may assume that each subset of the set of all nodes of $X[m, m]$ may be smoothing independently, i.e. that for every subset $\Gamma$ of $\mathrm{Sing}(X[m, m])$ the set of curves in $\prod(m+1, m+1)$ near $X[m, m]$ in which we smooth exactly the nodes in $\mathrm{Sing}(X[m, m])\backslash\Gamma$ has, near $Y$, codimension $\mathrm{card}(\Gamma)$ in $\mathrm{Hilb}(\prod(m + 1, m + 1))$. The same assertion for $Y$ follows from this, $\mathrm{card}(S) = 2m + 2$, that every line bundle appearing in a decomposition of $N_{Y/\prod(m+1,m+1)}|D$ has degree at least $2m + 1$ and a Mayer–Vietoris exact sequence as in the proof of [11], Th. 4.1. Hence we obtain the case $q = 0$ of 1.3. If $q > 0$ we just smooth $q$ nodes and apply semicontinuity.

## 3. Line bundles on singular curves

For any triple $g, r, d$ of integers, let $\rho(g, r, d) := g - (r + 1)(g + r - d)$ be the so-called Brill–Noether number associated to $g, r$ and $d$. For any smooth projective curve $X$, set $W^r_d(X) := \{L \in \mathrm{Pic}(X) : h^0(X, L) \geq r + 1\}$. On a general smooth curve $X$ of genus $g \geq 2$ we have $W^r_d(X)) \neq \emptyset$ if and only if $\rho(g, r, d) \geq 0$; if $\rho(g, r, d) \geq 0$, then $W^r_d(X)$ is non-empty, smooth outside $W^{r+1}_d(X)$ and of pure dimension $\rho(g, r, d)$; $W^r_d(X)$ is irreducible if $\rho(g, r, d) > 0$ ([1], chs V and VII, and in particular the references [9] and for the smoothness and irreducibility in arbitrary characteristic). If $\rho(g, r, d) \geq 0$ this implies that a general $L \in W^r_d(X)$ has no base points and $h^0(X, L) = r + 1$; here and in the statements of 3.1, 3.2 and 3.5 if $\rho(g, r, d) = 0$ (i.e. if $W^r_d(X)$ is finite) the word 'general $L \in W^r_d(X)$' means 'every $L \in W^r_d(X)$'; if $C$ is singular (i.e. $q \neq g$) in the statement of 3.1, 3.2 and 3.3 the word 'general' means only 'general in a smooth component with the expected dimension $\rho(g, x - 1, a)$ and $\rho(g, y - 1, b)$' because we do not claim any irreducibility result for the schemes $W^r_d(C)$ when $C$ is a singular curve. In the smooth case ($q = g$) when $\rho(g, x - 1, b) = 0$ to have 'for all $L \in W^r_d(X)$' we need to use [6] and hence we need to assume $\mathrm{char}(\mathbf{K}) = 0$.

**Theorem 3.1.** *Fix integers $g$ and $q$ with $g \geq q \geq 0$ and $g \geq 3$. Let $\pi : Y \to C$ be a birational morphism with $Y$ general curve of genus $q$ and $C$ general nodal curve with $g - q$ nodes and $Y$ as normalization, i.e. assume that $\pi^{-1}(\mathrm{Sing}(C))$ is formed by $2g - 2q$ general points of $Y$. Fix integers $a, b, x$ and $y$ with $2 \leq x \leq g - 2, 2 \leq y \leq g + x - a - 1, \rho(g, x - 1, a) \geq 0, 0 \leq a \leq 2g - 2$ and $g + y - x - 1 \leq b \leq g + y - 1$. Let $L \in W_a^{x-1}(C)$ and $M \in W_b^{y-1}(X)$ be general elements. Then the multiplication map $\mu_{L,M} : H^0(C, L) \otimes H^0(C, M) \to H^0(C, L \otimes M)$ is injective.*

