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Limits of commutative triangular systems on locally compact groups

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Abstract. On a locally compact group G, if $\nu_n^{k_n} \to \mu$, $(k_n \to \infty)$, for some probability measures ν_n and μ on G, then a sufficient condition is obtained for the set $A = \{\nu_n^m | m \le k_n\}$ to be relatively compact; this in turn implies the embeddability of a shift of μ. The condition turns out to be also necessary when G is totally disconnected. In particular, it is shown that if G is a discrete linear group over $\mathbf{R}$ then a shift of the limit μ is embeddable. It is also shown that any infinitesimally divisible measure on a connected nilpotent real algebraic group is embeddable.

Keywords. Embeddable measures; triangular systems of measures; infinitesimally divisible measures; totally disconnected groups; real algebraic groups.

1. Introduction

Commutative triangular systems of probability measures on locally compact groups have been studied extensively and recently the embedding of the limit μ (or a translate $x\mu$, $x \in G$) have been shown on a large class of groups under certain conditions like infinitesimality of triangular system and/or 'fullness' of the limit μ (see [S4] for the latest results and the literature cited therein for earlier results). Generalizing the techniques developed in [S3,S4], we extend our earlier result to some particular triangular systems on algebraic groups. We also discuss special triangular systems of identical measures, i.e. limit theorems. In particular if $\nu_n^{k_n} \to \mu$ on G then we give a sufficient condition for the set $A = \{\nu_n^m \mid m \le k_n\}$ to be relatively compact; this in turn would imply the embeddability of a shift of the limit μ. The condition turns out to be also necessary if G is totally disconnected. We hereby generalize our earlier results on limit theorems on Lie groups to general locally compact groups. We also show the embedding of a shift of the limit μ if G is a discrete linear group over $\mathbf{R}$.

Let G be a locally compact (Hausdorff) group and let $M^1(G)$ be the topological semigroup of probability measures with weak topology and convolution as the semigroup operation. Let μ, ν be any measures in $M^1(G)$. Let the convolution product of μ and ν be denoted by $\mu\nu$. For any compact subgroup H of G let ω_H denote the normalized Haar measure of H. Let $M_H^1(G) = \omega_H M^1(G)\omega_H$, then $M_H^1(G)$ is a closed subsemigroup of $M^1(G)$ with identity ω_H. For any $x \in G$, let δ_x denote the Dirac measure at x and let $x\mu = \delta_x\mu$, (similarly, $\mu x = \mu\delta_x$). Let $I_\mu = \{x \in G \mid x\mu = \mu x\}$ and let $I(\mu) = \{x \in G \mid x\mu = \mu x = \mu\}$, then I_μ (resp. $I(\mu)$) is a closed (resp. compact) subgroup of G. Let $J_\mu = \{\lambda \in M^1(G) \mid \lambda\mu = \mu\lambda = \mu\}$. Clearly, J_μ is a compact semigroup and for any $\lambda \in M^1(G)$, $\lambda \in J_\mu$ if and only if $\operatorname{supp}\lambda \subset I(\mu)$. Let $G(\mu)$ be the smallest closed subgroup of G containing $\operatorname{supp}\mu$. Let $N(\mu)$ (resp. $Z(\mu)$) be the normalizer (resp. centralizer) of $G(\mu)$ in

G. Let $\tilde{\mu}$ denote the *adjoint of* μ, defined by $\tilde{\mu}(B) = \mu(B^{-1})$, for all Borel subsets B of G. μ is said to be *symmetric* if $\mu = \tilde{\mu}$. Let G^0 denote the connected component of the identity in G. For a set $A \subset M^1(G)$ and a normal subgroup $C \subset G$, we denote $A/C = \pi(A)$, where $\pi : G \to G/C$ is the natural projection.

A measure $\mu \in M^1(G)$ is said to be *infinitely divisible* (resp. *weakly infinitely divisible*) if for every $n \in \mathbf{N}$, there exists $\mu_n \in M^1(G)$ such that $\mu_n^n = \mu$ (resp. $\mu_n^n x_n = \mu$ for some $x_n \in G$); and it is said to be *embeddable* if there exists a continuous one-parameter convolution semigroup $\{\mu_t\}_{t\geq 0}$ such that $\mu_1 = \mu$. Since we aim to prove the embeddability of a given measure under various conditions, the reader is referred to [M2], a survey article on the embedding problem of infinitely divisible measures.

Let S be a Hausdorff semigroup with identity e and let $s \in S$. Let T_s denote the set of two sided factors of s, that is, $T_s = \{t \in S \mid tr = rt = s \text{ for some } r \in S\}$. Elements $s, t \in S$ are said to be *associates* if s and t are two sided factors of each other, i.e. $s \in T_t$ and $t \in T_s$. A subset A of S is said to be *associatefree* if $s, t \in A$ are associates then $s = t$. An element h in S is said to be an *idempotent* if $h^2 = h$. An element s is said to be *bald* (in S) if e is the only idempotent contained in T_s. For a subset A of S, a decomposition of s as $s = s_1 \cdots s_n$, for some $n \in \mathbf{N}$, where $s_i \in A$ and $s_i s_j = s_j s_i$ for all i, j, is called an *A-decomposition of s*. An element s (in S) is said to be *infinitesimally divisible* if s has a U-decomposition for every neighbourhood U of e in S. A set $\Delta = \{s_{ij} \in S \mid i \in \mathbf{N},\ 1 \leq j \leq n_i, n_i \to \infty \text{ as } i \to \infty\}$ is said to be a *triangular system* in S; we will sometimes write $\Delta = (s_{ij})_{i\in\mathbf{N}, j=1}^{n_i}$. Δ is said to be commutative if for every fixed i, s_{ij} commute with each other, it is said to be *infinitesimal* if as $i \to \infty$, $s_{ij} \to e$ uniformly in j. We say that Δ converges to μ if $s_{i1} \cdots s_{in_i} = s_i \to \mu$.

