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Obstructions to Clifford system extensions of algebras

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Abstract. In this paper we do phrase the obstruction for realization of a generalized group character, and then we give a classification of Clifford systems in terms of suitable low-dimensional cohomology groups.

Keywords. Clifford system; character; cohomology groups; obstructions.

1. Introduction

The problem of Clifford system extensions resides in the classification and the construction of the manifold of all Clifford systems over a commutative ring k, $S = \bigoplus_{\sigma \in G} S_\sigma$, the type being given group G and with 1-component S_1 isomorphic to a given k-algebra R. Each such G-graded Clifford system extension realizes a *generalized collective character* of G in R, that is a group homomorphism $\Phi : G \to \mathrm{Pic}_k(R)$ of G into the group of isomorphism classes of invertible left $R \otimes_k R^\circ$-modules, and this leads to a problem of obstruction. When a generalized collective character is specified, it is possible that no Clifford system extensions realizing the specified homomorphism can exist. The main result in this paper is to obtain a necessary and sufficient condition for the existence of such a Clifford system extension, formulated in terms of a certain 3-dimensional group cohomology class $T(\Phi)$, referred to here as the *Teichmüller obstruction* of Φ. The construction of $T(\Phi)$ is closely analogous to a construction by Kanzaki [9], for a description of the Chase-Harrison-Rosenberg seven term exact sequence [2] about the Brauer group. In the case where a generalized collective character Φ has an extension, the manifold of such strongly graded extensions is shown as a principal and homogeneous space under a 2nd cohomology group.

This paper has been strongly influenced by the work on the classification of crossed-product rings by Hacque in [7, 8], where he makes a systematic analysis of the important phenomenon bound to the existence of obstructions. Clifford systems, also called *strongly graded algebras*, are a direct generalization of crossed product algebras and they were introduced and applied by Dade in several important papers [3, 4], where he develops Clifford's theory axiomatically, and which can be referred to for general background.

In §2, we state a minimum of needed notation and terminology. Section 3 contains the main results of the paper, namely the construction of the Teichmüller obstruction map and the obstruction theorems. We conclude in §4 by exhibiting a non-realizable collective character.

2. Clifford system extensions and generalized collective characters

Throughout the paper k is a commutative ring with identity and G is a group.

A *G-graded Clifford system over k S* is a k-algebra with identity, also denoted by S, together with a family of k-submodules S_σ, $\sigma \in G$, such that $S = \bigoplus_{\sigma \in G} S_\sigma$ and $S_\sigma S_\tau = S_{\sigma\tau}$ for all $\sigma, \tau \in G$, where the product $S_\sigma S_\tau$ consists of all finite sums of ring products xy of elements $x \in S_\sigma$ and $y \in S_\tau$. Note that the 1-component S_1 is a k-subalgebra of S and each σ-component S_σ, $\sigma \in G$, is a two-sided S_1-submodule of S.

By a Clifford system extension of a k-algebra R we mean a Clifford system k-algebra S whose 1-component S_1 is isomorphic to R. More precisely, we have the following:

DEFINITION 2.1

Let R be a k-algebra and G a group. A G-graded Clifford system extension of R is a pair $(S,\ j)$, where $S = \bigoplus_{\sigma \in G} S_\sigma$ is a G-graded Clifford system k-algebra and $j : R \hookrightarrow S$ is a k-algebra embedding with $j(R) = S_1$.

If $(S,\ j)$, $(S',\ j')$ are two G-graded Clifford system extensions of R, by a morphism between them $f : (S,\ j) \to (S',\ j')$, we mean a grade-preserving k-algebra homomorphism $f : S \to S'$ that respects the embeddings of R, that is, such that $f\,j = j'$.

The most striking example is the group algebra $R[G]$, but also crossed products of R and G yield examples of G-graded Clifford system extensions of a k-algebra R.

From ([4], Corollary 2.10) it follows that any Clifford system extension morphism $f : (S,\ j) \to (S',\ j')$ is necessarily an isomorphism. Therefore the existence of a morphism is an equivalence relation between G-graded Clifford system extensions of R and, in this case, we usually say that the extensions are equivalent. Then

$$\mathrm{Cliff}_k(G, R) \tag{1}$$

denotes the set of equivalence classes of G-graded Clifford system extensions of the k-algebra R.

If $(S,\ j)$ is a G-graded Clifford system extension of R, then each S_σ, $\sigma \in G$, is an invertible $R \otimes_k R^\circ$-module and, for every $\sigma, \tau \in G$, the canonical morphism $S_\sigma \otimes_R S_\tau \to S_{\sigma\tau}$, $x_\sigma \otimes x_\tau \mapsto x_\sigma x_\tau$, is an $R \otimes_k R^\circ$-isomorphism. Hence, there is a canonical map

$$\chi : \mathrm{Cliff}_k(G, R) \longrightarrow \mathrm{Hom}_{Gp}(G, \mathrm{Pic}_k(R)), \tag{2}$$

where $\mathrm{Hom}_{Gp}(G, \mathrm{Pic}_k(R))$ is the set of group homomorphisms of G into $\mathrm{Pic}_k(R)$, the group of isomorphism classes of invertible $R \otimes_k R^\circ$-modules, which carries the class of a G-graded Clifford system extension $(S,\ j)$ to the group homomorphism $\chi_{[S,\ j]} : G \to \mathrm{Pic}_{k(R)}$, given by

$$\chi_{[S,\,j]}(\sigma) = [S_\sigma], \quad \sigma \in G. \tag{3}$$

We have the Baer notion of *Kollectivcharakter* in mind, and we define a *generalized collective character* of the group G in the k-algebra R as a group homomorphism $\Phi : G \to \mathrm{Pic}_k(R)$. Let us recall the exact group sequence ([1], Chapter II, (5.4)),

$$1 \to \mathrm{InAut}(R) \longrightarrow \mathrm{Aut}_k(R) \stackrel{\delta}{\longrightarrow} \mathrm{Pic}_k(R), \tag{4}$$

in which δ maps a k-algebra automorphism of R, $\alpha \in \mathrm{Aut}_k(R)$, to the class of the invertible $R \otimes_k R^\circ$-module R_α, which is the same left R-module as R with right action given by $x \cdot y = x\alpha(y)$, $x, y \in R$. Then, there is a canonical embedding $\mathrm{Out}_k(R) \stackrel{\delta}{\hookrightarrow} \mathrm{Pic}_k(R)$, of the group of *outer automorphisms* of the k-algebra R, $\mathrm{Out}_k(R) = \mathrm{Aut}_k(R)/\mathrm{InAut}(R)$, into the Picard group $\mathrm{Pic}_k(R)$. A group homomorphism $\Phi : G \to \mathrm{Out}_k(R)$ has been called a *collective character* (cf. Hacque [7, 8]); so that collective characters of G in R are those generalized ones factoring through the embedding $\mathrm{Out}_k(R) \hookrightarrow \mathrm{Pic}_k(R)$. Of course, by *character* we understand a group homomorphism $G \to \mathrm{Aut}_k(R)$.

Hence $\mathrm{Hom}_{Gp}(G, \mathrm{Pic}_k(R))$ is the set of generalized collective characters of G in R, and the map χ associates with each equivalence class of G-graded Clifford system extensions of R a generalized collective character. We refer to a generalized collective character $\Phi : G \to \mathrm{Pic}_k(R)$ as *realizable* if it is in the image of χ, that is, if it is induced as explained above from a G-graded Clifford system extension of R. The map χ produces a partitioning of the set of equivalence classes of G-graded Clifford system extensions of R,

$$\mathrm{Cliff}_k(G, R) = \coprod_{\Phi} \mathrm{Cliff}_k(G, R; \Phi), \tag{5}$$

where, for any generalized collective character $\Phi \in \mathrm{Hom}_{Gp}(G, \mathrm{Pic}_k(R))$, we denote by $\mathrm{Cliff}_k(G, R; \Phi) = \chi^{-1}(\Phi)$ the fiber of χ over Φ . Thus a generalized collective character Φ is realizable if the set $\mathrm{Cliff}_k(G, R; \Phi)$ is not empty. We refer to $\mathrm{Cliff}_k(G, R; \Phi)$ as the *set of equivalence classes of realizations* of the generalized collective character Φ.

3. The Teichmüller cocycle and the obstruction theorems

If R is a k-algebra, let $C(R) = \{r \in R | rx = xr, x \in R\}$ denote its center. Then $C(R)$ is a k-algebra whose group of units we denote by $C(R)^*$.

We will often use the following elementary fact, which is a consequence of ([1], Chapter II, (3.5)).

Lemma 3.1. *If P, Q are invertible $R \otimes_k R^\circ$-modules, then for any two $R \otimes_k R^\circ$-isomorphisms $\alpha, \beta : P \to Q$, there exists a unique $u \in C(R)^*$ such that $\beta = u\alpha = \alpha u$ (i.e., $\beta(x) = u\alpha(x) = \alpha(ux)$ for all $x \in P$).*

Proof. Given $\alpha : P \to Q$, an $R \otimes_k R^\circ$-isomorphism, the map $C(R)^* \to \mathrm{Isom}_{R\otimes_k R^\circ}(P, Q)$, $u \mapsto u\alpha$, is bijective since it can be obtained as the composite map of the canonical group isomorphism $C(R)^* \cong \mathrm{Aut}_{R\otimes_k R^\circ}(R)$, the group isomorphism $- \otimes_R P : \mathrm{Aut}_{R\otimes_k R^\circ}(R) \cong \mathrm{Aut}_{R\otimes_k R^\circ}(P)$ and the bijection induced by α, $\alpha_* : \mathrm{Aut}_{R\otimes_k R^\circ}(P) \cong \mathrm{Isom}_{R\otimes_k R^\circ}(P, Q)$. □

If P is any invertible $R \otimes_k R^\circ$-module and $u \in C(R)^*$, since $x \mapsto xu$ is an $R \otimes_k R^\circ$-automorphism of P, there exists a unique element $\alpha_P(u) \in C(R)^*$ such that $\alpha_P(u)x = xu$ for all $x \in P$. Clearly $\alpha_P : C(R)^* \to C(R)^*$ is an automorphism and

$$\mathrm{Pic}_k(R) \xrightarrow{\rho} \mathrm{Aut}(C(R)^*), \quad \rho([P]) = \alpha_P \tag{6}$$

is a group homomorphism (note that ρ is the restriction to $C(R)^*$ of Bass' homomorphism $h : \mathrm{Pic}_k(R) \to \mathrm{Aut}_k(C(R))$ ([1], Chap. II, (5.4)). Hence $C(R)^*$ is a $\mathrm{Pic}_k(R)$-module.

By composition with the homomorphism (6) we have for any group G a map

$$\mathrm{Hom}_{Gp}(G, \mathrm{Pic}_k(R)) \xrightarrow{\rho_*} \mathrm{Hom}_{Gp}(G, \mathrm{Aut}(C(R)^*)) \tag{7}$$

that to each generalized collective character of G in the k-algebra R, $\Phi : G \to \mathrm{Pic}_k(R)$, associates a character $\Phi^* = \rho\Phi : G \to \mathrm{Aut}(C(R)^*)$ from group G in the abelian group $C(R)^*$. Of course, the set of characters $\mathrm{Hom}_{Gp}(G, \mathrm{Aut}(C(R)^*))$ is the set of G-module structures on $C(R)^*$. Hence every generalized collective character $\Phi : G \to \mathrm{Pic}_k(R)$, of

G in R, determines a G-module structure on $C(R)^*$ for which the corresponding G-action of an element $\sigma \in G$ on an element $u \in C(R)^*$ is given by ${}^{\sigma}u = \alpha_P(u)$ for any $P \in \Phi(\sigma)$. In particular,

$$xu = {}^{\sigma}ux \tag{8}$$

for any $\sigma \in G$, $u \in C(R)^*$, $x \in P$ and $P \in \Phi(\sigma)$. We will denote by $H^n_{\Phi}(G, C(R)^*)$, $n \geq 0$, the nth cohomology group of G with coefficients in this G-module.

We will now show how every generalized collective character $\Phi : G \to \mathrm{Pic}_k(R)$ has a cohomology class $T(\Phi) \in H^3_{\Phi}(G, C(R)^*)$ canonically associated with it, whose construction has several precedents: the *Teichmüller cocycle* homomorphism $H^0(G, Br(R)) \to H^3(G, R^*)$ [10,6], defined when R/k is a field Galois extension with group G; the *Eilenberg-Mac Lane obstruction* defined by a G-kernel, defined in [5] for the study of group extensions with a non-abelian kernel; the description by Kanzaki [9] of the homomorphism $H^1(G, \mathrm{Pic}_R(R)) \to H^3(G, R^*)$, in the Chase-Harrison-Rosenberg seven term exact sequence [2], about the Brauer group relative to a Galois extension of commutative rings R/k; the *Teichmüller obstruction* associated to a collective character $\Phi : G \to \mathrm{Out}(R)$, by Hacque in [7,8] for the study of obstructions to the existence of crossed product rings.