*Proof.* By [9], Prop. 1.2, there is a nodal curve $D$ with $p_a(D) = g$ and exactly $g$ ordinary nodes such that for every $L \in W_a^{x-1}(D)$ with $h^0(D, L) = x$ the multiplication map $\mu_{L,\omega_D \otimes L^*} : H^0(D, L) \otimes H^0(D, \omega_D \otimes L^*) \to H^0(D, \omega_D)$ is injective; we will only use that this is true just for one $L \in W_a^{x-1}(D)$ with $h^0(D, L) = x$. By semicontinuity for a general nodal curve, $C$, with $p_a(C) = g$ and with exactly $g - q$ nodes as only singularities there is $L \in \mathrm{Pic}(C)$ with $\deg(C) = a$, $h^0(C, L) = x$ and such that the multiplication map $\mu_{L,\omega_C \otimes L^*} : H^0(C, L) \otimes H^0(C, \omega_C \otimes L^*) \to H^0(C, \omega_C)$ is injective. By Riemann–Roch we have $h^1(C, L) = g + x - a - 1$. By assumption we have $g - a - 1 \leq y \leq g + x - a - 1$ and $b \geq g + y - x - 1$. Let $D$ (resp. $D'$) be the union of $g + x - a - 1 - y$ (resp. $b - g - y + x + 1$) general points of $C$. Set $R := \omega_C \otimes L^*(-D)$ and $A := R(D')$. Hence $\deg(R) = g + y - x + 1 \leq b = \deg(A)$. Since $D$ is general, we have $h^0(C, R) = y$. Adding $D$ as a base locus we may see the vector space $H^0(C, R)$ as a subspace of $H^0(C, \omega_C \otimes L^*)$. Thus the multiplication map $\mu_{L,R} : H^0(C, L) \otimes H^0(C, R) \to H^0(C, L \otimes R)$ is injective. By Rieman–Roch we have $h^1(C, R) = x$. Thus $h^1(C, R) \geq \deg(D')$. Hence $h^0(C, A) = h^0(C, R)$ by the generality of $D'$, i.e. $A \in W_b^{y-1}(C)$ and the complete linear system associated to $A$ has $D'$ in its base locus. Thus the multiplication map $\mu_{L,A} : H^0(C, L) \otimes H^0(C, A) \to H^0(C, L \otimes A)$ is injective. Hence by semicontinuity for general $M \in W_b^{y-1}(C)$ the multiplication map $\mu_{L,M} : H^0(C, L) \otimes H^0(C, M) \to H^0(C, L \otimes M)$ is injective. Quoting [5] instead of [9] we have the following result.

**Theorem 3.2.** *Fix integers $g$ and $q$ with $g \geq q \geq 0$ and $g \geq 3$. Let $\pi : Y \to C$ be a birational morphism with $Y$ general curve of genus $q$ and $C$ general cuspidal curve with $g - q$ nodes and $Y$ as normalization, i.e. assume that $\pi^1(\mathrm{Sing}(C))$ is formed by $g - q$ general points of $Y$. Fix integers $a, b, x$ and $y$ with $2 \leq x \leq g - 2, 2 \leq y \leq g + x - a - 1, \rho(g, x - 1, a) \geq 0, 0 \leq a \leq 2g - 2$ and $g + y - x - 1 \leq b \leq g + y - 1$. Let $L \in W_a^{x-1}(C)$ and $M \in W_b^{y-1}(X)$ be general elements. Then the multiplication map $\mu_{L,M} : H^0(C, L) \otimes H^0(C, M) \to H^0(C, L \otimes M)$ is injective.*

*Remark* 3.3. Theorem 3.2 is true with the same proof for every rational cuspidal curve, not just the general one ([5]).