In § 2, we prove a limit theorem for general locally compact groups, (see Theorem 2.1). In § 3, we show that if $\nu_n^{k_n} \to \mu$, $(k_n \to \infty)$, on a discrete linear group over $\mathbf{R}$, then $x\mu$ is embeddable for some $x \in G$ (see Theorem 3.1). In § 4, we show that any infinitesimally divisible probability measure μ on a connected nilpotent real algebraic group is embeddable, (more generally see Theorem 4.1).

2. Limit theorems on locally compact groups

Theorem 2.1. *Let G be a locally compact group and let $\pi : G \to G/G^0$ be the natural projection. Let $\{\nu_n\}$ be a relatively compact sequence in $M^1(G)$ such that for any limit point ν of it, $G(\pi(\nu))$ is a compact group in G/G^0 and $\nu_n^{k_n} \to \mu$ for some $\mu \in M^1(G)$ and for some unbounded sequence $\{k_n\} \subset \mathbf{N}$. Suppose that for some connected nilpotent normal subgroup N of G, the closed subgroup generated by* $\operatorname{supp}\mu$ *and N contains G^0. Then the set $A = \{\nu_n^m \mid m \leq k_n\}$ is relatively compact and there exists $x \in I_\mu$ such that $x\mu$ is embeddable.*

Remarks. (1) The above theorem generalizes Theorem 1.7(1) of [S4]. (2) If G is totally disconnected then $G^0 = \{e\}$ and hence the above theorem implies that if $\nu_n^{k_n} \to \mu$ and if $\{\nu_n\}$ is relatively compact and for any limit point ν of it, $G(\nu)$ is compact then A is relatively compact. Conversely, if A is relatively compact then so are $\{\nu_n\}$ and $\{\nu_n^{k_n}\}$, and for any limit point ν of $\{\nu_n\}$, $G(\nu)$ is compact as $\{\nu^n\} \subset A$. Thus, for totally disconnected groups we get a necessary and sufficient condition for the set A as above to be relatively compact.

We first prove a more general theorem for totally disconnected locally compact groups.

Theorem 2.2. *Let G be a totally disconnected locally compact group and let $\{\nu_n\} \subset M^1(G)$ be such that $\nu_n \to \nu$ where $G(\nu)$ is compact and $\nu_n^{k_n}\nu'_n \to \mu$ for some sequence $\{\nu'_n\}$ in $M^1(G)$ such that $\nu_n\nu'_n = \nu'_n\nu_n$ for all n. Then given any neighbourhood U of e and an $\epsilon > 0$ there exists an l, such that for all large n, $\nu_n^m(G(\nu)I(\mu)U) > (1-\epsilon)^l$, for all $m \le k_n$. In particular $A = \{\nu_n^m \mid m \le k_n\}$ and $\{\nu'_n\}$ are relatively compact.*

Proof. As $G(\nu)$ is compact and $\nu^m \in T_\mu$, for all m, $\operatorname{supp}\nu \subset xI(\mu) = I(\mu)x$, for all $x \in \operatorname{supp}\nu$ (cf. [S4], Theorem 2.4). Therefore $G(\nu)I(\mu)$ is a compact group.

Let V be an open compact subgroup of G such that V is normalized by $G(\nu)I(\mu)$, and $V \subset U$. Since $\nu_n \to \nu$, $\nu_n(G(\nu)I(\mu)V) > 1-\epsilon$, for all large n. Let $V' = \{\lambda \mid \lambda(G(\nu)I(\mu)V) > (1-\delta)^{1/2}\}$ and let $U' = \{\lambda \mid \lambda(G(\nu)I(\mu)V) \ge 1-\delta\}$ for some positive $\delta < \epsilon$. Then $V'V' \subset U'$. Let $J = \{\lambda \in M^1(G) \mid \operatorname{supp}\lambda \subset G(\nu)I(\mu)V\}$. Clearly, J is a compact semigroup and $JV' = V'$. Let $\lambda \in U' \setminus V'$. If possible, suppose that $\lambda^n \in T_\mu$ for all n, then by Theorem 2.4 of [S4], $\operatorname{supp}\lambda \subset xI(\mu) = I(\mu)x$, for all $x \in \operatorname{supp}\lambda$. Since $\lambda \in U'$, $\operatorname{supp}\lambda \subset G(\nu)I(\mu)V$, i. e. $\lambda \in J \subset V'$, a contradiction. Hence for $\lambda \in T_\mu \cap U' \setminus V'$, there exists $n = n(\lambda)$, such that $\lambda^n \notin T_\mu$. By Lemma 2.1 of [S4], $T_\mu \cap U' \setminus V'$ is compact. As in the proof of Lemma 2.5 in [S4], one can find l, such that for any $\lambda \in T_\mu \cap U' \setminus V'$, μ cannot be expressed as $\mu = \lambda^l\lambda'$, for any λ' which commutes with λ.

Since $\nu_n \to \nu$, $\nu_n \in V'$, for all large n. Let such a large n be fixed. