Proc. Indian Acad. Sci. (Math. Sci.), Vol. 111, No. 1, February 2001, pp. 65–94.

Topological *-algebras with C^*-enveloping algebras II

S J BHATT

Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar 388 120, India
E-mail: sjb@spu.ernet.in

MS received 17 October 1997; revised 14 March 2000

Abstract. Universal C^*-algebras $C^*(A)$ exist for certain topological *-algebras called algebras with a C^*-enveloping algebra. A Frechet *-algebra A has a C^*-enveloping algebra if and only if every operator representation of A maps A into bounded operators. This is proved by showing that every unbounded operator representation π, continuous in the uniform topology, of a topological *-algebra A, which is an inverse limit of Banach *-algebras, is a direct sum of bounded operator representations, thereby factoring through the enveloping pro-C^*-algebra $E(A)$ of A. Given a C^*-dynamical system (G, A, α), any topological *-algebra B containing $C_c(G, A)$ as a dense *-subalgebra and contained in the crossed product C^*-algebra $C^*(G, A, \alpha)$ satisfies $E(B) = C^*(G, A, \alpha)$. If $G = \mathbb{R}$, if B is an α-invariant dense Frechet *-subalgebra of A such that $E(B) = A$, and if the action α on B is m-tempered, smooth and by continuous *-automorphisms: then the smooth Schwartz crossed product $S(\mathbb{R}, B, \alpha)$ satisfies $E(S(\mathbb{R}, B, \alpha)) = C^*(\mathbb{R}, A, \alpha)$. When G is a Lie group, the C^∞-elements $C^\infty(A)$, the analytic elements $C^\omega(A)$ as well as the entire analytic elements $C^{e\omega}(A)$ carry natural topologies making them algebras with a C^*-enveloping algebra. Given a non-unital C^*-algebra A, an inductive system of ideals I_α is constructed satisfying $A = C^*\text{-ind}\lim I_\alpha$; and the locally convex inductive limit ind $\lim I_\alpha$ is an m-convex algebra with the C^*-enveloping algebra A and containing the Pedersen ideal K_A of A. Given generators G with weakly Banach admissible relations R, we construct universal topological *-algebra $A(G, R)$ and show that it has a C^*-enveloping algebra if and only if (G, R) is C^*-admissible.

Keywords. Frechet *-algebra; topological *-algebra; C^*-enveloping algebra; unbounded operator representation; O^*-algebra; smooth Frechet algebra crossed product; Pedersen ideal of a C^*-algebra; groupoid C^*-algebra; universal algebra on generators with relations.

1. Statements of the results

In [5], a functor E has been considered that associates C^*-algebras $E(A)$ with certain topological *-algebras A, called algebras with a C^*-enveloping algebra. By a classic construction due to Gelfand and Naimark, a Banach *-algebra A admits a C^*-enveloping algebra $C^*(A) = E(A)$ ([14], 2.7, p. 47). By ([15], Theorem 2.1), a complete locally m-convex *-algebra has a C^*-enveloping algebra if and only if it admits a greatest continuous C^*-seminorm. The following extrinsic characterization of such algebras has been motivated by the simple observation that any *-homomorphism from a Banach *-algebra into the *-algebra of linear operators on an inner product space maps the algebra into bounded operators.

Theorem 1.1. *Let A be a Frechet *-algebra. Then A is an algebra with a C^*-enveloping algebra if and only if every *-representation of A is a bounded operator representation.*

The above theorem is false without the assumption that A is metrizable (see Remark 4.4). By a *-*representation* $(\pi, \mathcal{D}(\pi), H)$ of a *-algebra A [37] is meant a homomorphism π from A into linear operators (not necessarily bounded) all defined on a common dense invariant subspace $\mathcal{D}(\pi)$ of a Hilbert space H such that for all x in A, $\pi(x^*) \subset \pi(x)^*$. In the general theory of *-algebras, following Palmer [24], A is called a BG^*-*algebra* if every *-homomorphism from A into linear operators on a pre-Hilbert space maps A into bounded operators. The absence of a complete algebra norm on a non-Banach *-algebra A indicates that A may contain elements that fail to be bounded in any natural sense. Hence an appropriate framework for the representation theory of A is that of unbounded operator representations. However, this natural point of view was developed rather late, following [30, 20]. Prior to (and later, in spite of) this, bounded operator representations of A have been investigated in detail, especially when A is a locally m-convex *-algebra, i.e., $A = \text{proj lim} A_\alpha$, the inverse limit (also called the projective limit) of Banach *-algebras [9, 15], (see [16] for a summary of bounded operator representations of A). In fact, such an A, when *-semisimple, admits sufficiently many continuous irreducible bounded operator representations [9]. Then the enveloping pro-C^*-algebra (projective limit of C^*-algebras) $E(A)$ of A, discussed in [10], [19] and [15], turns out to be $E(A) = \text{proj lim } E(A_\alpha)$, $E(A_\alpha) = C^*(A_\alpha)$ being the enveloping C^*-algebra of the Banach *-algebra A_α ([15], Theorem 4.3). Thus A has a C^*-*enveloping algebra* if $E(A)$ is a C^*-algebra. By the construction, $E(A)$ is universal for norm-continuous bounded operator representations of A. Theorem 1.2, to be used to prove Theorem 1.1, shows desirably that $E(A)$ is also universal for representations into unbounded operators. The uniform topology ([37], p. 77, 78) on an unbounded operator algebra is defined at the end of this section.

Theorem 1.2. *Let A be complete locally m-convex *-algebra. Let $(\pi, D(\pi), H)$ be a closed *-representation of A continuous in the uniform topology on $\pi(A)$. Then there exists a unique *-representation $(\sigma, D(\sigma), H_\sigma)$ of $E(A)$ such that the following hold.*

(1) *$H_\sigma = H$ and $D(\sigma) = D(\pi)$.*
(2) *As a representation of $E(A)$, σ is closed and continuous in the uniform topology on $\sigma(E(A))$.*
(3) *σ is an 'extension' of π to $E(A)$ in the sense that for all x in A, $(\sigma \circ j)(x) = \pi(x)$, $j : A \to E(A)$ being the natural map, $j(x) = x + \text{srad}\,(A)$, $\text{srad}\,(A)$ denoting the star radical of A.*
(4) *On the unbounded operator algebra $\pi(A)$, the uniform topology $\tau_D^{\pi(A)}$ is a (not necessarily complete) pro-C^*-topology which coincides with the relative uniform topology $\tau_D^{\sigma(E(A))}$ from $\sigma(E(A))$.*

COROLLARY 1.3

Let π be a closed irreducible *-representation of a complete locally m-convex *-algebra A continuous in the uniform topology on $\pi(A)$. (In particular, let A be Frechet and π be irreducible). Then π maps A into bounded operators.

AO^*-algebras (abstract O^*-algebras) [36, 37] provide the unbounded operator algebra analogues of C^*-algebras. Starting with a topological (not necessarily m-convex)

*-algebra A, one can construct an enveloping AO^*-algebra $O(A)$ universal for *-representations continuous in the uniform topology, and declare A to have a C^*-*enveloping algebra* if the uniform topology on $O(A)$ is normable. On the other hand, by modifying the construction in [15], the pro-C^*-algebra $E(A)$ can also be considered as the universal object for norm-continuous bounded operator *-representations of more general locally convex, non-m-convex, *-algebras A. In general, the completion of $O(A)$ differs from $E(A)$. For a barrelled A, $O(A)$ is normable implies that $E(A)$ is a C^*-algebra, but the converse does not hold. In the present context, the following shows that both the approaches are consistent in the metrizable case.

Theorem 1.4. *Let A be a Frechet *-algebra. Then the pro-C^*-algebra $E(A)$ is the completion of the AO^*-algebra $O(A)$. Thus $O(A)$ is normable if and only if A is an algebra with a C^*-enveloping algebra.*

There are several situations in C^*-algebra theory in which topological *-algebras arise naturally [27]. Enveloping C^*-algebras provide a standard method of constructing C^*-algebras; and frequently, lurking behind such a construction is a topological *-algebra B such that $E(B) = A$. Let α be a strongly continuous action of a locally compact group G by *-automorphisms of a C^*-algebra A. The crossed product C^*-algebra $C^*(G, A, \alpha)$ is the enveloping C^*-algebra of the L^1-crossed product Banach *-algebra $L^1(G, A, \alpha)$. If B is a topological *-algebra such that $C_c(G, A) \subseteq B \subseteq C^*(G, A, \alpha)$ and $C_c(G, A)$ is dense in B, then $E(B) = C^*(G, A, \alpha)$. Let G be a Lie group. Then the *-subalgebra $C^\infty(A)$ of C^∞-elements of A is a Frechet *-algebra with an appropriate topology such that $E(C^\infty(A)) = A$. The *-algebras $C^\omega(A)$ and $C^{e\omega}(A)$ consisting of analytic elements and entire elements of A are shown to carry natural topologies making them algebras with C^*-enveloping algebras. We also consider the smooth crossed product [29, 34]. For simplicity, we take $G = \mathbb{R}$, and prove the following.

Theorem 1.5. *Let α be a strongly continuous action of $\mathbb{R}$ by *-automorphisms of a C^*-algebra A. Suppose that B is a dense Frechet *-subalgebra of A satisfying the following.*

(a) *A has a bounded approximate identity contained in B and which is a bounded approximate identity for B.*
(b) $E(B) = A$.
(c) *B is α-invariant; and the action α of $\mathbb{R}$ on B is smooth, m-tempered and by continuous *-automorphisms of B.*

Then the smooth Schwartz crossed product $S(\mathbb{R}, B, \alpha)$ is a Frechet *-algebra with a C^*-enveloping algebra, and $E(S(\mathbb{R}, B, \alpha)) = C^*(\mathbb{R}, A, \alpha)$. Further, if the action of $\mathbb{R}$ on B is isometric (see § 5), then the L^1-crossed product $L^1(\mathbb{R}, B, \alpha)$ is also a Frechet *-algebra with a C^*-enveloping algebra, and $E(L^1(\mathbb{R}, B, \alpha)) = C^*(\mathbb{R}, A, \alpha)$.

It follows that $E(S(\mathbb{R}, C^\infty(A), \alpha) = C^*(\mathbb{R}, A, \alpha)$. In particular, if α is a smooth action of $\mathbb{R}$ on a C^∞-manifold M, then $E(S(\mathbb{R}, C^\infty(M), \alpha) = C^*(\mathbb{R}, C(M), \alpha)$, the covariance C^*-algebra of the $\mathbb{R}$-space M.

For a locally compact Hausdorff space X, let $\mathcal{K}$ be the directed set consisting of all compact subsets of X. For $K \in \mathcal{K}$, let $C_K(X) = \{f \in C_c(X) : \text{supp} f \subseteq K\}$, $C_c(X)$ denoting the compactly supported continuous functions on X. It is well known that $\{C_K(X) : K \in \mathcal{K}\}$ forms an inductive system; and $C_0(X) = C^*\text{-ind} \lim C_K(X)$ (C^*-inductive limit), $C_c(X) = \text{ind} \lim C_K(X)$ (locally convex inductive limit). Further, $C_c(X)$ with the locally convex inductive limit topology is a complete locally m-convex Q-algebra and $E(C_c(X)) = C_0(X)$. The following provides a non-commutative analogue of this. We refer to the last paragrapgh in this section for the relevant definitions pertaining to topological algebras.

Theorem 1.6. *Let A be a non-unital* C^**-algebra. Let* K_A *denote its Pedersen ideal. For* $a \in K_A^+$*, let* I_a *denote the closed two sided ideal of A generated by* aa^**. Let* $K_A^{nc} = \mathrm{U}\{I_a : a \in K_A^+\}$*. Then the following hold.*

(1) $\{I_a : a \in K_A^+\}$ *forms an inductive system,* $A = C^* - \text{ind lim}\{I_a : a \in K_A^+\}$, and $K_A^{nc} = \text{ind lim}\{I_a : a \in K_A^+\}$.
(2) K_A^{nc} *with the locally convex inductive limit topology t is a locally m-convex Q-algebra satisfying* $E(K_A^{nc}) = E(K_A) = A$.
(3) *If A has a countable bounded approximate identity, then* (K_A^{nc}, t) *is an LFQ-algebra.*

In general $K_A \neq K_A^{nc}$, though $K_A \subseteq K_A^{nc}$. Now K_A has been interpreted as a non-commutative analogue of $C_c(X)$. Then K_A^{nc} may be interpreted as continuous functions on a non-commutative space vanishing at infinity in 'commutative directions' and having compact supports in 'non-commutative directions'. This interpretation is suggested by the remarks preceeding ([28], Theorem 8).

The universal C^*-algebra $C^*(G,R)$ on a C^*-admissible set of generators G with relations R provides another method of constructing C^*-algebras. Motivated by some problems in C^*-algebras, Phillips introduced more general weakly C^*-admissible generators with relations (G,R) leading to the construction of the universal pro-C^*-algebra $C^*(G,R)$ on (G,R) [27]. In § 8, we construct a universal topological *-algebra $A(G,R)$ on (G,R) with weakly Banach admissible relations R, and prove the following.

Theorem 1.7. *Let* (G,R) *be weakly Banach admissible.*

(1) $E(A(G,R)) = C^*(G,R)$.
(2) $A(G,R)$ *has a* C^**-enveloping algebra if and only if* (G,R) *is* C^**-admissible.*

The paper is organized as follows. Proofs of Theorems 1.1, 1.2 and 1.4 are presented in § 3. The preliminary lemmas and constructions in the locally convex, non-m-convex set up more general than in [5], are discussed in § 2. Section 4 contains a couple of remarks including some corrections in [5]. The smooth crossed product is discussed in § 5 culminating in the proof of Theorem 1.5. Section 6 contains the proof of Theorem 1.6. This is followed by a brief discussion on the C^*-algebra of a groupoid in § 7. Universal C^*-algebras on generators with relations are discussed in § 8. In what follows, we briefly recall the relevant ideas in unbounded operator representations.

For the basic theory of unbounded operator *-representations $(\pi, \mathcal{D}(\pi), H)$ of a *-algebra A, we refer to [37, 30]. Let A^1 denote the unitization of A. The graph topology $t_\pi = t_{\pi(A^1)}$ on $\mathcal{D}(\pi)$ is defined by seminorms $\xi \rightarrow ||\xi|| + ||\pi(x)\xi||$, where $x \in A$. The *closure* $\bar{\pi}$ of π is the *-representation $(\bar{\pi}, D(\bar{\pi}), H)$, where $D(\bar{\pi}) = \bigcap\{D(\overline{\pi(x)}) : x \in A^1\}$, $D(\overline{\pi(x)})$ being the domain of the closure $\overline{\pi(x)}$ of $\pi(x)$; and $\bar{\pi}(x) = \overline{\pi(x)}|_{D(\bar{\pi})}$ for all x in A^1. Throughout, π is assumed *non-degenerate*, i.e., the norm closure $(\pi(A)H)^- = H$ and the t_π-closure $\overline{(\pi(A)\mathcal{D}(\pi))}^{t_\pi} = \mathcal{D}(\bar{\pi})$. If $\pi = \bar{\pi}$, then π is *closed*. The *hermitian adjoint* π^* of π is the representation (not necessarily a *-representation) $(\pi^*, D(\pi^*), H)$, where $D(\pi^*) = \bigcap\{D(\pi(x)^* : x \in A^1\}$, and $\pi^*(x) = \pi(x^*)^*|_{D(\pi^*)}$ for all $x \in A^1$. If $\pi = \pi^*$, then π is *self-adjoint*. Further, π is *standard* if $\pi(x^*)^* = \overline{\pi(x)}$ for all x in A^1. If each $\pi(x)$ is a bounded operator, then π is *bounded*. If π is a direct sum of bounded representations, then π is *weakly unbounded*. An *O^*-algebra* is a collection $\mathcal{U}$ of linear operators T all defined on a dense subspace D of a Hilbert space H such that for all $T \in \mathcal{U}$, one has $TD \subseteq D$, and $T^*D \subseteq D$; and $\mathcal{U}$ is a *-algebra with the pointwise linear operations, composition as the multiplication, and $T \to T^+ := T^*|_D$ as the involution. Given a *-representation $(\pi, D(\pi), H)$ of a *-algebra A, the *uniform topology* [20], ([37], p. 77–78) $\tau_D = \tau_{D(\pi)}^{\pi(A)}$ on the O^*-algebra $\pi(A)$ is the locally convex topology defined by the seminorms $\{q_K : K$ is a bounded subset of $(D(\pi), t_\pi)\}$, where

$$q_K(\pi(x)) = \sup\{|\langle \pi(x)\xi, \eta\rangle| : \xi, \eta \text{ in } K\}.$$

A vector ξ in $D(\pi)$ is *strongly cyclic* [30] (called cyclic in [37]) if $D(\pi) = (\pi(A)\xi)^{-t_\pi}$ the closure of $(\pi(A)\xi)$ in $(D(\pi), t_\pi)$. By a *cyclic vector*, we mean ξ in $D(\pi)$ such that the norm closure $(\pi(A)\xi)^- = H$. For topological *-algebras, we refer to [21]. A *Q-algebra* is a topological algebra whose quasi-regular elements form an open set. An *LFQ-algebra* is a Q-algebra which is an LF-space [41]. The topology of a locally convex (respectively locally m-convex) *-algebras A is determined by the family $K(A)$ (respectively $K_s(A)$), or a separating subfamily $\mathcal{P}$ thereof, consisting of continuous *-seminorms (repsectively continuous submultiplicative *-seminorms) p. If A has a bounded approximate identity (e_i), then it is assumed that $p(e_i) \leq 1$ for all i and all p. A *pro-C^*-algebra* is a complete locally m-convex *-algebra whose topology is determined by a family of C^*-seminorms. A *Frechet *-algebra* (respectively *locally convex F^*-algebra*) is a complete metrizable locally m-convex (respectively locally convex) *-algebra. A *σ-C^*-algebra* means a Frechet pro-C^*-algebra. For pro-C^*-algebras, we refer to [26, 27].