Let $\Phi : G \to \mathrm{Pic}_k(R)$ be a generalized collective character of a group G in a k-algebra R. In each isomorphism class $\Phi(\sigma) \in \mathrm{Pic}_k(R)$, choose an invertible $R \otimes_k R^{\circ}$-module $P_{\sigma} \in \Phi(\sigma)$; in particular, select $P_1 = R$. Since Φ is a homomorphism, the modules $P_{\sigma} \otimes_R P_{\tau}$ and $P_{\sigma\tau}$ must be $R \otimes_k R^{\circ}$-isomorphic for each pair $\sigma, \tau \in G$. Then we can select $R \otimes_k R^{\circ}$-isomorphisms

$$\Gamma_{\sigma,\tau} : P_{\sigma} \otimes_R P_{\tau} \to P_{\sigma\tau} \tag{9}$$

with $\Gamma_{\sigma,1}(x \otimes r) = xr$ and $\Gamma_{1,\sigma}(r \otimes x) = rx$, $r \in R$, $x \in P_{\sigma}$.

For any three elements $\sigma, \tau, \gamma \in G$, the diagram

$$\begin{array}{ccc} P_{\sigma} \otimes_R P_{\tau} \otimes_R P_{\gamma} & \xrightarrow{\Gamma_{\sigma,\tau} \otimes P_{\gamma}} & P_{\sigma\tau} \otimes_R P_{\gamma} \\ {\scriptstyle P_{\sigma} \otimes \Gamma_{\tau,\gamma}} \downarrow & & \downarrow {\scriptstyle \Gamma_{\sigma\tau,\gamma}} \\ P_{\sigma} \otimes_R P_{\tau\gamma} & \xrightarrow[\Gamma_{\sigma,\tau\gamma}]{} & P_{\sigma\tau\gamma} \end{array} \tag{10}$$

need not be commutative but, by Lemma 3.1, there exists a unique element $T^{\Phi}_{\sigma,\tau,\gamma} \in C(R)^*$ such that

$$\Gamma_{\sigma\tau,\gamma}(\Gamma_{\sigma,\tau} \otimes P_{\gamma}) = T^{\Phi}_{\sigma,\tau,\gamma}(\Gamma_{\sigma,\tau\gamma}(P_{\sigma} \otimes \Gamma_{\tau,\gamma})). \tag{11}$$

Clearly $T^{\Phi}_{1,\tau,\gamma} = T^{\Phi}_{\sigma,1,\gamma} = T^{\Phi}_{\sigma,\tau,1} = 1$ so that the choices of P_{σ} and $\Gamma_{\sigma,\tau}$ determine a normalized 3-dimensional cochain of G with coefficients in $C(R)^*$.

Lemma 3.2. *The cochain $T = T^{\Phi} : G^3 \to C(R)^*$ is a 3-cocycle of G with coefficients in the G-module $C(R)^*$.*

Proof. We must prove the identity

$$T_{\sigma,\tau,\gamma} T_{\sigma,\tau\gamma,\delta}\, {}^{\sigma}T_{\tau,\gamma,\delta} = T_{\sigma\tau,\gamma,\delta} T_{\sigma,\tau,\gamma\delta} \tag{12}$$

for any $(\sigma, \tau, \gamma, \delta) \in G^4$. To see this, we compute the isomorphism

$$J = (\Gamma_{\sigma\tau\gamma,\delta})(\Gamma_{\sigma\tau,\gamma} \otimes 1)(\Gamma_{\sigma,\tau} \otimes 1 \otimes 1) : P_{\sigma} \otimes_R P_{\tau} \otimes_R P_{\gamma} \otimes_R P_{\delta} \longrightarrow P_{\sigma\tau\gamma\delta}$$

in two ways. On one hand, for all $x \in P_\sigma$, $y \in P_\tau$, $z \in P_\gamma$ and $t \in P_\delta$, we have

$$\begin{aligned} J(x\otimes y\otimes z\otimes t) &= \Gamma_{\sigma\tau\gamma,\delta}(\Gamma_{\sigma\tau,\gamma}(\Gamma_{\sigma,\tau}(x\otimes y)\otimes z)\otimes t)\\ &\overset{(11)}{=} T_{\sigma,\tau,\gamma}\Gamma_{\sigma\tau\gamma,\delta}(\Gamma_{\sigma,\tau\gamma}(x\otimes\Gamma_{\tau,\gamma}(y\otimes z))\otimes t)\\ &\overset{(11)}{=} T_{\sigma,\tau,\gamma}T_{\sigma,\tau\gamma,\delta}\Gamma_{\sigma,\tau\gamma\delta}(x\otimes\Gamma_{\tau\gamma,\delta}(\Gamma_{\tau,\gamma}(y\otimes z)\otimes t))\\ &\overset{(11)}{=} T_{\sigma,\tau,\gamma}T_{\sigma,\tau\gamma,\delta}\Gamma_{\sigma,\tau\gamma\delta}(x\otimes T_{\tau,\gamma,\delta}\Gamma_{\tau,\gamma\delta}(y\otimes\Gamma_{\gamma,\delta}(z\otimes t))\\ &\overset{(8)}{=} T_{\sigma,\tau,\gamma}T_{\sigma,\tau\gamma,\delta}\ {}^{\sigma}T_{\tau,\gamma,\delta}\Gamma_{\sigma,\tau\gamma\delta}(x\otimes(\Gamma_{\tau,\gamma\delta}(y\otimes\Gamma_{\gamma,\delta}(z\otimes t))), \end{aligned}$$

and on the other hand

$$\begin{aligned} J(x\otimes y\otimes z\otimes t) &= T_{\sigma\tau,\gamma,\delta}\Gamma_{\sigma\tau,\gamma\delta}(\Gamma_{\sigma,\tau}(x\otimes y)\otimes\Gamma_{\gamma,\delta}(z\otimes t))\\ &\overset{(11)}{=} T_{\sigma\tau,\gamma,\delta}T_{\sigma,\tau,\gamma\delta}\Gamma_{\sigma,\tau\gamma\delta}(x\otimes(\Gamma_{\tau,\gamma\delta}(y\otimes\Gamma_{\gamma,\delta}(z\otimes t))) \end{aligned}$$

and comparing the two expressions together with Lemma 3.1 gives (12). □

We now observe the effect of different choices of P_σ and $\Gamma_{\sigma,\tau}$ in the construction of the 3-cocycle T^Φ for a given generalized collective character $\Phi : G \to \mathrm{Pic}_k(R)$.

Lemma 3.3. (i) *If the choice of* Γ *in* (9) *is changed, then* T^Φ *is changed to a cohomologous cocycle. By suitably changing* Γ, T^Φ *may be changed to any cohomologous cocycle.*

(ii) *If the choice of the invertible* $R\otimes_k R^\circ$*-modules* P *is changed, then a suitable new selection of* Γ *leaves cocycle* T^Φ *unaltered.*

Proof. (i) By Lemma 3.1, any other choice of $\Gamma_{\sigma,\tau}$ in (9) has the form $\Gamma'_{\sigma,\tau} = h_{\sigma,\tau}\Gamma_{\sigma,\tau}$, where $h : G^2 \to C(R)^*$ is a normalized 2-cochain of G in $C(R)^*$.

For any $\sigma, \tau, \gamma \in G$ we have the following expressions for the isomorphism $J = \Gamma'_{\sigma\tau,\gamma}(\Gamma'_{\sigma,\tau}\otimes P_\gamma)$ from $P_\sigma\otimes_R P_\tau\otimes_R P_\gamma$ onto $P_{\sigma\tau\gamma}$:

$$\begin{aligned} J(x\otimes y\otimes z) &= \Gamma'_{\sigma\tau,\gamma}(\Gamma'_{\sigma,\tau}(x\otimes y)\otimes z)\\ &\overset{(8)}{=} h_{\sigma\tau,\gamma}\Gamma_{\sigma\tau,\gamma}(h_{\sigma,\tau}\Gamma_{\sigma,\tau}(x\otimes y)\otimes z)\\ &= h_{\sigma\tau,\gamma}h_{\sigma,\tau}\Gamma_{\sigma\tau,\gamma}(\Gamma_{\sigma,\tau}(x\otimes y)\otimes z)\\ &\overset{(11)}{=} h_{\sigma\tau,\gamma}h_{\sigma,\tau}f_{\sigma,\tau,\gamma}\Gamma_{\sigma,\tau\gamma}(x\otimes\Gamma_{\tau,\gamma}(y\otimes z)) \end{aligned}$$

and

$$\begin{aligned} J(x\otimes y\otimes z) &\overset{(11)}{=} T'_{\sigma,\tau,\gamma}\Gamma'_{\sigma,\tau\gamma}(x\otimes\Gamma'_{\tau,\gamma}(y\otimes z))\\ &= T'_{\sigma,\tau,\gamma}h_{\sigma,\tau\gamma}\Gamma_{\sigma,\tau\gamma}(x\otimes h_{\tau,\gamma}\Gamma_{\tau,\gamma}(y\otimes z))\\ &\overset{(11)}{=} T'_{\sigma,\tau,\gamma}h_{\sigma,\tau\gamma}\ {}^{\sigma}h_{\tau,\gamma}\Gamma_{\sigma,\tau\gamma}(x\otimes\Gamma_{\tau,\gamma}(y\otimes z)) \end{aligned}$$

and comparing the two expressions together with Lemma 3.1 yield

$$T'_{\sigma,\tau,\gamma}h_{\sigma,\tau\gamma}\ {}^{\sigma}h_{\tau,\gamma} = h_{\sigma\tau,\gamma}h_{\sigma,\tau}T_{\sigma,\tau,\gamma}, \tag{13}$$

an identity that asserts that the 3-cocycles T and T' are cohomologous.

(ii) If $P'_\sigma \in \Phi(\sigma)$, $\sigma \in G$, is another selection of invertible $R\otimes_k R^\circ$-modules, then we can select $R\otimes_k R^\circ$-isomorphisms $\varphi_\sigma : P'_\sigma \to P_\sigma$ and choose $\Gamma'_{\sigma,\tau} : P'_\sigma\otimes_R P'_\tau \to P'_{\sigma\tau}$, the isomorphism making the following diagram commutative:

$$\begin{array}{ccc} P_\sigma \otimes_R P_\tau \otimes_R P_\gamma & \xrightarrow{\Gamma_{\sigma,\tau}\otimes P_\gamma} & P_{\sigma\tau} \otimes_R P_\gamma \\ {\scriptstyle P_\sigma\otimes\Gamma_{\tau,\gamma}}\downarrow & & \downarrow{\scriptstyle \Gamma_{\sigma\tau,\gamma}} \\ P_\sigma \otimes_R P_{\tau\gamma} & \xrightarrow[\Gamma_{\sigma,\tau\gamma}]{} & P_{\sigma\tau\gamma} \end{array} \tag{14}$$

for each $\sigma, \tau \in G$. Thus we have

$$\begin{aligned} \varphi_{\sigma\tau\gamma}(\Gamma'_{\sigma\tau,\gamma}(\Gamma'_{\sigma,\tau}(x\otimes y)\otimes z)) &= \Gamma_{\sigma\tau,\gamma}(\varphi_{\sigma\tau}(\Gamma'_{\sigma,\tau}(x\otimes y))\otimes\varphi_\gamma(z)) \\ &= \Gamma_{\sigma\tau,\gamma}((\Gamma_{\sigma,\tau}(\varphi_\sigma(x)\otimes\varphi_\tau(y))\otimes\varphi_\gamma(z)) \\ &= T_{\sigma,\tau,\gamma}\Gamma_{\sigma,\tau\gamma}(\varphi_\sigma(x)\otimes\Gamma_{\tau,\gamma}(\varphi_\tau(y))\otimes\varphi_\gamma(z))) \\ &= T_{\sigma,\tau,\gamma}\Gamma_{\sigma,\tau\gamma}(\varphi_\sigma)(x)\otimes\varphi_{\tau\gamma}(\Gamma'_{\tau,\gamma}(y\otimes z)) \\ &= T_{\sigma,\tau,\gamma}\varphi_{\sigma\tau\gamma}(\Gamma'_{\sigma,\tau\gamma}(x\otimes\Gamma'_{\tau,\gamma}(y\otimes z)) \\ &= \varphi_{\sigma\tau\gamma}(T_{\sigma,\tau,\gamma}\Gamma'_{\sigma,\tau\gamma}(x\otimes\Gamma'_{\tau,\gamma}(y\otimes z))), \end{aligned}$$

for all $x \in P'_\sigma$, $y \in P'_\tau$ and $z \in P'_\gamma$.

Hence $\Gamma'_{\sigma\tau,\gamma}(\Gamma'_{\sigma,\tau}(x\otimes y)\otimes z) = T_{\sigma,\tau,\gamma}\Gamma'_{\sigma,\tau\gamma}(x\otimes\Gamma'_{\tau,\gamma}(y\otimes z))$ and the 3-cocycle T is unchanged. □

These lemmas show that each generalized collective character $\Phi : G \to \mathrm{Pic}_k(R)$ determines in invariant fashion a 3-dimensional cohomology class $T(\Phi) = [T^\Phi] \in H^3_\Phi$ $(G, C(R)^*)$. We refer to the map $\Phi \mapsto T(\Phi)$ as the *Teichmüller obstruction map* (see [8] for background).

Next we prove the main objective of this paper.

Theorem 3.4. *A generalized collective character $\Phi : G \to \mathrm{Pic}_k(R)$ is realizable if and only if its Teichmüller obstruction $T(\Phi) \in H^3_\Phi(G, C(R)^*)$ vanishes.*

Proof. Suppose first that $(S = \bigoplus_{\sigma\in G} S_\sigma,\ j)$ is a realization of Φ. Then, in the construction of the Teichmüller 3-cocycle T^Φ of G with coefficients in the G-module $C(R)^*$, one can take just the invertible $R\otimes_k R^\circ$-modules S_σ, $\sigma \in G$, $\sigma \neq 1$, and the canonical $R \otimes_k R^\circ$-isomorphisms $\Gamma_{\sigma,\tau} : S_\sigma \otimes_R S_\tau \to S_{\sigma\tau}$, $\Gamma_{\sigma,\tau}(x\otimes y) = xy$, $\Gamma_{\sigma,1}(x\otimes r) = xj(r)$ and $\Gamma_{1,\sigma}$ $(r\otimes x) = j(r)x$ for each $\sigma, \tau \in G$. Since multiplication in the k-algebra S is associative $\Gamma_{\sigma\tau,\gamma}(\Gamma_{\sigma,\tau}\otimes S_\gamma) = \Gamma_{\sigma,\tau\gamma}(S_\sigma\otimes\Gamma_{\tau,\gamma})$ for all $\sigma, \tau, \gamma \in G$, and then $T^\Phi_{\sigma,\tau,\gamma} = 1$ in (11). Therefore, $T(\Phi) = [T^\Phi]$ is the zero cohomology class.

Conversely, suppose that the generalized collective character Φ has a vanishing cohomology class $f(\Phi)$. Select any invertible $R\otimes_k R^\circ$-modules $P_\sigma \in \Phi(\sigma)$, $\sigma \in G$, with $P_1 = R$. By Lemma 3.3(i), there is a choice of $R\otimes_k R^\circ$-isomorphisms $\Gamma_{\sigma,\tau} : P_\sigma \otimes_R P_\tau \to P_{\sigma\tau}$ with $\Gamma_{1,\sigma}$ and $\Gamma_{\sigma,1}$ the canonical ones, such that the Teichmüller 3-cocycle T^Φ is identically 1. This means that (10) is commutative for any $\sigma, \tau, \gamma \in G$. Hence, the family $(P_\sigma, \Gamma_{\sigma,\tau})$ gives rise to a *generalized crossed product algebra* in the sense of Kanzaki [9] $\Delta = \bigoplus_{\sigma\in G} P_\sigma$, where the product of elements $x \in P_\sigma$ and $y \in P_\tau$ is defined by $xy = \Gamma_{\sigma,\tau}(x\otimes y)$, which is a G-graded Clifford system over k, extension of R by the canonical injection $j : R = P_1 \hookrightarrow \Delta$. Since $\chi_{[\Delta,\ j]}(\sigma) = [P_\sigma] = \Phi(\sigma)$, Φ is realized, that is, $\mathrm{Cliff}_k(G, R; \Phi) \neq \emptyset$. □

Now, to complete the classification of G-graded Clifford system extensions of a k-algebra R, we have the following result.

Theorem 3.5. *If a generalized collective character $\Phi : G \to \mathrm{Pic}_k(R)$ is realizable, then the set of isomorphism classes of realizations of Φ, $\mathrm{Cliff}_k(G, R; \Phi)$, is a principal homogeneous space under the abelian group $H^2_\Phi(G, C(R)^*)$. In particular, there is a (non-canonical) bijection*

$$\mathrm{Cliff}_k(G, R; \Phi) \cong H^2_\Phi(G, C(R)^*).$$

Proof. We will describe an action

$$H^2_\Phi(G, C(R)^*) \times \mathrm{Cliff}_k(G, R; \Phi) \longrightarrow \mathrm{Cliff}_k(G, R; \Phi) \qquad (15)$$

below.

Let $h : G^2 \to C(R)^*$ be a normalized 2-cocycle representative of an element $[h] \in H^2_\Phi$ $(G, C(R)^*)$ and $(S = \bigoplus_{\sigma\in G} S_\sigma,\ j : R \cong S_1)$ be a G-graded Clifford system extension of R, representative of an element $[S,\ j] \in \mathrm{Cliff}_k(G, R; \Phi)$. A new G-graded Clifford system extension of R, $({}^hS,\ j)$ is defined by considering the k-algebra ${}^hS$ which is the same G-graded k-algebra as $S = \bigoplus_{\sigma\in G} S_\sigma$, where the product of elements $x \in S_\sigma$ and $y \in S_\tau$ is now defined by

$$x \star y = j(h_{\sigma,\tau})xy.$$

Since for any $x \in S_\sigma$, $y \in S_\tau$ and $z \in S_\gamma$ we have

$$\begin{aligned} x \star (y \star z) &= x \star (j(h_{\sigma,\tau})yz) = j(h_{\sigma,\tau\gamma})xj(h_{\tau,\gamma})yz \\ &\overset{(8)}{=} j(h_{\sigma,\tau\gamma})j({}^\sigma h_{\tau,\gamma})xyz = j(h_{\sigma\tau,\gamma})j(h_{\sigma,\tau})xyz \\ &= j(h_{\sigma\tau,\gamma})(x \star y)z = (x \star y) \star z, \end{aligned}$$

the multiplication is associative and so ${}^hS$ is a k-algebra. Furthermore, $S_\sigma \star S_\tau = j(h_{\sigma,\tau})S_\sigma S_\tau = j(h_{\sigma,\tau})S_{\sigma\tau} = S_{\sigma\tau}$, since $h_{\sigma,\tau}$ is invertible for all $\sigma, \tau \in G$.