## 4. Rank 1 torsion free sheaves

Let $C$ be an integral projective curve and $F$ and $G$ rank 1 torsion free sheaves on $C$. The sheaf $F \otimes G$ may have torsion, but the sheaf $F \otimes G/\mathrm{Tors}(F \otimes G)$ is a rank 1 torsion free sheaf. Call $\beta_{F,G} : H^0(C, F) \otimes H^0(C, G) \to H^0(C, F \otimes G/\mathrm{Tors}(F \otimes G))$ the composition of the multiplication map $\mu_{F,G} : H^0(C, F) \otimes H^0(C, G) \to H^0(C, F \otimes G)$ with the map $H^0(C, F \otimes G) \to H^0(C, F \otimes G/\mathrm{Tors}(F \otimes G))$ induced by the quotient map $F \otimes G \to F \otimes G/\mathrm{Tors}(F \otimes G)$. We believe that the linear map $\beta_{F,G}$ is more significant and has better behaviour than the plain multiplication map $\mu_{F,G}$. In this section we study $\beta_{F,G}$ and $\mu_{F,G}$ in the case of nodal curves. The general set-up works for curves with only ordinary nodes and ordinary cusps as singularities (see (4.1)). The restriction to nodal curves come from the use of 1.3. In many interesting cases the map $\beta_{F,G}$ is induced from a multiplication map for line bundles on a partial normalization of $C$ (see (4.2)). Here is the general set-up. Let $f : Y \to C$ be a birational morphism between integral projective curves. Set $\delta := p_a(Y) - p_a(C)$. We have $\delta \geq 0$ and $\delta = 0$ if and only if $f$ is an isomorphism. For every rank 1 torsion free sheaf $A$ on $Y$ the coherent sheaf $f_*(A)$ is a rank 1 torsion free sheaf on $C$. If $A \cong f^*(B)$ for some rank 1 torsion free sheaf $B$ on $C$, then $f_*(A) \cong B \otimes f^*(\boldsymbol{O}_Y)$ (projection formula) and hence $\deg(f_*(A)) = \deg(A) + \delta$. By the very definition of the direct image functor we have $h^0(C, f_*(A)) = h^0(Y, A)$. Since $f$ is finite, we have $h^1(C, f_*(A)) = h^1(Y, A)$. It is easy to check that for every rank 1 torsion free sheaf $B$ on $C$ the natural map $f_B{}^* : H^0(C, B) \to H^0(Y, f^*(B)/\mathrm{Tors}(f^*(B)))$ is injective. Let $L$, $M$ be rank 1 torsion free sheaves on $Y$. Since $H^0(Y, L) \cong H^0(C, f_*(L))$, $H^0(Y, M) \cong H^0(C, f_*(M))$, the multiplication map $\mu_{L,M} : H^0(Y, L) \otimes H^0(Y, M) \to H^0(Y, L \otimes M)$ induces a morphism $\beta_{L,M} : H^0(Y, L) \otimes H^0(Y, M) \to H^0(Y, L \otimes M/\mathrm{Tors}(L \otimes M))$, a morphism $\beta_{L,M,f} : H^0(C, f_*(L)) \otimes H^0(C, f_*(M)) \to H^0(C, f_*(L \otimes M))$ and a morphism $\alpha_{L,M,f} : H^0(C, f_*(L)) \otimes H^0(C, f_*(M)) \to H^0(C, f_*(L \otimes M)/\mathrm{Tors}(f_*(L \otimes M)))$. A section of a torsion free sheaf on a reduced curve is uniquely determined by its restriction to a Zariski open dense subset of the curve. Hence if $\mu_{L,M}$ is injective, then $\alpha_{L,M,f}$ is injective.

(4.1) Let $R$ be the local ring either of an ordinary node (i.e. of an $A_1$ singularity), $P$, of an irreducible curve or of an ordinary cusp (i.e. of an $A_2$ singularity). Let $\mathbf{m}$ be the maximal ideal of $R$. If $R$ is an ordinary node will say that a coherent sheaf on Spec($R$) is torsion free near $P$ if its completion has no nonzero element killed by an element of $R$ which is not a zero-divisor of $R$; this is the definition used in [4]. With this convention every finitely generated torsion free $R$-module $M$ (up to a completion) is of the form $R^{\oplus a} \oplus \mathbf{m}^{\oplus b}$ for some integer $a \geq 0$, $b \geq 0$, $a + b > 0$, with $a + b = \mathrm{rank}(M)$ ([4], Th. 2.4.2 and Remark 1 after that, or [14], Prop. 2 at p. 162). The same is true if $R$ is an ordinary cusp. We will need only the case $\mathrm{rank}(M) = 1$; hence either $M \cong R$ or $M \cong \mathbf{m}$. It is easy to check that $\mathbf{m}$ contains a rank 1 submodule $M$ with $M \cong R$ and $\mathbf{m}/M \cong \mathbf{K}$; obviously $\mathbf{m}$ is contained in the rank 1 free module $R$ and $R/\mathbf{m} \cong \mathbf{K}$.

For any coherent sheaf $F$ on an integral projective curve $X$ with pure rank $r$ the degree $\deg(F)$ of $F$ is defined by the Riemann–Roch formula $\deg(F) := \chi(F) + r(g - 1)$. If $F$ is a torsion free sheaf on $X$, set $\mathrm{Sing}(F) := \{P \in X : F \text{ is not locally free at } P\} = \{P \in \mathrm{Sing}(X) : F \text{ is not locally free at } X\}$.

(4.2) Let $C$ be an integral projective curve whose only singularities are ordinary nodes and ordinary cusps. Let $F$ be a rank 1 torsion free sheaf on $C$. Set $S := \mathrm{Sing}(F) = \{P \in C : F \text{ is not locally free at } P\}$. Hence by 4.1 for every $P \in \mathrm{Sing}(F)$ near $P$ the sheaf $F$ is formally equivalent to the maximal ideal of $\boldsymbol{O}_{C,P}$. Set $\delta := \mathrm{card}(S)$. Let $\pi : Y \to C$ be the partial normalization of $C$ in which we normalize only the points of $S$. We have $p_a(C) = p_a(Y) + \delta$. Set $L := \pi^*(F)/\mathrm{Tors}(\pi^*(F))$. By 4.1 we have $L \in \mathrm{Pic}(C)$, $F \cong \pi_*(L)$ and $\deg(F) = \deg(L) + \delta$. Let $M(C; x, S)$ be the set of all rank 1 torsion free sheaves, $G$, on $C$ with $\deg(G) = x$ and $\mathrm{Sing}(G) = S$. Now we will use the following observation.

*Remark* 4.3. Let $C$ be an integral projective curve whose only singularities are ordinary nodes and ordinary cusps. Fix $S \subseteq \mathrm{Sing}(C)$ and let $\pi : Y \to C$ be the partial normalization of $C$ in which we normalize only the points of $S$. $\mathrm{Pic}^0(Y)$ is a $q$-dimensional algebraic group, $q := p_a(C) - \mathrm{card}(S) = p_a(Y)$, which is an extension of an abelian variety of dimension $p_a(C) - \text{card}(\text{Sing}(C))$ by a connected affine group $G$; $G$ is the product of some copies of the additive group (the number of copies being the number of cusps of $Y$, i.e. of the cusps in $\text{Sing}(C)\setminus S$) and some copies of the multiplicative group (the number of copies being the number of nodes of $Y$). In particular $\text{Pic}^0(Y)$ is an irreducible $q$-dimensional variety. Hence for every integer $x$ the set $M(C; x, S)$ has a natural structure of $q$-dimensional irreducible algebraic variety. Hence we are allowed to consider the general element of $M(C; x, S)$.

Take another rank 1 torsion free sheaf $G$ with $S = \text{Sing}(G)$. Set $M := \pi^*(G)/\text{Tors}(\pi^*(G))$. Hence $G \cong \pi_*(M)$ and $\deg(G) = \deg(M)+\delta$. By (4.1) we have $F\otimes G/\text{Tors}(F\otimes G) \cong \pi_*(L\otimes M)$. Since $H^0(C, F) \cong H^0(Y, L)$, $H^0(C, G) \cong H^0(Y, M)$ and $H^0(Y, L\otimes M) \cong H^0(C, \pi_*(L \otimes M))$, the linear maps $\mu_{L,M}$ and $\alpha_{L,M,\pi}$ have kernel and cokernel with the same dimension. In particular $\mu_{L,M}$ is surjective (resp. injective) if and only if $\alpha_{L,M,f}$ is surjective (resp. injective). Hence by Theorem 1.3 for the integer $q := g - \delta$ we obtain the following result.

PROPOSITION 4.4

*Let C be an integral projective curve whose only singularities are ordinary nodes. Fix a set* $S \subseteq \text{Sing}(C)$ *and set* $g := p_a(C)$ *and* $\delta := \text{card}(S)$. *Let* $\pi : Y \to C$ *be the partial normalization of C in which we normalize only the points of S. Fix integers* $a, b$ *with* $a \geq g$ *and* $b \geq g$. *Then for general element* $\pi_*(L) \in M(C; a, S)$ *and* $\pi_*(M) \in M(C; b, S)$ *the map* $\alpha_{L,M,\pi}$ *has maximal rank.*

*Remark* 4.5. Let $C$ be an integral projective curve whose only singularities are ordinary nodes or ordinary cusps. Set $g := p_a(C)$. Fix $S \subseteq \text{Sing}(C)$ and set $s := \text{card}(S)$. Let $\pi : Y \to C$ be the partial normalization of $C$ in which we normalize the set $S$. For every $L \in \text{Pic}(C)$ we have $\pi_*(L) \in M(C; x, S)$ with $x = \deg(L)+s = \deg(L)+p_a(C)-p_a(Y)$ and $h^0(Y, L) = h^0(C, \pi_*(L))$, $h^1(Y, L) = h^1(C, \pi_*(L))$. Hence taking a general $L \in \text{Pic}^{x-s}(Y)$ we obtain that for every integer $x \geq g - 1$ a general $F \in M(C; x, S)$ has $h^1(C, F) = 0$, i.e. $h^0(C, F) = \deg(F) + 1 - g$.

(4.6) Let $C$ be an integral projective curve whose only singularities are ordinary nodes and ordinary cusps. Let $F$ and $G$ be rank 1 torsion free sheaves on $C$ with $\text{Sing}(F) \cap \text{Sing}(G) = \emptyset$. This condition is equivalent to the torsion freeness of $F \otimes G$. We have $\deg(F \otimes G) = \deg(F) + \deg(G)$ and $\text{Sing}(F \otimes G) = \text{Sing}(F) \cup \text{Sing}(G)$. Since $F \otimes G$ has no torsion, here we will consider the usual multiplication map $\mu_{F,G}$. For the injectivity of $\mu_{F,G}$ it is usually not restrictive to assume $F$ spanned (otherwise we reduce to the study of the subsheaf $F'$ of $F$ spanned by $H^0(C, F)$, although $\text{Sing}(F') \neq \text{Sing}(F)$ in general). Usually we will consider a range in which $F \otimes G$ is spanned and hence to obtain the surjectivity of $\mu_{F,G}$ it is necessary to assume that $F$ and $G$ are spanned.

(4.7) Let $C$ be an integral projective curve whose only singularities are ordinary nodes and ordinary cusps. Let $F$ be a rank 1 spanned torsion free sheaf on $C$ and $\pi : Y \to C$ the partial normalization of $C$ in which we normalize exactly the points of $\text{Sing}(F)$. Set $L := \pi^*(F)/\text{Tors}(\pi^*(F))$. By 4.1 we have $L \in \text{Pic}(C)$, $F \cong \pi_*(L)$ and $\deg(F) = \deg(L) + \delta$ and $h^0(Y, L) = h^0(C, F)$. Since $F$ is spanned, $\pi^*(L)$ is spanned and hence $L$ is spanned.