2. Preliminary constructions and lemmas

Let A be a *-algebra, not necessarily having an identity element. Let f be a positive linear functional on A. Then f is *representable* if there exists a closed strongly cyclic *-representation $(\pi, D(\pi), H)$ of A having a strongly cyclic vector $\xi \in D(\pi)$ such that $f(x) = \langle \pi(x)\xi, \xi\rangle$ for all $x \in A$. If π can be chosen to be a bounded operator representation, then f is *boundedly representable*. The first half of the following is an unbounded representation theoretic analogue of ([39], Theorem 1), whereas the remaining half improves a part of ([39], Theorem 1) even in the bounded case. The proof exhibits the unbounded analogue of the GNS construction in the case of non-unital algebras. This provides a useful supplement to ([37], § 8.6). It is well-known that a representable functional is boundedly representable if and only if it is *admissible* in the sense that for each $x \in A$, there exists $k > 0$ such that $f(y^*x^*xy) \leq kf(y^*y)$ for all $y \in A$. In the following, Lemma 2.1(3) is very close to ([39], Theorem 1) in which a C^*-seminorm p is taken.

Lemma 2.1. *Let f be a positive linear functional on a *-algebra A. The following are equivalent.*

(1) *f is representable.*
(2) *There exists $m > 0$ such that $|f(x)|^2 \leq mf(x^*x)$ for all $x \in A$.*
Further, f is boundedly representable if and only if f satisfies (2) *above and the following.*
(3) *There exists a submultiplicative *-seminorm p on A and $M > 0$ such that $|f(x)| \leq Mp(x)$ for all $x \in A$.*

When A is a Banach *-algebra, Lemma 2.1 is given in ([7], Theorem 37.11, p. 199). In the framework of unbounded representation theory, it is discussed in [2]. There is a gap in the proof in ([7], Theorem 37.11) in that hermiticity of f has been implicitly used. Regrettably it remained unnoticed in [2]. This was rectified in [39] in the formalism of bounded representations. The following proof provides an analogous correction in the context of unbounded representations.

Proof. Suppose (1) holds with $f(x) = \langle \pi(x)\xi, \xi \rangle$ for all $x \in A$. Then for all x in A.

$$|f(x)|^2 \leq ||\pi(x)\xi||^2 ||\xi||^2 \leq ||\xi||^2 \langle \pi(x)\xi, \pi(x)\xi \rangle$$
$$= ||\xi||^2 \langle \pi(x)^* \pi(x)\xi, \xi \rangle = ||\xi||^2 \langle \pi(x^*)\pi(x)\xi, \xi \rangle$$

as $\pi(A)\xi \subseteq D(\pi) = D(\pi(x)) = D(\pi(x^*))$ and $\pi(x^*) \subseteq \pi(x)^*$. Thus

$$|f(x)|^2 \leq ||\xi||^2 \langle \pi(x^*x)\xi, \xi \rangle = ||\xi||^2 f(x^*x)$$

for all $x \in A$, giving (2).

Conversely, assume (2). We adopt the GNS construction. Let $N_f = \{x \in A : f(x^*x) = 0\}$. By the Cauchy-Schwarz inequality, N_f is a left ideal of A. Let $X_f = A/N_f$, and $\lambda_f : A \to X_f$ be $\lambda_f(x) = x + N_f$. Then $\langle \lambda_f(x), \lambda_f(y) \rangle = f(y^*x)$ defines an inner product on X_f. Let H_f be the Hilbert space obtained by completing X_f. Let $\varphi : X_f \to \mathbb{C}$ be $\varphi(\lambda_f(x)) = f(x)$, a linear functional. Then for all $x \in A$,

$$|\varphi(\lambda_f(x))|^2 = |f(x)|^2 \leq mf(x^*x) = m\langle \lambda_f(x), \lambda_f(x) \rangle = m||\lambda_f(x)||^2.$$

Thus φ extends uniquely to H_f as a bounded linear functional; and by Riesz theorem, there exists a $\xi \in H_f$ such that for all $x \in A, f(x) = \varphi(\lambda_f(x)) = \langle \lambda_f(x), \xi \rangle$. Further, if m is the minimum possible constant in the assumed inequality, then $||\xi|| = m^{1/2}$. The idea of using Riesz theorem at this stage is borrowed from [39]. Define a *-representation $(\pi_0, D(\pi_0), H_f)$ of A by: $D(\pi_0) = X_f$; and for any x in A, $\pi_0(x)\lambda_f(y) = \lambda_f(xy)$ for all y in A. Let π be the closure of π_0. Then for all x, y in A,

$$\langle \lambda_f(x), \lambda_f(y) \rangle = f(y^*x) = \langle \lambda_f(y^*x), \xi \rangle = \langle \pi_0(y^*)\lambda_f(x), \xi \rangle. \tag{i}$$

Assertion 1. $X_f = \pi_0^*(A)\xi$.

Let $x \in A$. For all $y \in A$,

$$|\langle \pi_0(x)\lambda_f(y), \xi \rangle| = |\langle \lambda_f(xy), \xi \rangle| = |f(xy)| \leq f(xx^*)^{1/2} f(y^*y)^{1/2}$$
$$= f(xx^*)^{1/2} ||\lambda_f(y)||$$

showing that the linear functional $\lambda_f(y) \to \langle \pi_0(x)\lambda_f(y), \xi \rangle$ on $D(\pi_0)$ is $||\ \ ||$-continuous. Hence $\xi \in D(\pi_0(x)^*)$ for all $x \in A$. It follows, by the definition of $D(\pi_0^*)$, that $\xi \in D(\pi_0^*)$. Now (i) becomes $\langle \lambda_f(x), \lambda_f(y) \rangle = \langle \lambda_f(x), \pi_0(y^*)^* \xi \rangle$ for all $x \in A$. Since X_f is dense in H_f, we obtain $\lambda_f(y) = \pi_0(y^*)^* \xi = \pi_0^*(y)\xi$ for all y in A. Thus $X_f = \pi_0^*(A)\xi$.

Assertion 2. $\xi \in D(\pi)$.

Since $\overline{\pi_0(x)} = \pi_0(x)^{**}$, we show that $\xi \in D(\pi_0(x)^{**})$ for all $x \in A$, i.e., for all x, the functional on $D(\pi_0(x)^*)$ given by $\eta \to \langle \pi_0(x)^* \eta, \xi \rangle$ is $||\ \ ||$-continuous. Fix an $x \in A$. Now $\xi \in D(\pi_0^*)$, hence $\xi \in D(\pi_0(x^*)^*)$ so that the functional g on $D(\pi_0(x^*)) = X_f$ defined by $g(\eta) = \langle \pi_0(x^*)\eta, \xi \rangle$ is $||\ \ ||$-continuous, and extends continuously to H_f. Now let $\psi \in D(\pi_0(x)^*)$. Let (η_k) be a sequence in X_f such that $\eta_k \to \psi$ in $||\ \ ||$. Then $\xi \in D(\pi_0^*)$ implies that for any $x \in A$,

$$\langle \pi_0(x)^* \psi, \xi \rangle = \langle \psi, \pi_0(x^*)^* \xi \rangle = \langle \psi, \pi_0^*(x)\xi \rangle$$
$$= \lim \langle \eta_k, \pi_0^*(x)\xi \rangle = \lim \langle \pi_0(x^*)\eta_k, \xi \rangle = g(\psi)$$

showing that $\psi \to \langle \pi_0(x)^*\psi, \xi\rangle$ is $\|\ \|$-continuous on $D(\pi_0(x)^*)$. This proves the assertion 2.

Now by the proof of assertions 1 and 2 above, it follows that for any $x \in A$,

$$\begin{aligned} f(x) = \varphi(\lambda_f(x)) &= \langle \lambda_f(x), \xi\rangle = \langle \pi_0(x^*)^*\xi, \xi\rangle = \langle \pi_0^*(x)\xi, \xi\rangle \\ &= \langle \bar{\pi}_0(x)\xi, \xi\rangle = \langle \pi(x)\xi, \xi\rangle. \end{aligned}$$

Clearly ξ is a strongly cyclic vector for π. Thus (2) implies (1).

Now assume (2) and (3). Let $N_p = \{x \in A : p(x) = 0\}$, a *-ideal in A. Let A_p be the Banach *-algebra obtained by completing A/N_p in the norm $\|x_p\|_p = p(x)$ where $x_p = x + N_p$. By (3), $F(x_p) = f(x)$ gives a well-defined continuous positive functional on A_p. By standard Banach *-algebra theory, for all x, y in A,

$$\begin{aligned} \|\pi(x)\pi(y)\xi\|^2 &= \langle \pi(y^*x^*xy)\xi, \xi\rangle = f(y^*x^*xy) = F(y_p^*x_p^*x_py_p) \\ &\leq \|x_p^*x_p\|F(y_p^*y_p) \leq p(x)^2 f(y^*y) = p(x)^2\|\pi(y)\xi\|^2. \end{aligned}$$

Since $\pi(A)\xi$ is dense in H_f, π is a bounded operator representation.

COROLLARY 2.2

Let A be a *-algebra.

(1) *A positive functional f on A is representable if and only if f is extendable as a positive functional on the unitization A^1 of A.*
(2) *A representable positive functional on A satisfies $f(x^*) = f(x)^-$ for all x in A.*
(3) *Let A be a topological *-algebra having a bounded approximate identity. Then every continuous positive functional on A is representable.*

COROLLARY 2.3

Let A be a complete locally m-convex *-algebra with a bounded approximate identity (e_γ) satisfying $p(e_\gamma) \leq 1$ for all ρ in a defining family of seminorms.

(1) *Let f be a continuous positive functional on A. Then f is boundedly representable and there exists $p \in K_s(A)$ such that $|f(x)| \leq (\limsup f(e_\gamma e_\gamma^*))p(x)$ for all $x \in A$.*
(2) *Let $(\pi, \mathcal{D}(\pi), H)$ be a *-representation of A. Then each $\pi(e_\gamma)$ is a bounded operator and $\|\pi(e_\gamma)\| \leq 1$ for all γ. Further, if π is strongly continuous (in particular, if π is continuous in the unifrom topology, which is the case if A is locally convex F^** ([37], *Theorem 3.6.8, p. 99)), then $\|\pi(e_\gamma)\xi - \xi\| \to 0$ for each ξ.*

Proof. (1) By continuity, there exist $p \in K_s(A)$ and $m > 0$ such that $|f(x)| \leq mp(x)$ for all $x \in A$. Now Lemma 2.1 applies by Corollary 2.2(3). Let $l = \limsup f(e_\gamma e_\gamma^*)$, which is finite. Let $c = \sup\{|f(x)| : p(x) = 1\}$. Choose a sequence (x_n) in A such that $f(x_n) \to c$ and $p(x_n) = 1$ for all n. Then, by the Cauchy-Schwarz inequality,

$$|f(x_n)|^2 = \lim_\gamma |f(e_\gamma x_n)|^2 \leq (\limsup f(e_\gamma e_\gamma^*)) f(x_n^*x_n) \leq lc,$$

as $p(x_n^*x_n) \leq p(x_n)^2 = 1$. Hence $c^2 \leq lc$, i.e. $c \leq l$, and the assertion follows.
(2) Let $P = (p_\alpha)$ be a cofinal subset of $K_s(A)$ determining the topology of A. Let $A_p = \{x \in A : \sup_\alpha p_\alpha(x) < \infty\}$. Then A_p is a *-subalgebra of A containing each e_γ. As A is complete, A_p is a Banach *-algebra with norm $p(x) = \sup_\propto p_\propto(x)$. For any $\xi \in H$,

consider the positive functional $\omega_\xi(x) = \langle \pi(x)\xi, \xi \rangle$ on A. Then for all $x \in A$, $|\omega_\xi(x)|^2 \leq ||\xi||^2\, \omega_\xi(x^*x)$. By Lemma 2.1, ω_ξ is representable, hence extends as a positive functional ω on the unitization A^1 of A. In view of the inclusion map $(A_p)^1 \to A^1$, ω is a positive functional on $(A_p)^1$. By ([7], Corollary 37.9, p. 198), ω is continuous in the norm of $(A_p)^1$. It follows that ω_ξ restricted to A_p is continuous in the norm of A_p and $||\omega_\xi|| \leq ||\xi||^2$. For each γ,

$$||\pi(e_\gamma)\xi||^2 = \omega_\xi(e_\gamma e_\gamma^*) \leq ||\omega_\xi|| p(e_\gamma)^2 \leq ||\xi||^2$$

showing that $||\pi(e_\gamma)|| \leq 1$. Now suppose that π is strongly continuous. Let $\eta \in \mathcal{D}(\pi)$ and $\varepsilon > 0$. There exists $x \in A$ and $\eta' \in \mathcal{D}(\pi)$ such that $||\pi(x)\eta' - \eta|| \leq \varepsilon/3$. Since $e_\gamma x \to x$, there exists γ_0 such that for all $\gamma \geq \gamma_0$,

$$\begin{aligned} ||\eta - \pi(e_\gamma)\eta|| &\leq ||\eta - \pi(x)\eta'|| + ||\pi(x)\eta' - \pi(e_\gamma x)\eta'|| \\ &\quad + ||\pi(e_\gamma)||\; ||\pi(x)\eta' - \eta|| < \varepsilon \end{aligned}$$

showing that $\pi(e_\gamma)\eta \to \eta$ for each $\eta \in \mathcal{D}(\pi)$. This completes the proof of Corollary 2.3.

The enveloping pro-C^*-algebra $E(A)$

We construct the enveloping pro-C^*-algebra $E(A)$ for a locally convex *-algebra A with jointly continuous multiplication. This extends the consideration in [10, 15, 19] in which A is additionally assumed m-convex. The added generality will include several constructions relevant in C^*-algebra theory (like the C^*-algebra of a groupoid). Let $R(A)$ denote the set of all continuous bounded operator *-representations $\pi : A \to B(H_\pi)$ of A into the C^*-algebras $B(H_\pi)$ of all bounded linear operators on Hilbert spaces H_π. Let $R'(A) = \{\pi \in R(A) : \pi \text{ is topologically irreducible}\}$. For $p \in K(A)$, let

$$R_p(A) = \{\pi \in R(A) : \text{ for some } k > 0, ||\pi(x)|| \leq kp(x) \text{ for all } x\},$$

and $R'_p(A) = R_p(A) \cap R'(A)$. Then

$$R(A) = \bigcup\{R_p(A) : p \in K(A)\}, \quad R'(A) = \bigcup\{R'_p(A) : p \in K(A)\}.$$

Let $r_p(x) = \sup\{||\pi(x)|| : \pi \in R_p(A)\}$.

Lemma 2.4. *Let A be as above, $p \in K(A)$. Then $r_p(\ )$ is a continuous C^*-seminorm on A satisfying $r_p(x) \leq p(x^*x)^{1/2}$. If $p \in K_s(A)$, then $r_p(x) = \sup\{||\pi(x)|| : \pi \in R'_p(A)) \leq p(x)$ for all $x \in A$.*

Proof. Let $s_p(x) = p(x^*x)^{1/2}$. Let $h = h^* \in A$ and $\pi \in R_p(A)$. Then $||\pi(h^n)|| \leq kp(h^n)$ for all $n \in \mathbb{N}$. By standard Banach algebra arguments, the spectral radius satisfies

$$r(\pi(h)) = \liminf ||\pi(h^n)||^{1/n} = \inf ||\pi(h^n)||^{1/n} \leq \inf p(h^n)^{1/n} \leq p(h).$$

Hence, for any $x \in A$,

$$||\pi(x)||^2 = ||\pi(x^*x)|| = r(\pi(x^*x)) \leq p(x^*x),$$

so that $r_p(x) \leq s_p(x)$. We use the joint continuity of multiplication to conclude the continuity of the C^*-seminorm $x \to r_p(x)$. Now suppose $p \in K_s(A)$. Then

$$r_p(x) \leq s_p(x) \leq (p(x^*)p(x))^{1/2} \leq p(x).$$

Further, let $N_p = \{x \in A : p(x) = 0\}$, a closed *-ideal in A. Let A_p be the Banach *-algebra obtained by completing A/N_p in the norm $||x + N_p||_p = p(x)$. Then $R_p(A)$ (respectively $R'_p(A)$) can be identified with $R(A_p)$ (respectively $R'(A_p)$). The assertion follows from the fact that for all $z \in A_p$,

$$\sup\{||\varphi(z)|| : \varphi \in R(A_p)\} = \sup\{||\varphi(z)|| : \varphi \in R'(A_p)\}$$

([14], 2.7, p. 47). This completes the proof of the lemma.

Define the *star radical* to be

$$\begin{aligned} \text{srad}\,(A) &= \{x \in A : r_p(x) = 0 \text{ for all } p \in K(A)\} \\ &= \{x \in A : \pi(x) = 0 \text{ for all } \pi \in R(A)\}. \end{aligned}$$

For each $p \in K(A)$, $q_p(x + \text{srad}\,(A)) = r_p(x)$ defines a continuous C^*-seminorm on the quotient locally convex *-algebra $A/\text{srad}\,(A)$ with the quotient topology. Let τ be the Hausdorff topology on $A/\text{srad}\,(A)$ defined by $\{q_p : p \in K(A)\}$. The *enveloping pro-C^*-algebra* $E(A)$ of A is the completion of $(A/\text{srad}\,(A), \tau)$. When A is metrizable, $E(A)$ is metrizable. In view of Corollary 2.2, when A is m-convex, this coincides with the enveloping l.m.c. *-algebra defined in [10, 19, 15].

Lemma 2.5. *Let A be a locally convex *-algebra with jointly continuous multiplication.*

(a) *Let $\bar{A}$ be the completion of A. Then $E(\bar{A}) = E(A)$.*
(b) $E(A^1) = E(A)^1$.

Proof. Since A has jointly continuous multiplication, $\bar{A}$ is a complete locally convex *-algebra. The map $i : A/\text{srad}\,(A) \to \bar{A}/\text{srad}\,(\bar{A})$, where $i(x + \text{srad}\,(A)) = x + \text{srad}\,(\bar{A})$, is a well defined *-isomorphism into $E(\bar{A})$. Note that for any $p \in K(\bar{A})$, $R_p(A) = R_p(\bar{A})$ via the restriction (in fact, also $K(\bar{A}) = K(A)$), hence $\text{srad}\,(A) = A \cap \text{srad}\,(\bar{A})$. For any $p \in K(A)$, let $\tilde{p} \in K(\bar{A})$ be the unique extension of p. Then, for any $x \in A$,

$$q_p(x + \text{srad}(A)) = r_p(x) = q_{\tilde{p}}(x + \text{srad}(\bar{A}));$$

and for any $\tilde{p} \in K(\bar{A})$, $q_{\tilde{p}}(x + \text{srad}(A)) = q_{\tilde{p}|A}(x + \text{srad}(A))$. Thus i is a homeomorphism for the respective pro-C^*-topologies. On the other hand, i has dense range in $\bar{A}/\text{srad}(\bar{A})$. Indeed, let $z \in \bar{A}$. Choose a net (x_i) in A such that $x_i \to z$ in the topology t of $\bar{A}$. Then

$$\begin{aligned} q_{\tilde{p}}(x_i - z + \text{srad}(A)) &= r_{\tilde{p}}(x_i - z) = \sup\{||\pi(x_i - z)|| : \pi \in R_{\tilde{p}}(\bar{A})\} \\ &\le k\tilde{p}(x_i - z) \to 0 \end{aligned}$$

for all $\tilde{p} \in K(\bar{A})$. Thus $E(\bar{A})$, which is the completion of $\bar{A}/\text{srad}(\bar{A})$, coincides with the completion $E(A)$ of $A/\text{srad}(A)$. This completes the proof of (a). We omit the proof of (b).

A representation $(\pi, D(\pi), H)$ of A is *countably dominated* if there exists a countable subset B of A such that for any $x \in A$, there exists $b \in B$ and a scalar $k > 0$ such that $||\pi(x)\xi|| \le k||\pi(b)\xi||$ for all $\xi \in D(\pi)$ ([22], p. 419).

Lemma 2.6. (a) *Let A be a locally convex *-algebra. Let $j : A \to E(A)$, $j(x) = x + \text{srad}(A)$.*

(1) *If $\pi : A \to B(H)$ is a continuous bounded operator *-representation, then there exists a unique continuous *-representation $\sigma : E(A) \to B(H)$ such that $\pi = \sigma \circ j$. Further, π is irreducible if and only if σ is irreducible.*

(2) *Let $(\pi, \mathcal{D}(\pi), H)$ be a closed *-representation of A continuous in the uniform topology. Let π be weakly unbounded. Then there exists a closed weakly unbouned *-representation $(\sigma, \mathcal{D}(\sigma), H)$ of $E(A)$ such that $\pi = \sigma \circ j$ and $\mathcal{D}(\sigma)$ is dense in the locally convex space $(\mathcal{D}(\pi), t_\pi)$.*

(3) *Let A be unital and symmetric. Assume that A is separable or nuclear (as a locally convex space). Let $(\pi, \mathcal{D}(\pi), H)$ be a separably acting, countably dominated *-representation of A continuous in the uniform topology. Then there exists a closed *-representation $(\sigma, \mathcal{D}(\sigma), H)$ of $E(A)$ such that $\pi = \sigma \circ j$.*

(b) (1) *There exists a unital, locally convex, non-m-convex, F^*-algebra A such that A admits a faithful family of unbounded operator *-representations, but admits no non-zero bounded operator *-representation.*

(2) *There exists a unital non-locally-convex F^*-algebra that admits no non-zero *-representation.*

Proof. (a) (1) follows by the definition of $E(A)$.
(2) Let $\pi = \oplus\pi_i$, where each π_i is a norm continuous bounded operator *-representation $\pi_i : A \to B(H_i)$ on a Hilbert space H_i. We take $D(\pi_i) = H_i$. Let $E_i : H \to H_i$ be the orthogonal projection. By (1), there exist continuous *-homomorphisms $\sigma_i : E(A) \to B(H_i)$, $\sigma_i \circ j = \pi_i$. Let $\sigma = \oplus\sigma_i$ on the Hilbert direct sum $\oplus H_i = H$ having the domain

$$\begin{aligned}\mathcal{D}(\sigma) &= \{\eta = \Sigma E_i\eta \in H : \Sigma||\sigma_i(z)E_i\eta||^2 < \infty \text{ for all } z \in E(A)\}\\ &\subset \mathcal{D}(\pi) = \{\eta = \Sigma E_i\eta \in H : \Sigma||\pi_i(x)E_i\eta||^2 < \infty \text{ for all } x \in A\}.\end{aligned}$$

On $\mathcal{D}(\sigma)$, the σ-graph topology $t_{\sigma(E(A))}$ is finer than the relativized π-graph topology $t_\pi|_{\mathcal{D}(\sigma)}$. Being closed and weakly unbounded, both σ and π are standard representations. Hence, for all $h = h^*$ in A, the operators $\sigma(j(h))$ having domain $\mathcal{D}(\sigma)$ and $\pi(h)$ with domain $\mathcal{D}(\pi)$ are essentially self-adjoint. Since self-adjoint operators are maximally symmetric, $\mathcal{D}(\sigma)$ is dense in $\mathcal{D}(\overline{\pi(h)})$ for the graph topology defined by $\xi \to ||\xi|| + ||\overline{\pi(h)}\xi||$. Thus $\mathcal{D}(\sigma)$ is dense in the locally convex space $\mathcal{D}(\pi) = \cap\{\mathcal{D}(\overline{\pi(h)}) : h = h^*$ in $A\}$.
(3) By ([22], Theorem 3.2 and remark on p. 422) and ([37], Theorem 12.3.5, p. 343), there exists a compact Hausdorff Z with a positive measure μ such that

$$\pi = \int_Z^{\oplus} \pi_\lambda \mathrm{d}\mu(\lambda), \quad \mathcal{D}(\pi) = \int_Z^{\oplus} \mathcal{D}(\pi_\lambda)\mathrm{d}\mu(\lambda), \quad H = \int_Z^{\oplus} H_\lambda \mathrm{d}\mu(\lambda)$$

and each π_λ is irreducible. Since A is symmetric, each $\bar{\pi}$ and $\bar{\pi}_\lambda$ are standard ([37], Corollary 9.1.4, p. 237) (the commutativity assumption in this reference is not required, as the arguments in ([2], Theorem 3.5) shows); and by [3], each π_λ is a bounded operator representaion, being irreducible. Then we can proceed as in (2).
(b) (1) Take $A = L^\omega[0,1] = \bigcap_{1 \le p < \infty} L^p[0,1]$ (the Arens algebra) with pointwise operations, complex conjugation, and the topology of L^p-convergence for each p, $1 \le p < \infty$. The algebra A is a unital, symmetric, locally convex F^*-algebra, admitting a faithful standard *-representation $(\pi, \mathcal{D}(\pi), H)$ such that $\overline{\pi(A)}$ is an extended C^*-algebra with a common dense domain [13]. However, there exists no non-zero bounded operator representation of A, as A admits no non-zero multiplicative linear functional; and hence no non-zero submultiplicative *-seminorm. Thus srad $(A) = A$ and $E(A) = (0)$. (2) Take $A = \mathcal{M}$ [0,1], the algebra of all Lebesgue measurable functions on [0,1] with the topology of

convergence in measure. It admits no non-zero positive linear functional, and hence no non-zero *-representation.

Remark. 2.7. We call a *-representation $(\pi, \mathcal{D}(\pi), H)$ of a *-algebra A *boundedly decomposable* if it can be disintegrated as $\pi = \int_Z^{\oplus} \pi_\lambda \mathrm{d}\mu(\lambda)$ with each π_λ a bounded operator *-representation. One may show that $E(A)$ is universal for all closed boundedly decomposable *-representations of a locally convex F^*-algebra A. We do not know whether in (2) and (3) of Corollary (2.4) (a), σ is continuous in the uniform topology.

The *bounded vectors* [4] for a *-representation π of a *-algebra A are $B(\pi) = \bigcap\{B(\pi(x)) : x \in A\}$, where, for an operator T, the bounded vectors for T are

$$B(T) = \{\xi \in \mathcal{D}(T) : \text{ there exists a } > 0, c > 0 \text{ such that } ||T^n\xi|| \leq ac^n \text{ for all } n \in \mathbb{N}\}.$$

The following is motivated by [35]. It shows that unbounded representations of locally *m*-convex *-algebras cannot be wildly unbounded,

Lemma 2.8. *Let $(\pi, \mathcal{D}(\pi), H)$ be a closed *-representation of a complete locally m-convex *-algebra A continuous in the uniform topology on $\pi(A)$. Then the following hold.*

(1) *$\mathcal{D}(\pi) = B(\pi)$; and π is a direct sum of norm-continuous cyclic bounded operator *-representations.*
(2) *π is standard. For commuting normal elements x, y of A, the normal operators $\overline{\pi(x)}$ and $\overline{\pi(y)}$ have mutually commuting spectral projections.*
(3) *The uniform topology $\tau_{\mathcal{D}}$ on $\pi(A)$ is a pro-C^*-topology, i.e., it is determined by a family of C^*-seminorms.*
(4) *If A is Frechet, then $\tau_{\mathcal{D}}$ is metrizable and π is direct sum of a countable number of cyclic bounded-operator *-representations.*

Proof. Let $\xi \in \mathcal{D}(\pi)$. Let ω_ξ on A be the positive functional $\omega_\xi(x) = \langle \pi(x)\xi, \xi\rangle$ for $x \in A$. By Lemma 2.1, ω_ξ is representable and admissible. Hence the closed GNS representation $(\pi_{\omega_\xi}, \mathcal{D}(\pi_{\omega_\xi}), H_{\omega_\xi})$ associated with ω_ξ is a cyclic, norm-continuous bounded operator *-representation with $\mathcal{D}(\pi_{\omega_\xi}) = H_{\omega_\xi}$. Let ξ_ω denote the cyclic vector for π_{ω_ξ}. Let $\mathcal{D}(\pi_\xi) = (\pi(A)\xi)^{-t_\pi}$ and $H_\xi = [\pi(A)\xi]^-$. Since π is closed, $\mathcal{D}(\pi_\xi) \subset \mathcal{D}(\pi)$. The π-invariant subspace $\mathcal{D}(\pi_\xi)$ defines a closed subrepresentation $\langle \pi_\xi, \mathcal{D}(\pi_\xi), H_\xi\rangle$ of π as $\pi_\xi(x) = \pi(x)|_{\mathcal{D}(\pi_\xi)}$. Since $\langle \pi_{\omega_\xi}(x)\xi_{\omega_\xi}, \xi_{\omega_\xi}\rangle = \omega_\xi(x) = \langle \pi_\xi(x)\xi, \xi\rangle$ for all $x \in A$, it follows that π_{ω_ξ} and π_ξ are unitarily equivalent. Thus π_ξ is a bounded operator representation, and $\mathcal{D}(\pi_\xi) = H_\xi \subset B(\pi)$. This also implies that H_ξ is reducing in the sense of ([37], § 8.3). Thus the following is established.

Assertion I. For any ξ in $\mathcal{D}(\pi)$, $[\pi(A)\xi]^{-t_\pi} = [\pi(A)\xi]^- \subset B(\pi)$.

It follows that $\pi(A)\mathcal{D}(\pi) \subset B(\pi)$, hence $B(\pi)$ is dense in $(\mathcal{D}(\pi), t_\pi)$ and norm dense in H. Since $B(\pi)$ forms a set of common analytic vectors for $\pi(A)$, the conclusion (2) follows, using ([40], Theorem 2). Also, a standard Zorn's lemma argument gives $\pi = \oplus\pi_i$, with each π_i a cyclic, continuous, bounded operator representation.

Assertion II. For each bounded subset M of $(\mathcal{D}(\pi), t_\pi)$, there exists $p \in K_s(A)$ such that $||\pi(x)\eta|| \leq ||\eta|| p(x)$ for all $x \in A$, $\eta \in M$.

By continuity, given M as above, there is $k > 0$ and $p \in K_s(A)$ such that $q_M(\pi(x)) \leq kp(x)$ for all $x \in A$. Hence, for each $\eta \in M$ and $x \in A$, $||\pi(x)\eta||^2 \leq kp(x^*x) \leq kp(x)^2$. By Corollary 2.3, $||\pi(x)\eta||^2 \leq lp(x)^2$, where $l = \limsup \omega_\eta(e_\gamma e_\gamma^*) \leq ||\eta||^2$. Hence $||\pi(x)\eta|| \leq ||\eta|| p(x)$ for all $x \in A$, all $\eta \in M$.

Now let $\xi \in \mathcal{D}(\pi)$. By (II) above, there exists $p \in K_s(A)$ such that for all $n \in \mathbb{N}$,

$$||\pi(x)^n \xi||^2 = \langle \pi(x^{*^n} x^n)\xi, \xi\rangle \leq ||\xi||^2 p(x)^{2n}$$

showing that $\xi \in B(\pi(x))$. Thus $\mathcal{D}(\pi) = B(\pi)$ proving (1).

The proof of (3) is based on arguments in ([35], Theorem 1). Let $\mathcal{F}$ be the collection of all subspaces (linear manifolds) K of $\mathcal{D}(\pi)$ such that K is π-invariant, and $\pi|_K$ is a bounded operator *-representation. For $K \in \mathcal{F}$, let s_K be the C^*-seminorm

$$s_K(\pi(x)) = \sup\{||\pi(x)\eta|| : \eta \in K, ||\eta|| \leq 1\}.$$

Let τ_1 be the topology on $\pi(A)$ defined by $\{s_K : K \in \mathcal{F}\}$. We show that $\tau_{\mathcal{D}} = \tau_1$. Clearly $\tau_1 \leq \tau_{\mathcal{D}}$. Let M be a bounded subset of $(\mathcal{D}(\pi), t_\pi)$. Choose k and p as in assertion (II) above. By Corollary 2.3, $|\omega_\xi(x)| \leq ||\xi|| p(x)$ for all $x \in A$, all $\xi \in M$. Thus

$$M \subset \mathcal{D}_p := \{\eta \in \mathcal{D}(\pi) : |\langle \pi(x)\eta, \eta\rangle| \leq ||\eta||^2 p(x) \text{ for all } x \text{ in} A\}.$$

Then $\mathcal{D}_p \in \mathcal{F}$; $||\pi(x)\eta|| \leq ||\eta||^2 p(x)^2$ for all $\eta \in \mathcal{D}_p$; and, as π is closed, ([35], Lemma 3) implies that $\mathcal{D}_p$ is $||\ \ ||$-closed. Let $S = \{\xi \in \mathcal{D}_p : ||\xi|| \leq 1\}$. As M is also $||\cdot||$ bounded, $||\eta|| \leq r$ for all $\eta \in M$; and $M \subset rS$. Then, for all $x \in A$, $q_M(\pi(x)) \leq r^2 s_{\mathcal{D}_p}(\pi(x))$. Thus $\tau_{\mathcal{D}} \leq \tau$. This gives (3). Finally (4) is consequence of the fact that the topology of a metrizable A is determined by a countable cofinal subfamily of $K_s(A)$. This completes the proof of Lemma 2.8.

Now let A be commutative. Let $\mathcal{M}(A)$ be the Gelfand space consisting of all non-zero continuous multiplicative linear functionals on A. Let $\mathcal{M}^*(A) = \{\varphi \in \mathcal{M}(A) : \varphi = \varphi^*\}$ and $\varphi^*(x) = \overline{\varphi(x^*)}$. For each $x \in A$, let $\hat{x} : \mathcal{M}^*(A) \to \mathbb{C}$ be the map $\hat{x}(\varphi) = \varphi(x)$. The following, which incorporates the spectral theorem for unbounded normal operators, describes all unbounded *-representations of A. The proof can be constructed using Lemma 2.8 and ([9], Theorem 7.3), in which all bounded *-representations of A have been realized.

COROLLARY 2.9

Let A be a commutative complete locally m-convex *-algebra. Let $(\pi, \mathcal{D}(\pi), H)$ be a closed *-representation of A continuous in the uniform topology. Then there exist a positive regular Borel measure μ on $\mathcal{M}^*(A)$ and a spectral measure E on the Borel sets in $\mathcal{M}^*(A)$ with values in $B(H)$ such that the following hold.

(1) *π is a unitarily equivalent to the representation $(\sigma, \mathcal{D}(\sigma), H_\sigma)$ by multiplication operators in $H_\sigma = L^2(\mathcal{M}^*(A), \mu)$ with domain*

$$\mathcal{D}(\sigma) = \{f \in H_\sigma : \varphi \to \hat{x}(\varphi) f(\varphi) \text{is in} H_\sigma \text{ for all } x \in A\}$$

defined as $(\sigma(x)f)(\varphi) = \hat{x}(\varphi)f(\varphi)$.

(2) *For each $x \in A$, $\pi(x) = \int_{\mathcal{M}^*(A)} \hat{x}(\varphi) \mathrm{d}E(\varphi)$.*

We say that a locally convex *-algebra A is an *algebra with a C^*-enveloping algebra* if the pro-C^*-algebra $E(A)$ is a C^*-algebra. In view of Lemma 2.5, we do not need to assume A to be complete or unital. In [5], A is further assumed to be m-convex. The following extends the main results in ([5], § 2) to the present more general set up, and can be proved as in [5]. A is called an *sQ-algebra* if for some $k > 0$, $p \in K(A)$, the spectral radius r satisfies $r(x^*x)^{1/2} \leq kp(x)$ for all $x \in A$; A is *-*sb* if $r(x^*x) < \infty$ for each x, equivalently, $r(h) < \infty$ for all $h = h^*$. Thus $Q \Rightarrow sQ \Rightarrow {}^*\text{-}sb$.

Lemma 2.10. *Let A be a complete locally convex *-algebra with jointly continuous multiplication.*

(1) *A is an algebra with a C^*-enveloping algebra if and only if A admits greatest continuous C^*-seminorm.*
(2) *If A is sQ, then A admits a greatest C^*-seminorm, which is also continuous.*
(3) *Let A be an F^*-algebra. If A is *-sb, then A has a C^*-enveloping algebra; but the converse does not hold* (see ([5], Example 2.4)).

The enveloping AO^*-algebra $O(A)$

For a locally convex *-algebra (A, t) (t denoting the topology of A), let $P_c(A, t)$ (respectively $P_{ca}(A, t)$) be the set of all continuous (respectively continuous admissible) representable positive functionals on A. For each f in $P_c(A, t)$, let $(\pi_f, \mathcal{D}(\pi_f), H_f)$ denote the strongly cyclic GNS representation defined by f as in Lemma 2.1. Let $I = \cap\{\ker \pi_f : f \in P_{ca}(A, t)\}$ and $J = \cap\{\ker \pi_f : f \in P_c(A, t)\}$. Then I and J are closed *-ideal of A, $J \subset I$, and $I = \operatorname{srad}(A)$ in view of the cyclic decomposability of any $\pi \in R(A)$. The *universal representation of* (A, t) is $\pi_u = \oplus\{\pi_f : f \in P_c(A, t)\}$. This is a slight variation of ([37], p. 228). Then $\sigma_u(x + J) = \pi_u(x)$ define a one–one *-homomorphism of A/J into the maximal O^*-algebra $\mathcal{L}^+(\mathcal{D}(\pi_u))$. Let $\sigma_u(t)$ be the topology on A/J induced by the uniform topology on $\pi_u(A)$; viz. $\sigma_u(t)$ is determined by the seminorms $\{q_M : M$ is a bounded subset of $(\mathcal{D}(\pi_u), t_{\pi_u})\}$, where $q_M(x + J) = \sup\{|\langle \pi_u(x)\xi, \eta\rangle| : \xi, \eta \text{ in } M\}$. Then $(A/J,\ \sigma_u(t))$ is an AO^*-algebra [36] in the sense that it is algebraically and topologically *-isomorphic to an O^*-algebra with uniform topology [37]. We call $(A/J,\ \sigma_u(t)$ the *enveloping AO^*-algebra* of A, denoted by $O(A)$.

Lemma 2.11. *Let A be as above.*

(1) *Every *-representation of A which is continuous in the uniform topology and which is a direct sum of strongly cyclic representations factors through $O(A)$. When A is either complete and m-convex, or is countably dominated, every *-representation of A continuous in the uniform topology factors through $O(A)$.*
(2) *Let A be barrelled. Then $\sigma_u(t)$ is coarser than the quotient topology t_q on A/J.*
(3) *There exists a continuous *-homomorphism from $O(A)$ into the pro-C^*-algebra $E(A)$.*
(4) *The following are equivalent.*
    (i) *$\sigma_u(t)$ is normable.*
    (ii) *$\sigma_u(t)$ is C^*-normable.*
    (iii) *There exists a linear norm on A/J defining a topology finer than $\sigma_u(t)$.*

When any of these conditions hold, and if A is barrelled, then A has a C^*-enveloping algebra; but the converse does not hold.

Proof. (1) follows from the construction of $O(A)$ and Lemma 2.8. (2) Let A be barrelled. Since J is closed. $(A/J, t_q)$ is barrelled ([32], ch. II, §7, Corollary 1, p. 61). Further, σ_u is

weakly continuous. Hence, σ_u is continuous in the uniform topology ([20], Theorem 4.1).
(3) Since $J \subset \operatorname{srad} A$, the map

$$\phi : A/J \to A/\operatorname{srad} A \to E(A), \phi(x+J) = x + \operatorname{srad} A$$

is a well defined *-homomorphism. Now, as $E(A)$ is a pro-C^*-algebra, $E(A)/\ker q_p$ is a C^*-algebra for any $p \in K(A)$, denoted by $E_p(A)$, with the norm $||z + \ker q_p|| = q_p(z)$ and $E(A) = \text{proj lim}\, E_p(A)$, inverse limit of C^*-algebras [26]. Let $\varphi_p : E(A) \to E_p(A)$ be $\varphi_p(z) = z + \ker q_p$. For the continuity of $\phi : (O(A), \sigma_u(t)) \to (E(A), \tau)$, it is sufficient to show the continuity of the *-homomorphism $\phi_p = \varphi_p \circ \phi : O(A) \to E_p(A)$. Now the map

$$\psi : A \to A/\operatorname{srad}(A) \to E(A) \to E_p(A), \psi(x) = (x + \operatorname{srad}(A)) + \ker q_p$$

is a continuous bounded operator *-representation; and $\psi = \phi_p \circ j_u$, $j_u(x) = x + I$. Hence ϕ_p is continuous for each $p \in K(A)$.
(4) (i) if and only if (ii) if and only if (iii) follows from ([20], Theorems 3.2, 3.3). Let A be barrelled. Let $|\ |$ be a norm on A/J determining $\sigma_u(t)$. Since $t_q \geq \sigma_u(t)$, $p_\infty(x) = |x + J|$ defines a continuous C^*-seminorm on A. Let p be any continuous C^*-seminorm on A. Let A_p be the completion of $A/\ker p$ in the C^*-norm $|x + \ker p| = p(x)$. Then $\pi_p : A \to A_p$, $\pi_p(x) = x + \ker p$ defines a continuous bounded operator *-representation. By (1), there exists a continuous *-homomorphism σ_p such that $\sigma_p \circ j_u = \pi_p$. Since the uniform topology on A_p is the $|\ |_p$-topology, and since $\sigma_u(t)$ is determined by $|\ |$, it follows that for some $k > 0$, $|\sigma_p(z)| \leq k|z|$ for all $z \in A/J$. Thus $p(x) \leq kp_\infty(x)$; and so $p(x) \leq p_\infty(x)$ for all $x \in A$, both being C^*-seminorms. Thus p_∞ is the greatest continuous C^*-seminorm on A. By Lemma 2.8, $E(A)$ is a C^*-algebra. That the converse does not hold is illustrated by Arens' algebra $A = L^\omega[0, 1]$, wherein $E(A) = (o)$, $O(A) = A$ topologically as well.

3. Proofs of theorems 1.1, 1.2 and 1.4

Proof of Theorem 1.2. First we prove the following.

Assertion I. Given a bounded subset M of $(\mathcal{D}(\pi), t_\pi)$, there exists $p \in K_s(A)$ and $k > 0$ such that $q_M(\pi(x)) \leq kr_p(x)$ for all $x \in A$.

By the continuity of π, given M, there exists $k > 0$ and $p \in K_s(A)$ such that $q_M(\pi(x)) \leq k\, p(x)$ for all $x \in A$. Let $\xi \in M$. Then

$$|\omega_\xi(x)| = |\langle \pi(x)\xi, \xi \rangle| \leq q_M(\pi(x)) \leq k\, p(x)$$

for all x. Since ω_ξ is representable, it is extendable to A^1. The arguments in the proof of Corollary 2.3(1) applied to the extension of ω_ξ to A^1 give

$$\omega_\xi(x^* x) \leq ||\xi||^2 p(x^* x) \leq ||\xi||^2 p(x)^2$$

for all x in A. Thus $||\pi_{\omega_\xi}(x)\xi|| \leq ||\xi|| p(x)$: and by the definition of r_p, $||\pi_{\omega_\xi}(x)\xi|| \leq ||\xi|| r_p(x)$ for all x in A. Since M is $||\ ||$-bounded, there exists $l > 0$ such that for all ξ in M, all x in A,

$$|\omega_\xi(x^* x)| = ||\pi_{\omega_\xi}(x)\xi||^2 \leq l^2 r_p(x)^2.$$

It follows that for all x in A, and all ξ, η in M,

$$|\langle \pi(x)\xi, \eta \rangle| \leq ||\eta|| \omega_\xi(x^* x)^{1/2} \leq l^2 r_p(x).$$

Thus $q_M(\pi(x)) \leq l^2 r_p(x)$ for all x in A.

Now, by Lemma 2.8, $\pi = \oplus \pi_i$, with each $\pi_i : A \to B(H_i)$ norm continuous. By Lemma 2.6, there exists a closed representation $(\sigma', \mathcal{D}(\sigma'), H)$ $\sigma' = \oplus \sigma_i$ of $E(A)$, with each $\sigma_i : E(A) \to B(H_i)$ norm continuous, $\sigma_i \circ j = \pi_i$ for all i. We shall eventually show $\mathcal{D}(\pi) = \mathcal{D}(\sigma')$.

On the other hand, consider the *-representation $(\sigma, \mathcal{D}(\sigma), H)$ of $A/\mathrm{srad}\,(A)$ having domain $\mathcal{D}(\sigma) = \mathcal{D}(\pi)$, and given by $\sigma(j(x)) = \pi(x)$ for all $x \in A$. By ([37], Proposition 2.2.3, p. 39), on $\mathcal{D}(\pi)$, $t_\pi = t_{\mathcal{L}^+(\mathcal{D}(\pi))}$ which is the graph topology on $\mathcal{D}(\pi)$ due to the maximal O^*-algebra $\mathcal{L}^+(\mathcal{D}(\pi))$. Hence, on $\pi(A)$, the uniform topology $\tau_{\mathcal{D}}^{\pi(A)} = \tau_{\mathcal{D}}^{\mathcal{L}^+(\mathcal{D}(\pi))}|_{\pi(A)} = \tau_1$ (say), which, by lemma 2.8, is a pro-C^*-topology. By ([37], Proposition 3.3.20, p. 85), $\sigma(A/\mathrm{srad}\,(A))$ is contained in a $\tau_{\mathcal{D}}^{\mathcal{L}^+(\mathcal{D}(\pi))}$-complete *-subalgebra of $\mathcal{L}^+(\mathcal{D}(\pi))$; and σ can be extended as a continuous *-homomorphism $\sigma(E(A), \tau) \to [\mathcal{L}^+(\mathcal{D}(\pi)), \tau_{\mathcal{D}}^{\mathcal{L}^+(\mathcal{D}(\pi))}]$ giving a closed *-representation σ of $E(A)$ on H with domain $D(\sigma) = D(\pi)$. Next we prove the following.

Assertion II. As representations of $E(A)$, $\sigma = \sigma'$.

This, we do, in the following steps.

(a) σ is an extension of σ'.

Clearly, $\mathcal{D}(\sigma') \subset \mathcal{D}(\pi) = \mathcal{D}(\sigma)$. We show $\sigma(z)|_{\mathcal{D}(\sigma')} = \sigma'(z)$ for all $z \in E(A)$. Fix $z \in E(A)$. Let $\eta \in \mathcal{D}(\pi)$. Choose a net (x_r) in A such that for all $p \in K_s(A)$, $q_p(j(x_r) - z) \to 0$. Choose an appropriate p by (I) above. Then

$$\begin{aligned} ||\sigma(j(x_r))\eta - \sigma(j(x_{r'}))\eta||^2 &= ||\pi(x_r)\eta - \pi(x_{r'})\eta||^2 \\ &= \omega_\eta((x_r - x_{r'})^*(x_r - x_{r'})) \\ &\le k\, r_p(x_r - x_{r'}) \\ &= k\, q_p(j(x)_r) - j(x_{r'})) \to 0. \end{aligned}$$

Hence $\pi(x_r)\eta$ is norm Cauchy in $\mathcal{D}(\pi)$; and similarly, $\pi(x)\pi(x_r)\eta$ is norm Cauchy in $\mathcal{D}(\pi)$ for all $x \in A$. Thus $\pi(x_r)\eta$ is Cauchy in $(\mathcal{D}(\pi), t_\pi)$, which is complete as π is closed. Thus there exists $\xi \in \mathcal{D}(\pi)$ such that $\lim(x_r)\eta = \xi$ in t_π. This defines $\sigma(z)$ as $\sigma(z)\eta = \xi$, which gives $\sigma(z)|_{\mathcal{D}(\sigma')} = \sigma'(z)$.

(b) σ is a closed representation of $E(A)$.

Indeed, as π is closed.

$$\begin{aligned} \mathcal{D}(\sigma) = \mathcal{D}(\pi) &= \bigcap \{\mathcal{D}(\overline{\pi_e(x)}) : x \in A^1\} \\ &= \bigcap \{\mathcal{D}(\overline{\sigma_e(j(x))} : j(x) \in j(A^1) = (j(A)^1)\} \\ &\supset \{\mathcal{D}(\overline{\sigma_e(z)}) : z \in (E(A))^1\} \\ &= \mathcal{D}(\bar{\sigma}) \supset \mathcal{D}(\sigma), \end{aligned}$$

hence $\mathcal{D}(\sigma) = \mathcal{D}(\sigma')$. This also follows from the fact that π is closed: on $\mathcal{D}(\sigma) = \mathcal{D}(\pi)$, $t_\pi = t_{\mathcal{L}^+(\mathcal{D}(\pi))} = t_{\sigma((E(A)^1))}$; as well as $\pi(A) \subset \sigma(E(A)) \subset \mathcal{L}^+(\mathcal{D}(\pi))$. This further implies $\tau_{\mathcal{D}}^{\mathcal{L}^+(\mathcal{D}(\pi))}|_{\sigma(E(A))} = \tau_{\mathcal{D}}^{\sigma(E(A))}$; which, in turn gives the following.

(c) σ' is continuous in the uniform topology as a *-representation of $(E(A), \tau)$.

Now, by (c), Lemma 2.8 implies that the closed representation σ' is standard; hence self-adjoint, and so maximal hermitian ([31], (I), Lemma 4.2). Then (a) gives $\sigma' = \sigma$, thereby verifying (II). This completes the proof of Theorem 1.2.

If π is irreducible, then σ is irreducible, hence is a bounded operator representation by [3], ([6], Theorem 4.7). This gives Corollary 1.3.

Proof of Theorem 1.1. Let A be Frechet. Then $A = \text{proj lim} A_n$, an inverse limit of a sequence of Banach *-algebras A_n. Assume that each *-representation (and hence the universal representation π_u) of A is a bounded operator representation. Since A is Frechet, π_u is continuous. Let σ be the representation of $E(A)$ defined by Theorem 1.2 corresponding to π_u. Then σ is also a bounded operator *-representation. Further, as A is Frechet, $E(A) = \text{proj lim}\, C^*(A_n)$ is also Frechet. Thus σ is continuous and there exists a continuous C^*-seminorm $q_\circ$ on $E(A)$ such that $||\sigma(z)|| \leq q_\circ(z)$ for all $z \in E(A)$. Now the bounded part of $E(A)$

$$b(E(A)) = \{z \in E(A) : q(z) < \infty \text{ for all continuous } C^*\text{-seminorm } q\}$$

is a C^*-algebra with the norm

$$\begin{aligned} ||z||_\infty &= \sup\{q(z) : q \text{ is a continuous } C^*\text{-seminorm on } E(A)\} \\ &= \sup\{q_p(z) : p \in K_s(A)\}. \end{aligned}$$

Since σ is one–one, the restriction $\sigma' = \sigma|_{b(E(A))}$ is a *-isomorphism of the C^*-algebra $b(E(A))$ into $B(H_\sigma)$. Hence, for all $z \in b(E(A))$,

$$||\sigma'(z)|| = ||z||_\infty \geq q_\circ(z) \geq ||\sigma(z)||.$$

It follows that $b(E(A)) = E(A)$. As $E(A)$ is Frechet, the continuous inclusion map $(b(A), ||\ \ ||_\infty) \to (E(A), \tau)$ is a homeomorphism. The converse follows from Theorem 1.2.

Proof of Theorem 1.4. By Corollary 2.3, $I = J = \text{srad}\,(A)$ in the notations of Lemma 2.9. Let $K = A/J$, a Frechet *-algebra in the quotient topology from A. By Lemma 2.8, the uniform topology $\tau_{\mathcal{D}}$ on $\pi_u(A)$ is a σ-C^*-topology; and the topology $\sigma_u(t)$ on K is determined by the (continuous) C^*-seminorms $\{s_G(\cdot) : G \in \mathcal{F}\}$, where $\mathcal{F}$ is the collection of all subspaces $\mathcal{D}$ of $\mathcal{D}(\pi_u)$ such that $\mathcal{D}$ is π_u-invariant and $\pi_u|_{\mathcal{D}}$ is a bounded operator *-representation; and $s_G(z) = ||\pi_u|_G(x)||$ for all $z = x + J$, $x \in A$. Thus $\sigma_u(t) \leq \tau$ where τ is the relative topology from $E(A)$ defined by all C^*-seminorms on $E(A)$. To show that $\tau \leq s_u(t)$, let $z_n = x_n + J \in K$, $z_n \to 0$ in $\sigma_u(t)$. Let q be any C^*-seminorm on A. There exists $\pi \in R(A)$ such that $q(x) = ||\pi(x)||$, and $\pi = \oplus\{\pi_f | f \in F_\pi\}$ for a suitable $F_\pi \subset P_c(A, t)$. Now $H_\pi = \oplus_{f \in F_\pi} H_f \subset \mathcal{D}(\pi_u)$, $H_\pi \in \mathcal{F}$, and $||\pi(x_n)|| = s_{H_\pi}(z_n) \to 0$. Hence $z_n \to 0$ in τ. Thus $\tau = \sigma_u(t)$, and $E(A) = (O(A), \sigma_u(t))$, the completion. The remaining assertion follows from Lemma 2.11.

4. Remarks

PROPOSITION 4.1

Let A be a *-sb Frechet *-algebra. If A is hermitian, then A is a Q-algebra.

Proof. We can assume that A is unital. Let P be a sequence of submultiplicative *-seminorms defining the topology of A. Let $A = \text{proj lim} A_q$ be the Arens–Michael decomposition expressing A as an inverse limit of a sequence of Banach *-algebras; where, for $q \in P$, A_q is the Banach *-algebra obtained by completing $A/\ker q$ in the norm $||x + \ker q|| = q(x)$. Let $\pi_q : A \to A_q$ be $\pi_q(x) = x + \ker q$.

Case 1. Assume that A is commutative. By hermiticity, $sp_A(h) = \{\phi(h) : \phi \in \mathcal{M}(A)\} \subset \mathbb{R}$ for all $h = h^* \in A$. Note that since A is hermitian. $\mathcal{M}(A) = \mathcal{M}^*(A)$. Using ([23], Proposition 7.5), it follows that for each q, $\mathcal{M}(A_q) = \mathcal{M}^*(A_q)$; hence by ([7], Theorem 35.3, p. 188), each A_q is hermitian. Now by ([17], Lemma 41.2, p. 225), for each $z \in A_q$, the spectral radius satisfies

$$r_{A_q}(z) \leq r_{A_q}(z^*z)^{1/2} = |z_q|_q,$$

$|\ |_q$ denoting the Gelfand–Naimark pseudonorm on A_q. Then $m_q(x) = |\pi_q(x)|_q$ defines a continuous C^*-seminorm on A. By Lemma 2.10, there exists a greatest continuous C^*-seminorm $p_\infty(\quad)$ on A. By ([23], Corollary 5.3), for each $x \in A$,

$$r_A(x) = \sup\{r_{A_q}(\pi_q(x))\} \leq \sup\{m_q(x)\} \leq p_\infty(x).$$

By the continuity of p_∞, there exists a $p \in K_s(A)$ and $k > 0$ such that for all x in A, $r(x) \leq p_\infty(x) \leq kp(x)$. It follows from ([23], Proposition 13.5) that A is a Q-algebra.

Case 2. Let A be non-commutative. Let M be a maximal commutative *-subalgebra of A containing the identity of A. Since M is spectrally invariant in A, M is also hermitian. By *-spectral boundedness and hermiticity, each positive functional on M can be extended to a positive functional on A ([17], Theorem 9.3, p. 49). It follows from ([15], Corollary 2.8) and the continuity of positive functionals on unital Frechet *-algebras, that for all $z \in M$, $p_\infty(z) = p^M_\infty(z) \leq r_M(z^*z)^{1/2}$, p^M_∞ being the greatest C^*-seminorm on M and $r_M(\cdot)$ denoting the spectral radius in M. Thus M is a commutative hermitian algebra with a C^*-enveloping algebra. By case 1, M is a Q-algebra. Further, M being hermitian, the Ptak's function $x \rightarrow r_M(x^*x)^{1/2}$ is a C^*-seminorm on M ([17], Corollary 8.3, p. 38; Theorem 8.17, p. 45).

Now let $x \in A$, and take M to be the maximal commutative *-subalgebra containing x^*x. Let $r_K(\cdot)$ denote the spectral radius in an algebra K. Then by Ptak's inequality in hermitian Frechet *-algebras ([17], Theorem 8.17, p. 45)

$$r_A(x) \leq r_A(x^*x)^{1/2} = r_M(x^*x)^{1/2} = p^M_\infty(x^*x)^{1/2} = p_\infty(x^*x)^{1/2} \leq q(x),$$

q being a *-algebra seminorm on A depending on p_∞ only. It follows from ([23], Proposition 13.5) that A is a Q-algebra.

(4.2) (i) It is claimed in ([5], Corollary 2.4) that a complete hermitian m-convex *-algebra with a C^*-enveloping algebra is a Q-algebra. Regrettably, there is a gap in the proof. The author sincerely thanks Prof. M Fragoulopoulou for pointing out this. It is implicitely used in the 'proof' therein that the completion of a hermitian normed algebra is hermitian. By Gelfand theory, this is certainly true in the commutative case, but is not true in non-commutative case (see ([17], p. 18)). Thus ([15], Corollary 2.4) remains valid in commutative case; and the above proposition partially repairs the gap in the non-commutative case. Consequently ([15], Lemma 2.15, Theorem 2.14) remains valid for Frechet algebras. Is a hermitian Frechet algebra with a C^*-enveloping algebra a Q-algebra? (ii) The algebra $C(\mathbb{R})$ of continuous functions on $\mathbb{R}$ exhibits that the condition *-*sb* can not be omitted from the above proposition. It also follows from above that a *-*sb* σ-C^*-algebra is a C^*-algebra.

(4.3) In Theorem 1.2, the assumption that π is closed can not be omitted. Let $A = C^\infty(\mathbb{R})$, the Frechet *-algebra of C^∞ functions on $\mathbb{R}$, with pointwise operations and the topology of uniform convergence on compact subsets of $\mathbb{R}$ of functions as well as their derivatives. Then $E(A) = C(\mathbb{R})$, the algebra of continuous functions on $\mathbb{R}$ with the compact open topology. On the Hilbert space $H = L^2(\mathbb{R})$, the *-representation π of A with $\mathcal{D}(\pi) = C_c^\infty(\mathbb{R})$, $\pi(a)f = af$, cannot be extended to a *-representation of $C(\mathbb{R})$ with the same domain ([10], Example 4.7).

(4.4) Theorem 1.1 means that a Fechet *-algebra has a C^*-enveloping algebra if and only if it is a BG^*-algebra [24]. In the non-metrizable case, it follows from Theorem 1.2 that if A is a complete topological m-convex *-algebra with a C^*-enveloping algebra, then every *-representation of A which is continuous in the uniform topology is a bounded operator representation. However, the converse does not hold. This is exhibited by the BG^*-algebra $C[0,1]$ of continuous functions on [0,1] with the pro-C^*-topology τ of uniform convergence on all countable compact subsets of [0,1]. Thus Theorem 1.1 is false without the assumption that A is Frechet. It would be of interest to find an example of a topological algebra with a C^*-enveloping algebra which is not a BG^*-algebra.

(4.5) Yood [42] has shown that a *-algebra A admits a greatest C^*-seminorm if and only if $\sup|f(x)| < \infty$ for each x, where the sup is taken over all admissible states S; and by Lemma 2.10, this happens for a Frechet A if and only if A has a C^*-enveloping algebra. Yood's result is an algebraic version of ([5], Corollary 2.9) that states that a complete m-convex algebra has a C^*-enveloping algebra if and only if S is equicontinuous.

(4.6) (i) Let π be a *-representation of a complete locally m-convex *-algebra A with a bounded approximate identity. Let A have a C^*-enveloping algebra. Is π continuous in the uniform topology? In particular, let π be a bounded operator *-representation. Is π norm-continuous?

(ii) Let A be a pro-C^*-algebra (more generally, a complete m-convex *-algebra with a bounded approximate identity). Let f be a representable, not necessarily continuous, positive functional on A. Is the GNS representation π_f a bounded operator representation? Is every *-representation of A weakly unbounded?

These are motivated by the point of view ([5], Remark 2.11, p. 207) that a topological *-algebras with a C^*-enveloping algebra provide a hermitian analogue of a commutative Q-algebra. It is easy to see that a *-representation π of a locally convex Q-algebra is a bounded operator representation and is norm continuous.

5. Crossed product constructions

We recall the crossed product of a C^*-dynamical system (G, A, α). Let α be a strongly continuous action of a locally compact group G by *-automorphisms of a C^*-algebra A. Let $C_c(G, A)$ be the vector space of all continuous A-valued functions with compact supports. It is a *-algebra with twisted convolution

$$x * y(g) = \int_G x(h)\alpha_h(y(h^{-1}g))\mathrm{d}h$$

and the involution $x^*(g) = \Delta(g)^{-1}\alpha_g(x(g^{-1}))^*$. The Banach *-algebra $L^1(G, A)$ is the completion of $C_c(G, A)$ in the norm $||x||_1 = \int_G ||x(h)||\mathrm{d}h$; and the crossed product C^*-algebra $C^*(G, A, \alpha)$ is the completion of $L^1(G, A)$ in its Gelfand–Naimark pseudonorm $||x|| = \sup\{||\pi(x)|| : \pi \in R(L^1(G, A))\}$, which is, in fact, a norm. Thus it is the enveloping C^*-algebra of the Banach *-algebra $L^1(G, A)$. The C^*-algebra $C^*(G, A, \alpha)$

can also be realized as the enveloping C^*-algebra of non-normed topological *-algebras smaller than $L^1(G,A)$.

Let $\mathscr{K}$ be the collection of all compact, symmetric neighbourhoods of the identity in G. For $K \in \mathscr{K}$, let $C_K(G,A) = \{f \in C_c(G,A) : \operatorname{supp} f \subseteq K\}$, a Banach space with the norm $||f|| = \sup\{||f(x)|| : x \in K\}$. The inductive limit topology τ on $C_c(G,A)$ is the finest locally convex topology on $C_c(G,A)$ making each of the embeddings $C_K(G,A) \to C_c(G,A)$, for all $K \in \mathscr{K}$, continuous. Then $C_c(G,A)$ is a locally convex, non- m-convex, topological *-algebra with jointly continuous multiplication and continuous involution. From ([18], p. 203), $E(C_c(G,A)) = C^*(G,A,\alpha)$. This immediately leads to the following.

PROPOSITION 5.1

Let (G,A,α) be a C^*-dynamical system. Let B be any topological *-algebra containing $C_c(G,A)$ as a dense *-subalgebra and satisfying $C_c(G,A) \subseteq B \subseteq C^*(G,A,\alpha)$. Then $E(B) = C^*(G,A,\alpha)$.

For $1 \leq p < \infty$, let $A^p(G,A) = L^1(G,A) \cap L^p(G,A)$, a Banach *-algebra with the norm $|x|_p = ||x||_1 + ||x||_p$. The above applies to $B = \bigcap\{A^p(G,A) : 1 \leq p < \infty\}$, a locally m-convex Q-Frechet *-algebra with the topology of $|\ |_p$-convergence for each p.

Smooth elements of a Lie group action

Let A be a unital C^*-algebra and G be a Lie group acting on A. Let Δ denote the infinitesimal generators of actions of 1-parameter subgroups of G on A, viz.,

$$\Delta = \{(\mathrm{d}/\mathrm{d}t)\alpha_{u(t)}|_{t=0} : t \to u(t) \text{ is a continuous homomorphism of } \mathbb{R} \text{ into } G\}.$$

Then Δ consists of derivations and it is a finite dimensional vector space ([11], p. 40) having basis, say $\delta_1, \delta_2, \ldots, \delta_d$. Then *$C^n$-elements* ($1 \leq n < \infty$) and *$C^\infty$-elements* of A for the action α are defined as follows.

$$C^n(A) = \{x \in A : x \in \operatorname{Dom}(\delta_{i_1}\delta_{i_2}\ldots\delta_{i_n}) \text{ for all } n\text{-tuples } \{\delta_{i_1},\ldots,\delta_{i_n}\} \text{ in } \Delta\}$$
$$C^\infty(A) = \bigcap\{C^n(A) : n \in \mathbb{N}\}.$$

By ([11], Proposition 2.2.1), each $C^n(A)$ and $C^\infty(A)$ are dense *-subalgebras of A; and $C^n(A)$ is a Banach *-algebra with the norm

$$||x||_n = ||x|| + \sum_{k=1}^{n} \sum_{i_1,i_2,\ldots,i_k=1}^{d} ||\delta_{i_1}\delta_{i_2}\ldots\delta_{i_k}(x)||/k!.$$

Then $C^\infty(A) = \operatorname{proj}\lim C^n(A)$ is a Frechet *-algebra with the topology defined by the norms $\{||\ ||_n : n = 1,2,\ldots\}$.

Lemma 5.2. *$C^\infty(A)$ has a C^*-enveloping algebra and $E(C^\infty(A)) = A$.*

Proof. It is well known that $C^n(A)$ and $C^\infty(A)$ are spectrally invariant in A. Hence $(C^n(A), ||\ ||)$ and $(C^\infty(A), ||\ ||)$ are Q-algebras in the norm $||\ ||$ from the C^*-algebra A. Since $||\ || \leq ||\ ||_n$, $(C^\infty(A),\tau)$ is also a Q-algebra. By Lemma 2.10, $(C^\infty(A),\tau)$ is an algebra with a C^*-enveloping algebra. Let $\pi : B \to B(H)$, where $B = C^n(A)$ or $C^\infty(A)$, be a bounded operator *-representation on a Hilbert space H. Then for all $x \in B$,

$$\begin{aligned}||\pi(x)||^2 &= ||\pi(x^*x)|| = r_{B(H)}(\pi(x^*x)) \leq r_{\pi(B)}(\pi(x^*x)) \\ &\leq r_B(x^*x) \leq ||x||^2.\end{aligned}$$

Hence π is $||\ \ ||$-continuous; and by the density of $C^\infty(A)$ in A, π extends uniquely to a *-representation of A on H. It follows that $E(C^\infty(A)) = C^*(C^n(A)) = A$ for all n.

An element $x \in A$ is *analytic* if $x \in C^\infty(A)$ and there exists a scalar $t > 0$ such that

$$\sum_{k=0}^{\infty}\left(\sum_{i_1,i_2,\ldots,i_k=1}^{d} ||\delta_{i_1}\delta_{i_2}\cdots\delta_{i_k}(x)||/k!\right)t^k < \infty;$$

whereas x is *entire* if $x \in C^\infty(A)$ and for all $t > 0$, it holds that

$$\sum_{k=0}^{\infty}\left(\sum_{i_1,i_2,\ldots,i_k=1}^{d} ||\delta_{i_1}\delta_{i_2}\cdots\delta_{i_k}(x)||/k!\right)t^k < \infty.$$

Let $C^\omega(A)$ (respectively $C^{e\omega}(A)$) denote the set of all analytic (respectively entire) elements of A. Then each of $C^\omega(A)$ and $C^{e\omega}(A)$ is a *-subalgebra of A and $C^{e\omega}(A) \subset C^\omega(A) \subset C^\infty(A)$. For each $t > 0$ and $x \in C^n(A)$, define

$$p_n^t(x) = ||x|| + \sum_{k=1}^{n}\left(\sum_{i_1,i_2,\ldots,i_k=1}^{d} ||\delta_{i_1}\cdots\delta_{i_k}(x)||/k!\right)t^k.$$

Then $||\ \ ||_n$ and $p_n^t(\ \ )$ are equivalent norms. Hence $P^t = (p_n^t(\ \ ))$ and $p = (||\ \ ||_n)$ define the same C^∞-topology τ on $C^\infty(A)$. Let $A_t = \{x \in C^\infty(A) : p^t(x) = \sup_k p_k^t(x) < \infty\}$, a *-subalgebra of $C^\infty(A)$, which is a Banach *-algebra with norm $p^t(\ \ )$, and which consists of elements of $C^\infty(A)$ whose numerical ranges defined with respect to P^t are bounded. For $t < s$, the inclusion $A_s \to A_t$ is norm decreasing. Thus

$$C^{e\omega}(A) = \bigcap\{A_t : t > 0\} = \bigcap_{n=1}^{\infty} A_n = \text{proj lim} A_n,$$

a Frechet m-convex, *-algebra with the topology $\tau_{e\omega}$ defined by the family of norms $\{p^t(\ \ ) : t \in \mathbb{N}\}$ (setting $p^\circ(\ \ ) = ||\ \ ||$). Further,

$$C^\omega(A) = \bigcup_{t>0} A_t = \bigcup_{n=1}^{\infty} A_{1/n} = \text{ind lim} A_{1/n}$$

with the linear inductive limit topology τ_ω. By ([21], Corollary 10.2, Lemma 10.2, p. 317) and ([32], Proposition 6.6, p. 59), $(C^\omega(A), \tau_\omega)$ is a complete m-convex *-algebra which is a Q-algebra. Thus $C^\omega(A)$ is an algebra with a C^*-enveloping algebra. Further if each A_t is dense and spectrally invariant in $C^\infty(A)$, then $C^{e\omega}(A)$ is an algebra with a C^*-enveloping algebra and $E(C^{e\omega}(A)) = E(C^\omega(A)) = A$.

The smooth crossed product

We recall the smooth Frechet algebra crossed product [29]. Let B be a Frechet *-algebra. Let (p_n) be a sequence of submultiplicative *-seminorms defining the topology of B. Let β be a strongly continuous action of $\mathbb{R}$ by continuous *-automorphisms of B. Then β is called *m-tempered* (respectively *isometric*) if for each $m \in \mathbb{N}$, there exists a polynomial

$P(X)$ such that $p_m(\beta_r(x)) \leq P(r)p_m(x)$ for all $x \in B$, $r \in \mathbb{R}$ (respectively for each $m \in \mathbb{N}$, $p_m(\beta_r(x)) = p_m(x)$ for all $x \in B$, all $r \in \mathbb{R}$). Let $S(\mathbb{R})$ be the Schwartz space consisting of the rapidly decreasing C^∞-functions on $\mathbb{R}$. It is a Frechet space with the Schwartz topology. The completed projective tensor product $S(\mathbb{R}) \otimes B = S(\mathbb{R}, B)$ consists of B-valued Schwartz functions on $\mathbb{R}$. If β is m-tempered, then $S(\mathbb{R}, B)$ becomes an m-convex Frechet algebra with twisted convolution

$$(f * g)(r) = \int_{\mathbb{R}} f(s)\beta_s(g(r-s))\mathrm{d}s.$$

This Frechet algebra is called the *smooth Schwartz crossed product* of B by the action β of $\mathbb{R}$, and is denoted by $S(\mathbb{R}, B, \beta)$. In general, $S(\mathbb{R}, B, \beta)$ need not be a *-algebra ([34], §4). If β is isometric, then the completed projective tensor product

$$\begin{aligned} L^1(\mathbb{R}) \otimes B &= L^1(\mathbb{R}, B) \\ &= \{f : \mathbb{R} \to B \text{ measurable function} : \int_{\mathbb{R}} p_m(f(r))\mathrm{d}r < \infty \text{ for all } m \in \mathbb{N}\} \end{aligned}$$

is a Frechet *-algebra with twisted convolution and the involution $f^*(r) = \beta_r(f(-r)^*)$, denoted by $L^1(\mathbb{R}, B, \beta)$. One has $S(\mathbb{R}, B, \beta) \subset L^1(\mathbb{R}, B, \beta)$.

The following is closely related with ([29], Lemma 1.1.9).

Lemma 5.3. *Let A be a dense Frechet *-subalgebra of a Frechet *-algebra B. Assume that A and B can be expressed as inverse limits of Banach *-algebras A_n and B_n respectively, where A_n is dense in B_n for all n; the inclusions $A \to A_n$, $B \to B_n$ have dense ranges for all n; and each A_n is spectrally invariant in B_n. Then A is spectrally invariant in B and $E(A) = E(B)$.*

Proof. By ([15], Theorem 4.3), $E(A) = \text{proj lim}\, E(A_n)$ and $E(B_n) = \text{proj lim}\, E(B_n)$. Since $A_n \to B_n$ is spectrally invariant with dense range, A_n is a Q-normed algebra in the norm of B_n. Hence every C^*-seminorm on A_n is continuous in the norm of B_n; and extends uniquely to B_n. Thus A_n and B_n have the same collection of C^*-seminorms. It follows that $E(A_n) = E(B_n)$ for all n; and so $E(A) = E(B)$.

PROPOSITION 5.4

Let α be an m-tempered strongly continuous action of $\mathbb{R}$ by continuous *-automorphisms of a Frechet *-algebra B contained as a dense *-subalgebra of a C^*-algebra A such that $E(B) = A$. Then $E(C^\infty(B)) = A$.

Proof. Let $\|\ \|$ denote the C^*-norm on A. Let (p_n) be an increasing sequence of submultiplicative *-seminorms defining the topology of B. In view of the continuity of the inclusion $B \to A$, the increasing sequence $q_n(\ ) = p_n(\ ) + \|\ \|$ of norms also determines the topology of B. Let $B_n = (B, q_n)$ be the completion, which is a Banach *-algebra. Then $B = \text{proj.}\lim B_n = \bigcap B_n$. Now, for any $n \in \mathbb{N}$, $r \in \mathbb{R}$, and $x \in B$,

$$\begin{aligned} q_n(\alpha_r(x)) &= \|\alpha_r(x)\| + p_n(\alpha_r(x)) \\ &= \|x\| + \text{poly}\,(r)p_n(x) = \text{poly}'\,(r)q_n(x) \end{aligned}$$

for some polynomial poly'( ). It follows that α is m-tempered for $(q_n(\ ))$ also; and it induces an action $\alpha^{(n)}$ of $\mathbb{R}$ by continuous *-automorphisms of B_n. Let $B_{n,m}$ be the Banach

*-algebra consisting of all C^m-vectors in B_n for $\alpha^{(n)}$. By ([33], Theorem 2.2), $B_{n,m} \to B_n$ are spectrally invariant embeddings with dense ranges. Also, $C^\infty(B) = \text{proj lim}_{n,m} B_{n,m} = \text{proj lim}_n B_{n,n}$. Now Lemma 5.2 implies that $C^\infty(B)$ is spectrally invariant in B and $E(C^\infty(B)) = A$.

PROPOSITION 5.5

Let α be a strongly continuous action of $\mathbb{R}$ by *-automorphisms of a C^*-algebra A. The following hold.

(a) *The Frechet algebras $S(\mathbb{R}, A, \alpha)$ and $S(\mathbb{R}, C^\infty(A), \alpha)$ are Q-algebras.*
(b) *The embeddings $S(\mathbb{R}, C^\infty(A), \alpha) \to S(\mathbb{R}, A, \alpha) \to C^*(\mathbb{R}, A, \alpha)$ are continuous, spectrally invariant and have dense ranges.*
(c) *The Frechet algebra $S(\mathbb{R}, C^\infty(A), \alpha)$ is *-algebra and $E(S(\mathbb{R}, C^\infty(A), \alpha) = C^*(\mathbb{R}, A, \alpha)$.*

Proof. By ([34], Theorem A.2), α leaves $C^\infty(A)$ invariant. In ([34], Corollary 4.9), taking the scale σ to be the weight $w(r) = 1 + |r|$ on $G = \mathbb{R} = H$, it follows that $S(\mathbb{R}, C^\infty(A), \alpha)$, is a Frechet *-algebra. Now $\tilde{\alpha}_s(f)(r) = \alpha_s(f(r))$ defines an action $\tilde{\alpha}$ of $\mathbb{R}$ on the Frechet algebra $S(\mathbb{R}, A, \alpha)$ for which, by ([29], p. 189), $C^\infty(S(\mathbb{R}, A, \alpha)) = S(\mathbb{R}, C^\infty(A), \alpha)$ homeomorphically. Note that the embeddings

$$S(\mathbb{R}, C^\infty(A), \alpha) \to S(\mathbb{R}, A, \alpha) \to L^1(\mathbb{R}, A, \alpha) \to C^*(\mathbb{R}, A, \alpha)$$

are continuous; $S(\mathbb{R}, C^\infty(A), \alpha)$ is dense in $S(\mathbb{R}, A, \alpha)$ by ([34], Theorem A.2); and $S(\mathbb{R}, A, \alpha)$ is dense in $L^1(\mathbb{R}, A, \alpha)$; which, in turn, is dense in $C^*(\mathbb{R}, A, \alpha)$.

Now let $\{|\ |_n\}$ be an increasing sequence of submultiplicative seminorms defining the topology of $S(\mathbb{R}, A, \alpha)$. Let $(B_n, |\ |_n)$ be the Hausdorff completion of $S(\mathbb{R}, A, \alpha)$ in $|\ |_n$. Then B_n is a Banach algebra and $S(\mathbb{R}, A, \alpha) = \text{proj.} \lim B_n$. Since $||\alpha_r(x)|| = ||x||$, the action $\tilde{\alpha}$ of $\mathbb{R}$ on $S(\mathbb{R}, A, \alpha)$ extends to a strongly continuous action $\tilde{\alpha}^{(n)}$ of $\mathbb{R}$ by automorphisms of B_n. Let $C^m(B_n)$ be the Banach algebra of all C^m-vectors in B_n for the action of $\tilde{\alpha}^{(n)}$. As noted in ([29], p. 189), $C^n(B_n)$ is dense and spectrally invariant in B_n; and $S(\mathbb{R}, C^\infty(A), \alpha) = \text{proj lim}\, C^n(B_n)$. Let $x \in S(\mathbb{R}, C^\infty(A), \alpha)$, $x = (x_n)$ being a coherent sequence with $x_n \in C^n(B_n)$ for all $n \in \mathbb{N}$. Now

$$sp_{S(\mathbb{R}, C^\infty(A), \alpha)}(x) = \bigcup_n sp_{C^n(B_n)}(x_n) = \bigcup_n sp_{B_n}(x_n) = sp_{S(\mathbb{R}, A, \alpha)}(x).$$

Thus $S(\mathbb{R}, C^\infty(A), \alpha)$ is spectrally invariant in $S(\mathbb{R}, A, \alpha)$; which in turn is spectrally invariant in $C^*(\mathbb{R}, A, \alpha)$ by ([33], Corollary 7.16). Thus each of $S(\mathbb{R}, C^\infty(A), \alpha)$ and $S(\mathbb{R}, A, \alpha)$ are Q-normed algebras in the C^*-norm of $C^*(\mathbb{R}, A, \alpha)$; and hence are Q-algebras in their respective Frechet topologies. Using Lemma 2.10, $E(S(\mathbb{R}, C^\infty(A), \alpha)) = C^*(\mathbb{R}, A, \alpha)$.

Proof of Theorem 1.5. Since $C^\infty(B) = B$, the Frechet m-convex algebra $S(\mathbb{R}, B, \alpha)$ is a *-algebra by ([34], Corollary 4.9). Since B is Frechet and sits in the C^*-algebra A, B is *-semisimple. Similarly, since the inclusion $S(\mathbb{R}, B, \alpha) \to C^*(\mathbb{R}, A, \alpha)$ is continuous and one–one, $S(\mathbb{R}, B, \alpha)$ is also *-semisimple. To prove that $E(S(\mathbb{R}, B, \alpha)) = C^*(\mathbb{R}, A, \alpha)$, it is sufficient to prove that any *-representation $\sigma : S(\mathbb{R}, B, \alpha) \to B(H_\sigma)$ extends to a *-representation $(\tilde{\sigma}) : C^*(\mathbb{R}, A, \alpha) \to B(H_\sigma)$. This would imply that the C^*-norm on $S(\mathbb{R}, B, \alpha)$ induced by the C^*-algebra norm on $C^*(\mathbb{R}, A, \alpha)$ is the greatest (automatically continuous) C^*-seminorm on $S(\mathbb{R}, B, \alpha)$. This is shown below by arguments analogous to those in ([25], Proposition 7.6.4, p. 255).

Let (x_λ) be a bounded approximate identity for A contained in B and which is also a bounded approximate identity for B. For each $n \in \mathbb{N}$, let $f_n \in C_c^\infty(\mathbb{R})$ be such that $0 \leq f_n \leq 1$, $f_n(x) = 1$ for all $x \in [-n, n]$, and supp $f_n \subset [-n-1, n+1]$. Then (f_n) is a bounded approximate identity for $S(\mathbb{R})$ (pointwise multiplication) contained in $C_c^\infty(\mathbb{R})$. The inverse Fourier transforms g_n of f_n constitute a bounded approximate identity for $S(\mathbb{R})$ with convolution. Thus $y_{n,\lambda} = g_n \otimes x_\lambda$ constitute a bounded approximate identity for $S(\mathbb{R}, B, \alpha)$. Given a *-representation $\sigma : S(\mathbb{R}, B, \alpha) \to B(H_\sigma)$ automatically continuous, let $\mathcal{U}(H_\sigma)$ be the group of all unitary operators on H_σ. Define $\pi : B \to B(H_\sigma)$ and $U : \mathbb{R} \to \mathcal{U}(H_\sigma)$ by

$$\pi(x) = \lim_{(n,\lambda)} \sigma(xy_{(n,\lambda)}(\quad)),$$
$$U_t = \lim_{(n,\lambda)} \sigma(\alpha_t(y(\cdot - t))).$$

The limits are taken in the weak sense; and they exist. As in ([25], § 7.6, p. 256), it is verified that π is a *-representation of B; U is a unitary representation of $\mathbb{R}$; $U_t\pi(x)U_t^* = \pi(\alpha_t(x))$ for all $t \in \mathbb{R}$, all $x \in B$; and for all $y \in S(\mathbb{R}, B, \alpha)$, $\sigma(y) = \int \pi(y(t))$ $U_t\mathrm{d}t$. Now, since $E(B) = A, \pi$ extends to a *-representation $\tilde{\pi} : A \to B(H_\sigma)$ so that $(\tilde{\pi}, U, H_\sigma)$ is a covariant representation of the C^*-dynamical system $(\mathbb{R}, A, \alpha)$. Then $\tilde{\sigma}(y) = \int \tilde{\pi}(y(t))U_t\,\mathrm{d}t$ defines a non-degenerate *-representation of the Banach *-algebra $L^1(\mathbb{R}, B, \alpha)$; and hence extends uniquely to a *-representation $\tilde{\sigma}$ of $C^*(\mathbb{R}, B, \alpha)$. This $\tilde{\sigma}$ is the desired extension of σ. This shows that $E(S(\mathbb{R}, B, \alpha)) = C^*(\mathbb{R}, A, \alpha)$.