Therefore $({}^hS,\ j)$ is actually a G-graded Clifford system over k extension of R, clearly representing an element $[{}^hS,\ j] \in \mathrm{Cliff}_k(G, R; \Phi)$, which we maintain depends only on $[h]$ and $[S,\ j]$. To see this, let us suppose that h' is another representative of $[h]$ and $(S',\ j')$ is another representative of $[S,\ j]$. Then, there must exist a 1-cochain $\psi : G \to C(R)^*$ such that $h'_{\sigma,\tau}\,{}^\sigma\psi_\tau\psi_\sigma = \psi_{\sigma\tau}h_{\sigma,\tau}$, $\sigma, \tau \in G$, and a grade-preserving isomorphism $f : S \to S'$ such that $fj = j'$, from which we build the grade-preserving k-isomorphism ${}^\psi f : {}^hS \to {}^{h'}S'$, ${}^\psi f(x) = f(j(\psi_\sigma)x)$ if $x \in S_\sigma$. For each $x \in S_\sigma$ and $y \in S_\tau$, we have

$$\begin{aligned} {}^\psi f(x \star y) &= f(j(\psi_{\sigma\tau})j(h_{\sigma,\tau})xy) = f(j(h'_{\sigma,\tau})j(\psi_\sigma)j({}^\sigma\psi_\tau)xy) \\ &\overset{(8)}{=} f(j(h'_{\sigma,\tau})j(\psi_\sigma)xj(\psi_\tau)y)) = j'(h'_{\sigma,\tau})\,{}^\psi f(x)\,{}^\psi f(y) \\ &= {}^\psi f(x) \star {}^\psi f(y), \end{aligned}$$

so that ${}^\psi f : ({}^hS,\ j) \to ({}^{h'}S',\ j')$ is actually an isomorphism of Clifford system extensions of R, that is, $[{}^hS,\ j] = [{}^{h'}S',\ j']$.

Therefore, $([h], [S,\ j]) \mapsto [{}^hS,\ j]$ is a well-defined action of the abelian group H^2_Φ $(G, C(R)^*)$ on $\mathrm{Cliff}_k(G, R; \Phi)$, which furthermore is a principal one. In fact, if we suppose that $[{}^hS,\ j] = [S,\ j]$, there must exist a grade preserving k-algebra isomorphism $f : {}^hS \to S$ such that $fj = j'$. For each $\sigma \in G$, the restriction $f_{/S_\sigma} : S_\sigma \to S_\sigma$ is a $R \otimes_k R^\circ$-isomorphism, and, by Lemma 3.1, there exists a unique $\psi_\sigma \in C(R)^*$ such that $f(x) = j(\psi_\sigma)x$ for all $x \in S_\sigma$. Thus $\psi : G \to C(R)^*$ is a 1-cochain. Since $f(x \star y) = f(x)f(y)$, for any $x \in S_\sigma$, $y \in S_\tau$, $\sigma, \tau \in G$, we have $j(\psi_{\sigma\tau}h_{\sigma,\tau})xy = j(\psi_\sigma)xj(\psi_\tau)y \overset{(8)}{=} j(\psi_\sigma\,{}^\sigma\psi_\tau)xy$. Therefore, since $S_\sigma S_\tau = S_{\sigma\tau}$, Lemma 3.1 implies that $\psi_{\sigma\tau}h_{\sigma,\tau} = \psi_\sigma\,{}^\sigma\psi_\tau$, that is, $h = \partial(\psi)$ represents the zero class in $H^2_\Phi(G, C(R)^*)$.

Finally, we observe that action (15) is transitive. Let (S, j),(S', j') be any two G-graded Clifford system extensions of R representing elements in $\mathrm{Cliff}_k(G, R; \Phi)$. Since S_σ,$S'_\sigma \in \Phi(x)$ for any $\sigma \in G$, there must exist $R \otimes_k R^\circ$-isomorphisms $f_\sigma : S_\sigma \to S'_\sigma$, $\sigma \in G$, with $f_1 = j' j^{-1}$. For each pair $\sigma, \tau \in G$, the square

$$\begin{array}{ccc} S_\sigma \otimes_R S_\tau & \longrightarrow & S_{\sigma\tau} \\ {\scriptstyle f_\sigma \otimes f_\tau}\downarrow & & \downarrow{\scriptstyle f_{\sigma\tau}} \\ S'_\sigma \otimes_R S'_\tau & \longrightarrow & S'_{\sigma\tau} \end{array} ,$$

where the horizontal arrows represent the canonical isomorphisms $x \otimes y \mapsto xy$, need not be commutative. But, by Lemma 3.1, there exists a unique $h_{\sigma,\tau} \in C(R)^*$ such that $f_{\sigma\tau}(xy) = j'(h_{\sigma,\tau}) f_\sigma(x) f_\tau(y)$ for all $x \in S_\sigma$, $y \in S_\tau$. Thus $h : G \to C(R)^*$ is a normalized 2-cochain. For any $x \in S_\sigma$, $y \in S_\tau$ and $z \in S_\gamma$, we have

$$f_{\sigma\tau\gamma}(xyz) = j'(h_{\sigma\tau,\gamma}) f_{\sigma\tau}(xy) f_\gamma(z) = j'(h_{\sigma\tau,\gamma} h_{\sigma,\tau}) f_\sigma(x) f_\tau(y) f_\gamma(z),$$

and analogously,

$$\begin{aligned} f_{\sigma\tau\gamma}(xyz) &= j'(h_{\sigma,\tau\gamma}) f_\sigma(x) f_{\tau\gamma}(z) \\ &= j'(h_{\sigma,\tau\gamma}) f_\sigma(x) j'(h_{\tau,\gamma}) f_\tau(y) f_\gamma(z) \\ &\overset{(8)}{=} j'(h_{\sigma,\tau\gamma} {}^{\sigma}h_{\tau,\gamma}) f_\sigma(x) f_\tau(y) f_\gamma(z). \end{aligned}$$

Lemma 3.1 implies that $h_{\sigma\tau,\gamma} h_{\sigma,\tau} = h_{\sigma,\tau\gamma} {}^{\sigma}h_{\tau,\gamma}$, that is, h is a 2-cocycle of G on $C(R)^*$. Clearly $f = \bigoplus_{\sigma \in G} f_\sigma$ establishes a G-graded Clifford system extension isomorphism $(S, j) \to ({}^h S', j')$, and so action (15) is transitive. □

To end this section, we shall focus on that class of rings known as crossed-product group k-algebras. According to ([4], §5) an *extension of a k-algebra R by a group G* in the sense of Hacque [8], is the same as G-graded Clifford system extension of R satisfying the condition that in any component there is at least one unit. As in Hacque's paper [8], let $\mathrm{Ext}_k(G, R)$ denote the set of isomorphism classes of extensions of a k-algebra R by a group G. Then $\mathrm{Ext}_k(G, R) \subseteq \mathrm{Cliff}_k(G, R)$, and we shall characterize this subset of $\mathrm{Cliff}_k(G, R)$ by means of collective characters as in the following proposition, where we take into account the canonical group embedding $\mathrm{Out}_k(R) \overset{\delta}{\hookrightarrow} \mathrm{Pic}_k(R)$ induced by the group exact sequence (4), whose image is ([1], Chap. II, (5.3))

$$\mathrm{Img}(\delta) = \{[P] \in \mathrm{Pic}_k(R) | P \cong R \text{ as left } R\text{-module}\}.$$

PROPOSITION 3.6

For any k-algebra R and group G there is a cartesian square

$$\begin{array}{ccc} \mathrm{Ext}_k(G,R) & \rightarrowtail & \mathrm{Cliff}_k(G,R) \\ \downarrow & & \downarrow{\scriptstyle \chi} \\ \mathrm{Hom}_{Gp}(G,\mathrm{Out}_k(R)) & \overset{\delta_*}{\rightarrowtail} & \mathrm{Hom}_{Gp}(G,\mathrm{Pic}_k(R)) \end{array} \qquad (16)$$

that is, $\mathrm{Ext}_k(G, R) = \chi^{-1}(\mathrm{Hom}_{Gp})(G, \mathrm{Out}_k(R))$ *is the set of classes of those G-graded Clifford system extensions of R which realize collective characters* (*in the sense of* [7, 8]).

Proof. Let $(S = \bigoplus_{\sigma \in G} S_\sigma,\ j)$ be a G-graded Clifford system extension of R such that for any $\sigma \in G$, there exists $u_\sigma \in S^* \cap S_\sigma$, that is, an extension of R by G. Right multiplication by u_σ is an isomorphism of left R-modules $R \xrightarrow{\sim} Ru_\sigma = S_\sigma$, $\sigma \in G$, and therefore the generalized collective character realized by $[S,\ j]$, $\chi_{[S,\ j]} : G \to \mathrm{Pic}_k(R)$, $\chi_{[S,\ j]}(\sigma) = [S_\sigma]$ factors through $\mathrm{Out}_k(R)$. Conversely, suppose $(S = \bigoplus_{\sigma \in G} S_\sigma,\ j)$ is a G-graded Clifford system extension of R such that $\chi_{[S,\ j]} = \delta\Phi$ for some $\Phi : G \to \mathrm{Out}_k(R)$. Then, if we choose any k-automorphism $f(\sigma) \in \Phi(\sigma)$ for each $\sigma \in G$, there must exist an $R \otimes_k R^\circ$-isomorphism $\varphi_\sigma : R_{f(\sigma)} \cong S_\sigma$. If $u_\sigma = \varphi_\sigma(1)$, then $S_\sigma = Ru_\sigma = u_\sigma R$. From $S_\sigma S_{\sigma^{-1}} = R1_S = S_{\sigma^{-1}} S_\sigma$, it follows that $u_\sigma R u_{\sigma^{-1}} = R1_S = u_{\sigma^{-1}} R u_\sigma$. Then there exist $a, b \in R$ such that $1 = u_\sigma a u_{\sigma^{-1}} = u_{\sigma^{-1}} b u_\sigma$ so that $u_\sigma \in S^* \cap S_\sigma$ and therefore $(S,\ j)$ represents an extension of R by G. □

From the general results about Clifford system extensions of algebras, we deduce the following group cohomology classification of extensions of an algebra by a group, which was proved by Hacque in [8].

COROLLARY 3.7

Let G be a group and R be a k-algebra.

(i) *Each collective character of G in R,* $\Phi : G \to \mathrm{Out}_k(R)$ *determines in an invariant fashion a three-dimensional cohomology class* $T(\Phi) \in H^3_\Phi(G, C(R)^*)$ *of G with coefficients in the G-module* (*via* Φ) *of all units in the center of R.*

(ii) *There is a canonical partition of the set of equivalence classes of extensions of R by G,*

$$\mathrm{Ext}_k(G, R) = \coprod_{\Phi} \mathrm{Ext}_k(G, R; \Phi),$$

where, for any collective character $\Phi : G \to \mathrm{Out}_k(R)$, $\mathrm{Ext}_k(G, R; \Phi)$ *is the set of equivalence classes of those extensions realizing* Φ.

(iii) *A collective character* $\Phi : G \to \mathrm{Out}_k(R)$ *is realizable, that is,* $\mathrm{Ext}_k(G, R; \Phi) \neq \emptyset$ *if and only if its obstruction vanishes.*

(iv) *If the obstruction of a collective character* $\Phi : G \to \mathrm{Out}_k(R)$ *vanishes, then* $\mathrm{Ext}_k(G, R; \Phi)$ *is a principal homogeneous space under* $H^2_\Phi(G, C(R)^*)$. *In particular, there is a bijection*

$$\mathrm{Ext}_k(G, R; \Phi) \cong H^2(G, C(R)^*). \tag{17}$$

4. An obstructed collective character

It is very easy to find unobstructed generalized collective characters. Of course any Clifford system yields one of them. In this example we shall exhibit a non-realizable collective character, that is, a group homomorphism $\Phi : G \to \mathrm{Pic}_k(R)$, for particular group G and k-algebra R, such that there is no G-graded Clifford system extension of R, $(S = \bigoplus_{\sigma \in G} S_\sigma,\ j : R \cong S_1)$ such that $\Phi(\sigma) = [S_\sigma]$, $\sigma \in G$.