*Remark* 4.8. Let $U$ be a quasi-projective one-dimensional scheme with a unique singular point, $P$, which is either an ordinary node or an ordinary cusp. Let $F$ and $G$ be rank 1 torsion free sheaves on $U$ such that $F$ is not locally free at $P$, while $G$ is locally free at

$P$, i.e. with $G \in \mathrm{Pic}(U)$. Let $\mathbf{K}_P$ be the skyscraper sheaf on $U$ supported by $P$ and with length $l$. By the last part of (4.1) there exist rank 1 torsion free sheaves $F'$, $F''$, $G'$, $G''$ on $U$ with $F' \subset F \subset F''$, $G' \subset G \subset G''$, $F/F' \cong F''/F \cong G/G' \cong G''/G' \cong \mathbf{K}_P$ and such that $F'$ and $F''$ are locally free, while $G'$ and $G''$ are not locally free at $P$.

*Remark* 4.9. Let $C$ be an integral projective curve whose only singularities are ordinary nodes and ordinary cusps. Take $S \subseteq \mathrm{Sing}(C)$ and any spanned $R \in \mathrm{Pic}(C)$ with $h^1(C, R) = 0$. By 4.8 we obtain the existence of $F \in M(C; x, S)$, $x = \deg(R) + \mathrm{card}(S)$, such that $R$ is a subsheaf of $F$ and $F/R \cong \boldsymbol{O}_S$. Since $\boldsymbol{O}_S$ is a skyscraper sheaf and $h^1(C, R) = 0$, we obtain $h^1(C, F) = 0$. Hence $h^0(C, F) = h^0(C, R) + \mathrm{card}(S)$. Since $R$ is spanned, this implies the spannedness of $F$.

PROPOSITION 4.10

*Fix non-negative integers $g, q, s, s', a, b$ with $g \geq s + s' + q$, $a \geq g + s$, $b \geq g + s$ and $(a + l - g - s)(b + l - g - s') \geq a + b + l - g - s - s'$. Let $C$ be a general integral nodal curve with $p_a(C) = g$ and normalization of genus $q$. Fix $S \subseteq \mathrm{Sing}(C)$ and $S' \subseteq \mathrm{Sing}(C)$ with $\mathrm{card}(S) = s$, $\mathrm{card}(S') = s'$ and $S \cap S' = \emptyset$. Then for a general $F \in M(C; a, S)$ and a general $G \in M(C; b, S')$ the multiplication map $\mu_{F,G}$ is surjective.*

*Proof.* By Remark 4.9 for general $F$ and $G$ we have $h^1(C, F) = h^1(C, G) = h^1(C, F \otimes G) = 0$. Take general $L \in \mathrm{Pic}(C)$ and $M \in \mathrm{Pic}(C)$. By 1.3 and the assumptions on $g, a, b, s$ and $s'$ the linear map $\mu_{L,M}$ is surjective. Take as $F'$ (resp. $G'$) any element of $M(C; a, S)$ (resp. $M(C; b, S')$ containing $L$ (resp. $M$) and with $F'/L \cong \boldsymbol{O}_S$ (resp. $G'/M \cong \boldsymbol{O}_{S}$) (Remark 4.8). By Remark 4.9 we have $h^1(C, F') = h^1(C, G') = 0$, $h^0(C, F') = h^0(C, L) + s$, $h^0(C, G') = h^0(C, M) + s'$ and both $F'$ and $G'$ are spanned. See $L \otimes M$ as a subsheaf of $F' \otimes G'$ with $F' \otimes G'/L \otimes M \cong \boldsymbol{O}_{S \cup S'}$. Since both $F'$ and $G'$ are spanned $\mathrm{Im}(\mu_{F',G'})$ spans $F' \otimes G'$. Hence $\dim(\mathrm{Im}(\mu_{F',G'})) \geq \dim(\mathrm{Im}(\mu_{L,M})) + s + s' = a + b - s - s' + l - g + s + s' = h^0(C, F' \otimes G')$. Hence $\mu_{F',G'}$ is surjective and we conclude by semicontinuity.

PROPOSITION 4.11

*Fix non-negative integers $g, q, s, s', a, b$ with $g \geq s + s' + q$, $a \geq g$, $b \geq g + s$ and $(a + l - g + s)(b + l - g + s') \leq a + b + l - g + s + s'$. Let $C$ be a general integral nodal curve with $p_a(C) = g$ and normalization of genus $q$. Fix $S \subseteq \mathrm{Sing}(C)$ and $S' \subseteq \mathrm{Sing}(C)$ with $\mathrm{card}(S) = s$, $\mathrm{card}(S') = s'$ and $S \cap S' = \emptyset$. Then for a general $F \in M(C; a, S)$ and a general $G \in M(C; b, S')$ the multiplication map $\mu_{F,G}$ is injective.*

*Proof.* Since $(a + l - g + s)(b + l - g + s'') \leq a + b + l - g + s + s'$, Theorem 1.3 shows that for a general $L \in \mathrm{Pic}^{a+s}(C)$ and a general $M \in \mathrm{Pic}^{b+s'}(C)$ the multiplication map $\mu_{L,M}$ is injective. By Remark 4.9 there is $F' \in M(C; a, S)$ and $G' \in M(C; b, S')$ with $F' \subset L$, $G' \subset M$, $L/F' \cong \boldsymbol{O}_S$ and $M/G' \cong \boldsymbol{O}_{S'}$. Since $F'$ is a subsheaf of $L$ and $G'$ is a subsheaf of $M$ the map $\mu_{F',G'}$ is injective. Since $h^1(C, F' \otimes G') = 0$, we conclude using semicontinuity.

There is a geometrically important case in which iterations of the multiplication maps do occur. Let $X$ be a smooth projective curve of genus $q$ and $L \in \mathrm{Pic}^k(X)$ with $h^0(X, L) = 2$ and $L$ spanned. The ordered sequence of integers $\{h^0(X, L^{\otimes t})\}_{t \geq 0}$ uniquely determines the so-called scrollar invariants of the pencil $L$ (see e.g. [12], §2). If $2k \leq q$ and $X$ is a general $k$-gonal curve of genus $q$ we have $h^0(X, L^{\otimes t}) = t + 1$ if $0 \leq t \leq [q/(k - 1)]$, while $h^0(X, L^{\otimes t}) = kd + 1 - q$ (i.e. $h^1(X, L^{\otimes t}) = 0$) if $t > [q/(k-1)]$ ([2]). Fix an integer with $2 \leq a \leq g$. The equalities $h^0(X, L^{\otimes t}) = t + 1$ if $0 \leq t \leq a$ are equivalent to the surjectivity of all multiplication maps $\mu_{L^{\otimes b}, L}$ with $1 \leq b < a$. On singular curve when $L$ is not locally free the sheaf $L \otimes L$ has always torsion and hence it is more interesting to consider the associated map $\alpha_{L,L,f}$ and its iterations.

PROPOSITION 4.12

*Fix integers $g, q$ and $k$ with $g > q \geq 2k \geq 4$. Let $C$ be an integral projective curve with $p_a(C) = g$ and whose only singularities are ordinary nodes and ordinary cusps and $f : X \to C$ its normalization. Assume that $X$ is a general $k$-gonal curve of genus $q$ and call $L$ its degree $k$ pencil. For every integer $t \geq 1$ set $F_t := f(L^{\otimes t})$. For every integer $t \leq [q/(k-1)]$ we have $h^0(C, F_t) = t + 1$ and for every integer $a < [q/(k-1)]$ the map $\alpha_{F_a, F_1, f}$ is surjective.*

Proposition 4.12 follows at once from the next observation which also explain the meaning of the sheaves involved in the statement of 4.12.

*Remark* 4.13. By 4.2 each $F_t$ is a rank 1 torsion free sheaf on $X$ with $\deg(F_t) = tk + g - q$ and $\mathrm{Sing}(F_t) = \mathrm{Sing}(C)$. Since $h^0(C, F_t) = h^0(X, L^{\otimes t})$, the first assertion of 4.12 follows from [2]. If $t \geq 2$ we have $F_t \cong F_{t-1} \otimes F_1/\mathrm{Tors}(F_{t-1} \otimes F_1) \cong F_1^{\otimes t}/\mathrm{Tors}(F_1^{\otimes t})$ (4.1 and induction on $t$). Hence we obtain the last assertion of 4.12 from the first assertion of 4.12.

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