Further, suppose that the action α of $\mathbb{R}$ on B is isometric. Then by [29], $L^1(\mathbb{R}, B, \alpha)$ is a *-algebra, which is a Frechet m-convex *-algebra; and

$$S(\mathbb{R}, B, \alpha) \to L^1(\mathbb{R}, B, \alpha) \to L^1(\mathbb{R}, A, \alpha) \to C^*(\mathbb{R}, A, \alpha)$$

are continuous embeddings with dense ranges. It follows that $E(L^1(\mathbb{R}, B, \alpha) = C^*(\mathbb{R}, A, \alpha)$. This completes the proof of the theorem.

Actions on topological spaces

(a) Let M be a locally compact Hausdorff space. Let $\sigma : M \to [0, \infty)$ be a Borel function, $\sigma(m) \geq 1$ for all $m \in M$. Assume that σ is bounded on compact subsets of M. Following ([34], § 5), let

$$C^\sigma(M) = \{f \in C_0(M) : ||\sigma^d f|| < \infty \text{ for all } d \in \mathbb{N}\},$$

called the algebra of continuous functions on M vanishing at infinity σ-rapidly. It is shown in [34] that $C^\sigma(M)$ is a Frechet m-convex *-algebra with the topology defined by seminorms

$$||\sigma^d f|| = \sup\{|(\sigma(x))^d f(x)| : x \in M\}, \quad \mathrm{d} \in \mathbb{N};$$

and that $C_c(M) \to C^\sigma(M) \to C_0(M)$ are continuous embeddings with dense ranges. Thus $E(C^\sigma(M)) = C_0(M)$. In fact, $C^\sigma(M)$ is an ideal in $C_0(M)$; hence inverse closed in $C_0(M)$; and so is a Q-algebra.

(b) Let G be a Lie group acting on M. If $f \in C^\sigma(M)$, define $\alpha_g(f)(m) = f(g^{-1}m)$. By ([34], § 5), if σ is uniformly G-translationally equivalent (in the sense that for every

compact $K \subset G$, there exists $l \in \mathbb{N}$ and $C > 0$ such that $\sigma(gm) \leq C\sigma(m)^l$ for all $g \in G$, $m \in M$), then $g \to \alpha_g$ defines a strongly continuous action of G by continuous *-automorphisms of $C^\sigma(M)$. Then the space $C^\infty(C^\sigma(M))$ consisting of C^∞-vectors for the action α of G on $C^\sigma(M)$ is an m-convex Frechet *-algebra with a C^*-enveloping algebra and $E(C^\infty(C^\sigma(M)) = C_0(M)$.
(c) In particular, let $G = \mathbb{R}$, M be a compact C^∞-manifold, and let the action of $\mathbb{R}$ on M be smooth. Then the induced action α on $C(M)$ is smooth, so that $\alpha_r(C^\infty(M)) \subseteq C^\infty(M)$ for all $r \in \mathbb{R}$. It follows from Theorem 5.1 that $E(S(\mathbb{R}, C^\infty(M), \alpha) = C^*(\mathbb{R}, C(M), \alpha)$ the covariance C^*-algebra.

6. The Pedersen ideal of a C^*-algebra

Let A be a non-unital C^*-algebra. Let K_A be its Pedersen ideal. It is a hereditary, minimal dense *-ideal of A. For $a \in A$, let $L_a = (Aa)^-$, $R_a = (aA)^-$, I_a be the closed *-ideal of A generated by aa^*. Since $a \in L_a \bigcap R_a$, $aa^* \in I_a$. Let $K_A^+ = K_A \bigcap A^+$ be the positive part of K_A endowed with the order relation induced from that of A^+. Let $K_A^{nc} = \bigcup\{I_a : a \in K_A^+\}$.

Lemma 6.1. *K_A^{nc} is a dense *-ideal of A containing K_A; and $A = C^*$-ind lim $\{I_a : a \in K_A^+\}$.*

Proof. Let $a \in K_A^+$. Then $a^2 = aa^* \in I_a$; and I_a being a C^*-algebra, $a = (a^2)^{1/2} \in I_a$. Thus $K_A^+ \subseteq K_A^{nc}$. Observe that for any $x = x^* \in K_A$, $x \in I_x$. Indeed, $x^2 \in K_a^+$; hence $x^2 \in I_x$ and $|x| \in I_x$. But than taking the Jordan decomposition $x = x^+ - x^-$ in A, $(x^+)^2 = (x^+)^2 + x^+x^- = x^+|x| \in I_x$; so that $x^+ \in I_x$, $x^- \in I_x$, and $x \in I_x$. In particular, $x^2 \in I_{x^2}$ and $|x| \in I_{x^2}$. By repeating this argument, $x \in I_{x^2} \subset K_A^{nc}$ for any $x = x^* \in K_A$. It follows that $K_A \subset K_A^{nc}$. Now, by ([28], Lemma 1), $0 \leq a \leq b$ in A implies $L_a \subseteq L_b$, $R_a \subseteq R_b$ and $I_a \subseteq I_b$; and $K_A = \bigcup\{L_a : a \in K_A^+\} = \bigcup\{R_a : a \in K_A^+\}$. The family $\{I_a : a \in K_A^+\}$ forms an inductive system of C^*-algebras; and C^*-ind $\lim\{I_a : a \in K_A^+\} = (\bigcup\{I_a : a \in K_A^+\})^- = A$, $(\quad)^-$ denoting the norm closure. This proves the lemma.

Let t_1 (respectively t_2) be the finest locally convex linear topology (respectively finest locally m-convex topology) on K_A^{nc} making continuous the embeddings $I_a \to K_A^{nc}$, where $a \in K_A^+$. Then (K_A^{nc}, t_1) (respectively (K_A^{nc}, t_2)) is the *linear topological inductive limit* (respectively *topological algebraic inductive limit*) of $\{I_a : a \in K_A^+\}$ ([21], ch. IV).

Proof of Theorem 1.6. In the present set up, ([21], p. 115, 118, 125) implies that $t_1 = t_2$, equal to τ say, and (K_A^{nc}, τ) is a complete m-barrelled locally m-convex *-algebra; and the $\|\ \|$-topology on K_A^{nc} is coarser than τ. Since K_A^{nc} is an ideal, it is inverse closed in its $\|\ \|$-completion A, and hence $(K_A^{nc}, \|\ \|)$ and $(K_A, \|\ \|)$ are Q-algebras. This implies that any *-homomorphism from K_A^{nc} into $B(H)$ for a Hilbert space H is $\|\ \|$-continuous and extends uniquely to A. Thus $\|\ \|$ is the greatest C^*-seminorm on K_A^{nc}. To show that $\|\ \|$ is the greatest τ-continuous C^*-seminorm on K_A^{nc} so that $E(K_A^{nc}) = A$, it is sufficient to show that (K_A^{nc}, τ) is a Q-algebra. To that end, in view of ([23], Lemma E.2), we show that 0 is a τ-interior point of the set $(K_A^{nc})_{-1}$ of quasiregular elements of K_A^{nc}. Note that, by ([21], p. 114), basic τ-neighbourhoods of 0 in K_A^{nc} are precisely of the form $V = |co|$ $\{\bigcup(U_a : a \in K_A^+)\}$, where $|co|$ denotes the absolutely convex hull and U_a denotes a convex balanced neighbourhood of 0 in $(I_a, \|\ \|)$. For any $a \in K_A^+$, $(I_a, \|\ \|)$ is a Q-algebra, and being an ideal in A, $(I_a)_{-1} = A_{-1} \bigcap I_a$. Hence, for the zero neighbourhood $U_a = \{x \in I_a : \|x\| \leq 1\}$ in $(I_a, \|\ \|)$,

$$U_a \subseteq (I_a)_{-1} = (K_A^{nc})_{-1} \bigcap I_a \subset (K_A^{nc})_{-1}; \text{ and}$$
$$|c\, o|\Big\{\bigcup(U_a : a \in K_A^{+})\} = \{x \in K_A^{nc} : ||x|| \leq 1\Big\} = U \text{ (say)}$$

is a zero neighbourhood in (K_A^{nc}, τ) contained in $(K_A^{nc})_{-1}$. It follows that (K_A^{nc}, τ) and (K_A, τ) are Q-algebras. Now, as in the proof of ([28], Theorem 4), $K_A^{nc} = \cup\{I_{e_\lambda}\}$, (e_λ) being a bounded approximate identity for A contained in K_A. Thus if A has countable bounded approximate identity, then K_A^{nc} is an LFQ-algebra; and τ is the finest (unique) locally convex topology on K_A^{nc} such that for each λ, $\tau|_{I_{e_\lambda}}$ is the norm topology.

7. The groupoid C^*-algebra

We follow the terminology and notations of [31]. Let G be a locally compact groupoid, i.e., a locally compact space G with a specified subset $G^2 \subseteq G \times G$ so that two continuous maps $G \to G$, $x \to x^{-1}$, and $G^2 \to G$, $(x, y) \to xy$ are defined satisfying $(xy)z = x(yz)$, $x^{-1}(xy) = y$ and $(zx)x^{-1} = z$. The unit space of G is $G^o = \{xx^{-1} : x \in G\} = \{x^{-1}x : x \in G\}$. Let $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$. Assume that there exists a left Haar system $\{\lambda^u : u \in G^o\}$ on G, i.e., a family of measures λ^u on G such that $\operatorname{supp} \lambda^u = r^{-1}(u)$; for each $f \subseteq C_c(G)$, $u \to \int f \mathrm{d}\lambda^u$ is continuous; and for all $x \in G$ and $f \in C_c(G)$, $\int f(xy)\mathrm{d}\lambda^{d(x)}(y) = \int f(y)\mathrm{d}\lambda^{r(x)}(y)$. Let σ be a continuous 2-cocycle in $Z^2(G, T)$. Let t denote the usual inductive limit topology on $C_c(G)$. Then $(C_c(G), t)$ is a topological *-algebra with jointly continuous multiplication

$$f * g(x) = \int f(xy)g(y^{-1})\sigma(xy, y^{-1})\mathrm{d}\lambda^{d(x)}(y)$$

and the involution $f^*(x) = (f(x^{-1})\sigma(x, x^{-1}))^-$ ([31], Proposition II.1.1, p. 48). The I-norm on $C_c(G, \sigma)$ is $||f||_I = \max(||f||_{I,r}, ||f||_{I,l})$, where

$$||f||_{I,r} = \sup\left\{\int |f|\mathrm{d}\lambda^u : u \in G^o\right\}, \quad ||f||_{I,l} = \sup\left\{\int |f|\mathrm{d}\lambda_u : u \in G^o\right\},$$

$\lambda_u = (\lambda^u)^{-1}$ being the image of λ^u by the inverse map $x \to x^{-1}$ ([31], p. 50). Then $||\ \ ||_I$ is a submultiplicative *-norm on $C_c(G, \sigma)$. The L^1-algebra of (G, σ) is the completion $A = (C_c(G, \sigma), ||\ \ ||_I)$, a Banach *-algebra. For f in $C_c(G, \sigma)$, define $||f|| = \sup\{||\pi(f)||\}$, π running over all weakly continuous, non-degenerate *-representations $\pi : (C_c(G, \sigma), t) \to B(H_\pi)$ satisfying $||\pi(f)|| \leq ||f||_I$ for all f. Then $||\ \ ||$ defines a C^*-norm on $C_c(G, \sigma)$; and the groupoid C^*-algebra of (G, σ) is $C^*(G, \sigma) = (C_c(G, \sigma), ||\ \ ||)^-$, the completion. The following can be proved using cyclic decomposition and ([31], Corollary II.1.22, p. 72).

PROPOSITION 7.1

Let G be second countable having sufficiently many non-singular G-Borel sets. Then $E(C_c(G, \sigma)) = C^*(G, \sigma)$.

8. The universal *-algebra on generators with relations

Let G be any set. Let $F(G)$ be the free associative *-algebra on *generators* G, viz., the *-algebra of all polynomials in non-commuting variables $G \coprod G^*$ where $G^* = \{x^* : x \in G\}$. Let R be a collection of statements about elements of G, called *relations* on G, assumed throughout to be such that they make sense for elements of a locally m-convex *-algebra. A *Banach (respectively C^*-) representation* of (G,R) is a function ρ from G to a Banach *-algebra (respectively a C^*-algebra) $\rho : G \to A$ such that $\{\rho(g) : g \in G\}$ satisfies the relations R in A. Let $\mathrm{Rep}_B(G,R)$ (respectively $\mathrm{Rep}(G,R)$) be the set of all Banach representations (respectively C^*-representations) of (G,R). Motivated by ([27], Definition 1.3.4), it is assumed that R satisfies the following.

(i) The function $\rho : G \to \{0\}$ is a Banach representation of (G,R).
(ii) Let $\rho : G \to A$ be a representation of (G,R) in a Banach *-algebra A. Let B be a closed *-subalgebra of A containing $\rho(G)$. Then ρ is a representation of (G,R) in B.
(iii) Let ρ be a representation of (G,R) in a complete locally m-convex *-algebra A. Let $\phi : A \to B$ be a continuous *-homomorphism into a Banach *-algebra B. Then $\phi \circ \rho$ is a representation of (G,R) in B.
(iv) Let A be a complete locally m-convex *-algebra expressed as an inverse limit of Banach *-algebras viz. $A = \mathrm{proj.}\lim A_p$. Let $\pi_p : A \to A_p$ be the natural maps. Let $\rho : G \to A$ be a function such that for all p, $\pi_p \circ \rho$ is a representation of (G,R). Then ρ is a representation of (G,R).

DEFINITION 8.1

(a) (Blackadar) (G,R) is *C^*-bounded* if for each g in G, there exists a scalar $M(g)$ such that $||\rho(g)|| \leq M(g)$ for all $\rho \in \mathrm{Rep}\,(G,R)$.
(b) (Blackadar) (G,R) is *C^*-admissible* if it is C^*-bounded and the following holds.
(bC^*) If (ρ_α) is a family of representations $\rho_\alpha : G \to B(H_\alpha)$ of (G,R) on Hilbert spaces H_α, then $\oplus\rho_\alpha : G \to B(\oplus H_\alpha)$ is a representation of (G,R).
(c) (G,R) is *weakly Banach admissible* if given finitely many representations $\rho_i : G \to A_i$ $(1 \leq i \leq n)$ of G into Banach *-algebras, the map $g \to \rho_1(g) \oplus \rho_2(g) \oplus \ldots \oplus \rho_n(g)$ is a representation of (G,R) in $\oplus\, A_i$. (G,R) is *weakly C^*-admissible* [27] if this holds with Banach algebras replaced by C^*-algebras.

The class of relations making sense for elements of a Banach *-algebra is smaller than the class of relations making sense for elements of a C^*-algebra. The usual algebraic relations involving the four elementary arithmetic operations on elements of G and G^* do make sense for Banach *-algebras; but relations like $x^+ \geq x^-$ for $x = x^*$ in G, or like $|x| \geq |y|$ for elements x, y in G, which make sense for C^*-algebras, fail to make sense for Banach *-algebras. We refer to [27] for relations satisfying (i)–(iv) except (ii). The relation (suggested by the referee). "The elements a, b and c generate A" fails to satisfy Definition 8.1(c). Our definition of weakly Banach admissible relations is very much ad hoc aimed at exploring a method of constructing non-abelian locally m-convex *-algebras.