For example consider $G = C_2 = \langle t; t^2 = 1 \rangle$, the cyclic group of order two, $k = \mathbb{F}_5$, the Galois field with five elements and $R = \mathbb{F}_5[D_{10}]$, the group $\mathbb{F}_5$-algebra of the dihedral group $D_{10} = \langle r, s;\ r^{10} = 1 = s^2,\ srs = r^{-1} \rangle$.

Let $\beta : \mathbb{F}_5[D_{10}] \xrightarrow{\sim} \mathbb{F}_5[D_{10}]$ be the algebra automorphism defined by $\beta(r) = r^7$ and $\beta(s) = r^5 s$. Since $\beta^2(r) = r^{-1} = srs$ and $\beta^2(s) = sr^5(r^5)^7 = s$, the automorphism β^2

is simply conjugation by s. Therefore, the equations $\Phi(1) = 1$ and $\Phi(t) = [\beta]$ determine a homomorphism

$$\Phi : C_2 \longrightarrow \text{Out}\,(\mathbb{F}_5[D_{10}]) \overset{\delta}{\subseteq} \text{Pic}\,(\mathbb{F}_5[D_{10}]) \tag{18}$$

of the cyclic group C_2 into the Picard group of $\mathbb{F}_5[D_{10}]$, that is, a collective character of C_2 in $\mathbb{F}_5[D_{10}]$.

PROPOSITION 4.1

The Teichmüller obstruction $T(\Phi) \in H^3_\Phi(C_2, C(\mathbb{F}_5[D_{10}])^*)$ is non-zero.

Proof. First, let us observe that in this case, a (normalized) n-cochain $h : C_2 \times \cdots \times C_2 \longrightarrow C(\mathbb{F}_5[D_{10}])^*$ is determined by a single constant $h(t, \ldots, t) = h \in C(\mathbb{F}_5[D_{10}])^*$, whose coboundary is given by $\delta h = \beta(h)h^{-1}$ if n is even or $\delta h = \beta(h)h$ if n is odd. Since we easily see that the Teichmüller 3-cocycle is $T^\Phi = r^5$, the proof of the proposition amounts to checking that there is no unit h in the center of $\mathbb{F}_5[D_{10}]$ such that $\beta(h) = r^5 h$.

The center of $\mathbb{F}_5[D_{10}]$ can be described as the 8-dimensional space over $\mathbb{F}_5$ generated by the elements

$$c_1 = 1, \quad c_2 = r + r^9, \quad c_3 = r^2 + r^8, \quad c_4 = r^3 + r^7, \quad c_5 = r^4 + r^6,$$
$$c_6 = r^5, \quad c_7 = (c_1 + c_3 + c_5)s, \quad c_8 = (c_2 + c_4 + c_6)s,$$

with multiplication given by

$c_2^2 = c_3 + 2$	$c_2c_3 = c_2 + c_4$	$c_2c_4 = c_3 + c_5$	$c_2c_5 = c_4 + 2c_6$	$c_2c_6 = c_5$
$c_2c_7 = 2c_8$	$c_2c_8 = 2c_7$	$c_3^2 = c_5 + 2$	$c_3c_4 = c_2 + 2c_6$	$c_3c_5 = c_3 + c_5$
$c_3c_6 = c_4$	$c_3c_7 = 2c_7$	$c_3c_8 = 2c_8$	$c_4^2 = c_5 + 2$	$c_4c_5 = c_2 + c_4$
$c_4c_6 = c_3$	$c_4c_7 = 2c_8$	$c_4c_8 = 2c_7$	$c_5^2 = c_3 + 2$	$c_5c_6 = c_2$
$c_5c_7 = 2c_7$	$c_5c_8 = 2c_8$	$c_6^2 = 1$	$c_6c_7 = c_8$	$c_6c_8 = c_7$
$c_7^2 = 0$	$c_7c_8 = 0$	$c_8^2 = 0.$		

Let C_0 be the $\mathbb{F}_5$-subalgebra generated by c_2; that is, the span of $c_1, \ldots, c_6$ and note that the minimal polynomial of c_2 is $(t+2)^3(t-2)^3$. Then $C(\mathbb{F}_5[D_{10}]) = C_0 \oplus \mathbb{F}_5 c_7 \oplus \mathbb{F}_5 c_8$ with multiplication given by $c_7^2 = c_7c_8 = c_8^2 = 0$ and $c_2c_7 = 2c_8$, $c_2c_8 = 2c_7$ and we see that there is a homomorphism $\varphi : C(\mathbb{F}_5[D_{10}]) \to \mathbb{F}_5$ mapping c_2 to -2 and c_7, c_8 to 0. Hence, $\varphi(c_3) = 2$, $\varphi(c_4) = -2$, $\varphi(c_5) = 2$ and $\varphi(c_6) = -1$. Since $\beta(c_2) = c_4$, $\beta(c_7) = c_8$ and $\beta(c_8) = c_7$, this homomorphism φ satisfies that $\varphi(\beta(h)) = \beta(h)$ for all $h \in C\,(\mathbb{F}_5[D_{10}])$. Therefore, if $h \in C(\mathbb{F}_5[D_{10}])$ is such that $\beta(h) = r^5 h$, then comparing the image of each side under φ yields $\varphi(h) = -\varphi(h)$, whence $\varphi(h) = 0$, and it follows that h is not invertible. □

Remark 4.2. Proposition 4.1 is an effect of inseparability. If we consider the Galois field $\mathbb{F}_3$ instead of $\mathbb{F}_5$, the resulting collective character (18), $\Phi : C_2 \to \text{Pic}(\mathbb{F}_3[D_{10}])$ defined similarly by $\Phi(t) = [\beta]$, where β is the corresponding algebra automorphism determined by $\beta(r) = r^7$ and $\beta(s) = r^5 s$, is unobstructed. In this case the Teichmüller cocycle $T^\Phi = r^5 = \delta(h)$, is the coboundary of the order 4 element $h = c_4 + c_5 + c_7 + 2c_8 \in C\,(\mathbb{F}_3[D_{10}])^*$, and therefore $T(\Phi) = 0$.

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