Lemma 8.2. (a) *Let (G,R) be weakly Banach admissible. Then there exists a complete m-convex *-algebra $A(G,R)$ and a representation $\rho : G \to A(G,R)$ such that given any representation $\sigma : G \to B$ into a complete m-convex *-algebra B, there exists a continuous *-homomorphism $\phi : A(G,R) \to B$ satisfying $\phi \circ \rho = \sigma$.*
(b) ([27], Proposition 1.3.6). *Let (G,R) be weakly C^*-admissible. Then there exists a pro-C^*-algebra $C^*(G,R)$ and a representation $\rho_\infty : G \to C^*(G,R)$ such that given any representation $\sigma : G \to B$ of G into a pro-C^*-algebra B, there exists a continuous *-homomorphism $\phi : C^*(G,R) \to B$ such that $\phi \circ \rho_\infty = \sigma$.*

Proof. (a) Let $K = K(F(G))$ be the set of all submultiplicative *-seminorms p on $F(G)$ of the form $p(x) = ||\sigma(x)||$, σ running through all Banach representations of G. For $p \in K$, let $N_p = \{x \in F(G) : p(x) = 0\}$ and $N_a = \cap\{N_p : p \in K\}$ a *-ideal of $F(G)$. Let $B = F(G)/N_a$. Take $\tilde{p}(x + N_a) = p(x)$. Let t be the Hausdorff topology defined by $\{\tilde{p} : p \in K\}$. Let $A(G,R)$ be the completion of (B,t). Let $\rho : G \to A(G,R)$ be $\rho(g) = g + N_a$.

Claim 1. ρ is a representation of G in $A(G,R)$.

Let q be any t-continuous submultiplicative *-seminorm on $A(G,R)$. Let A_q be the Banach *-algebra obtained by the Hausdorff completion of $(A(G,R),q)$. By (iv) above, it is sufficient to prove that $\pi_q \circ \rho : G \to A_q$ is a representation of (G,R). Since q is t-continuous, there exists $p_1, p_2, \ldots, p_k$ in K such that $q(x) \leq c \max p_i(x)$ for all $x \in F(G)$; and each p_i is of form $p_i(x) = ||\sigma_i(x)||$, $\sigma_i : G \to A(i)$ being a representation into some Banach algebra $A(i)$. By (c) of Definition 8.1, there exists a Banach *-algebra B and a representation $\sigma : G \to B$ such that $q(x) \leq ||\sigma(x)||$ for all $x \in F(G)$. In view of (ii), we assume that B is generated by $\sigma(G)$. Let $\phi : B \to A_q$ be $\phi(\sigma(x)) = (x + N_a) + \ker q = \pi_q(\rho(x))$. Then ϕ is well defined, continuous and $\phi \circ \sigma = \pi_q \circ \rho$. By the assumption (iii) above, $\phi \circ \sigma$ is a representation of G.

Claim 2. Given any representation $\sigma : G \to C$ into a complete m-convex *-algebra C, there exists a unique continuous *-homomorphism $\phi : A(G,R) \to C$ such that $\phi \circ \rho = \sigma$.

Let $C = \text{proj.}\lim C_\alpha$, an inverse limit of Banach *-algebras C_α, $\pi_\alpha : C \to C_\alpha$ being the projection maps. By (iii) of above, $\pi \circ \sigma$ is a Banach representation of (G,R). By the construction of $A(G,R)$, there exist continuous *-homomorphisms $\phi_\alpha : A(G,R) \to C_\alpha$ such that $\phi_\alpha \circ \rho = \pi_\alpha \circ \sigma$. Hence by the definition of an inverse limit, there exists a continuous *-homomorphism $\phi : A(G,R) \to C$ such that $\phi \circ \rho = \sigma$.
(b) We only outline the (needed) construction of $C^*(G,R)$ from [27]. Let S be the set of all C^*-seminorms on $F(G)$ of form $q(x) = ||\sigma(x)||$, σ running over all representations of G into C^*-algebras. Let $N_q = \{x \in F(G) : q(x) = 0\}$ and $N = \cap\{N_q : q \in S\}$. Let τ be the pro-C^*-topology on $F(G)/N$ defined by $\tilde{q}(x + N) = q(x)$, $q \in S$. Then $C^*(G,R)$ is the completion of $(F(G)/N, \tau)$. The map $\rho_\infty : G \to C^*(G,R)$ where $\rho_\infty(x) = x + N$ is the canonical representation.

The following brings out the essential point in arguments in claim 1 above.

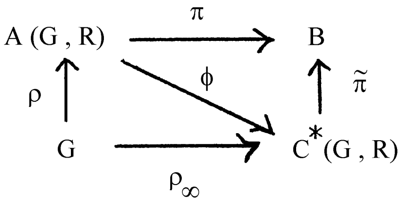
Lemma 8.3. *There exists a natural one-to-one correspondence between* $\text{Rep}_B(G,R)$ (*respectively* $\text{Rep}\,(G,R)$) *and t-continuous Banach *-representations (respectively C^*-algebra representations) of* $A(G,R)$.

Lemma 8.4. $\text{srad}\,(A(G,R)) \bigcap (F(G)/N_a) = \text{srad}\,(F(G)/N_a) = \{x + N_a : x \in N\}$.

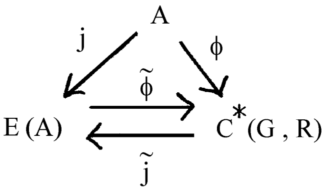
Proof. Let $C = F(G)/N_a$. Let $x + N_a \in C \bigcap \text{srad}\, A$. Then $\pi(x + N_a) = 0$ for all continuous *-homomorphisms $\pi : A \to B(H_\pi)$. By Lemma 8.3, $p(x) = 0$ for all $p \in S$. Hence $x \in N$, and $x + N_a \in \text{srad}\,(F(G)/N_a)$. Conversely, let $x \in N$. Then $q(x) = 0$ for all $q \in S$. Again by Lemma 8.3, $||\pi(x + N_a)|| = 0$ for all $\pi \in R(A)$, hence $x + N_a \in \text{srad}\, A$.

Proof of Theorem 1.7. (1) Let $A = A(G,R)$. Let $\phi : (F(G)/N_a, t) \to (F(G)/N_a, \tau)$ be $\phi(X + N_a) = x + N$. Then ϕ is a well defined, continuous *-homomorphism; hence extends as a continuous surjective *-homomorphism $\phi : A \to C^*(G,R)$. The universal property of $C^*(G,R)$, Lemma 8.3 and weak Banach admissibility of R imply the following whose proof we omit.

Assertion 1. Given any continuous *-homomorphism $\pi : A(G,R) \to B$ to a pro-C^*-algebra B, there exists a continuous *-homomorphism $\tilde{\pi} : C^*(G,R) \to B$ such that $\pi = \tilde{\pi} \circ \phi$.



By applying the above to the maps ϕ and $j : A \to E(A)$, $j(x) = x + \text{srad}\,(A)$, it follows that there exist continuous *-homomorphisms $\tilde{\phi} : E(A) \to C^*(G,R)$ and $\tilde{j} : C^*(G,R) \to E(A)$ such that the following diagrams commute.



Assertion 2. The maps $\tilde{\phi}$ and $\tilde{j}$ are inverse of each other.

Indeed, $\tilde{j}$ is one–one on $F(G)/N$. For given $x \in F(G)$,

$$0 = \tilde{j}(x+N) = \tilde{j} \circ \phi(x+N_a) = j(x+N_a)$$

which implies $(x+N_a) + \text{srad}\,(A) = 0$ and $(x+N_a) \in \text{srad}\,(A)$. Hence $x \in N$ by Lemma 8.4, so that $x + N = 0$. Similarly $\tilde{\phi}$ is one–one on $F(G)/N$. Also,

$$\begin{aligned}(\tilde{\phi} \circ \tilde{j})(x+N) &= \tilde{\phi} \circ \tilde{j} \circ \phi(x+N_a) \\ &= \tilde{\phi} \circ j(x+N_a) = \phi(x+N_a) = x+N,\end{aligned}$$

which implies that $\tilde{\phi} = \tilde{j}^{-1}$ on $F(G)/N_a$; and $\tilde{j} = \tilde{\phi}^{-1}$ on $F(G)/N_a + \text{srad}\,A$. By continuity and density, $\tilde{\phi}$ establishes a homeomorphic *-isomorphism $\tilde{\phi} : E(A) \to C^*(G,R)$ with $\tilde{\phi}^{-1} = \tilde{j}$.

(2) Let (G,R) be C^*-admissible. Then $\sup\{||\sigma(x)|| : \sigma \in \text{Rep}\,(G,R)\} < \infty$; and $\pi = \oplus\{\sigma : \sigma \in \text{Rep}\,(G,R)\} \in \text{Rep}\,(G,R)$. Thus $q(x) = ||\pi(x)||$ defines the greatest member of $S(F(G))$, q is a C^*-norm, and it is the greatest t-continuous C^*-seminorm on $F(G)/N$. Thus the topology τ on $C^*(G,R)$ is determined by q. Conversely suppose that $C^*(G,R)$ is a C^*-algebra so that $||z||_\infty = \sup\{q(z) : q$ is a continuous C^*-seminorm on $C^*(G,R)\} < \infty$ for all $z \in C^*(G,R)$, and τ is determined by the C^*-norm $||\ ||_\infty$. Let $p_\infty(x) = ||x+N||_\infty = \sup\{q(x) : q \in S\}$ for all $x \in F(G)$. Then $p_\infty \in S$ and $\ker p_\infty = N$. There exists a C^*-representation $\sigma : G \to C$ such that $p_\infty(g) = ||\sigma(g)||$ for all $g \in G$; and this defines a continuous C^*-representation $\sigma : C^*(G,R) \to C$. It is clear that R is C^*-bounded. We verify (bC^*) of Definition 8.1. Let $\{\rho_\alpha\} \subseteq \text{Rep}\,(G,R)$ with $\rho_\alpha : G \to B(H_\alpha)$

for some Hilbert space H_α. Let $H = \oplus H_\alpha$. For $x \in F(G)$, let $\lambda(x) = \oplus \rho_\alpha(x)$. By the C^*-boundedness of (G, R), $\lambda(x) \in B(H)$. This defines a *-homomorphism $\lambda : F(G) \to B(H)$ satisfying $||\lambda(x)|| = \sup ||\rho_\alpha(x)|| \leq p_\infty(x)$ for all $x \in F(G)$. Since ker $p_\infty = N$, λ factors to a *-representation $\tilde{\lambda} : F(G)/N \to B(H)$ satisfying $||\tilde{\lambda}(z)|| \leq ||z||_\infty$. As $||\ ||_\infty$ is τ-continuous, so is $\tilde{\lambda}$. By lemma 8.3, $\{\lambda(g) : g \in G\}$ satisfies the relations R in $B(H)$. Thus (G, R) is C^*-admissible.

Acknowledgements

The author sincerely thanks Prof. M Fragoulopoulou for fruitful correspondence on [5] and for providing [16,17]; as well as Prof. N C Phillips and Prof. L B Schweitzer for providing pre-publication versons of [29,33,34]. Section 5 has been significantly influenced by suggestions from the referee. Besides suggesting the reference [18], Proposition 5.1 and the actions on C^∞-manifolds, he has also suggested the problem discussed in Theorem 1.5. Further, he has critically read the manuscript and made several detailed suggestions on appropriately presenting the material in readable form. Thank you very much, referee.

References

[1] Allan G R, A spectral theory for locally convex algebras, *Proc. London Math. Soc.* **3** (1965) 399–421
[2] Bhatt S J, Representability of positive functionals on abstract *-algebra without identity with applications to locally convex *-algebras, *Yokohana Math. J.* **29** (1981) 7–16
[3] Bhatt S J, An irreducible representation of a symmetric *-algebra is bounded, *Trans. Am. Math. Soc.* **292** (1985) 645–652
[4] Bhatt S J, Bounded vectors for unbounded representations and standard representations of polynomial algebras, *Yokohama Math. J.* **41** (1993) 67–83
[5] Bhatt S J and Karia D J, Topological *-algebras with C^*-enveloping algebras, *Proc. Indian Acad. Sci. (Math. Sci.)* **102** (1992) 201–215
[6] Bhatt S J and Karia D J, On an intrinsic characterization of pro-C^*-algebras and applications, *J. Math. Anal. Appl.* **175** (1993) 68–80
[7] Bonsall F F and Duncan J, *Complete Normed Algebras* (New York: Springer-Verlag, Berlin Heidelberg) (1973)
[8] Bhatt S J and Dedania H V, On seminorm, spectral radius and Ptak's spectral function in Banach algebras, *Indian J. Pure. Appl. Maths.* **27(6)** (1996) 551–556
[9] Brooks R M, On locally m-convex *-algebras, *Pacific J. Math.* **23** (1967) 5–23
[10] Brooks R M, On representing F^*-algebras, *Pacific J. Math.* **39** (1971) 51–69
[11] Bratteli O, *Derivations, dissipations and group actions on C^*-algebras*, Lecture Notes in Mathematics 1229 (Springer-Verlag) (1986)
[12] Dixon P G, Automatic continuity of positive functionals on topological involution algebras, *Bull. Australian Math. Soc.* **23** (1981) 265–283
[13] Dixon P G, Generalized B^*-algebras, *Proc. London Math. Soc.* **21** (1970) 693–715
[14] Dixmier J, *C^*-algebras* (North Holland) (1977)
[15] Fragoulopoulou M, Spaces of representations and enveloping l.m.c. *-algebras, *Pacific J. Math.* **95** (1981) 16–73
[16] Fragoulopoulou M, *An introduction to the representation theory of topological *-algebras*, Schriften Math. Inst. Univ. Münster, 2 Serie, Heft 48, June 1988
[17] Fragoulopoulou M, *Symmetric topological *-algebras II: Applications*, Schriften Math. Inst. Univ. Münster, 3 Ser., Heft 9, 1993
[18] Green P, The local structure of twisted covariance algebras, *Acta. Math.* **140** (1978) 191–250
[19] Inoue A, Locally C^*-algebras, *Mem. Fac. Sci.* (Kyushu Univ.) (1972)
[20] Laβner G, Topological algebras of operators, *Rep. Math. Phys.* **3** (1972) 279–293

[21] Mallios A, *Topological algebras: Selected topics*, (Amsterdam: North Holland Publ. Co.) (1985)
[22] Mathot F, On decomposition of states of some *-algebras, *Pacific J. Math.* **90** (1980) 411–424
[23] Michael E A, Locally multiplicatively convex topological algebras, *Mem. Am. Math. Soc.* **11**, 1952
[24] Palmer T W, *Algebraic properties of *-algebras*, presented at the 13th. International Conference on Banach Algebras 97, Heinrich Fabri Institute, Blaubeuren, July–August 1997
[25] Pedersen G K, *C^*-algebras and their automorphism groups*, London Math. Soc. Monograph 14 (Academic Press) (1979)
[26] Phillips N C, Inverse limits of C^*-algebras, *J. Operator Theory* **19** (1988) 159–195
[27] Phillips N C, Inverse limits of C^*-algebras and applications in: Operator Algebras and Applications (eds.) D E Evans and M Takesaki) (1988) London Math. soc. Lecture Note 135 (Cambridge Univ. Press)
[28] Phillips N C, A new approach to the multipliers of Pedersen ideal *Proc. Am. Math. Soc.* **104** (1988) 861–867
[29] Phillips N C and Schweitzer L B, Representable K-theory for smooth crossed product by $\mathbb{R}$ and $\mathbb{Z}$, *Trans. Am. Math. Soc.* **344** (1994) 173–201
[30] Powers R T, Self adjoint algebras of unbounded operators I, II, *Comm. Math. Phys.* **21** (1971) 85–125; *Trans. Am. Math. Soc.* **187** (1974) 261–293
[31] Renault J, *A grouped approach to C^*-algebras*, Lecture Notes in Maths. 793 (Springer-Verlag) (1980)
[32] Schaefer H H, *Topological Vector Spaces* (MacMillan) (1967)
[33] Schweitzer L B, Spectral invariance of dense subalgebras of operator algebras, *Int. J. Math.* **4** (1993) 289–317
[34] Schweitzer L B, Dense m-convex Frechet algebras of operator algebra crossed product by Lie groups, *Int. J. Math.* **4** (1993) 601–673
[35] Schmudgen K, Lokal multiplikativ konvexe Op^*-Algebren, *Math. Nach.* **85** (1975) 161–170
[36] Schmudgen K, The order structure of topological *-algebras of unbounded operators, *Rep. Math. Phys.* **7** (1975) 215–227
[37] Schmudgen K, *Unbounded operator algebras and representation theory* OT 37, (Basel-Boston-Berlin: Birkhäuser-Verlag) (1990)
[38] Sebestyen Z, Every C^*-seminorm is automatically submultiplicative, *Period. Math. Hungar.* **10** (1979) 1–8
[39] Sebestyen Z, On representability of linear functionals on *-algebras, *Period. Math. Hungar.* **15(3)** (1984) 233–239
[40] Stochel J and Szafranieo F H, Normal extensions of unbounded operators, *J. Operator Theory* **14** (1985) 31–55
[41] Traves F, *Topological vector spaces, distributions and kernals* (New York London: Academic Press) (1967)
[42] Yood B, C^*-seminorms, *Studia Math.* **118** (1996) 19–26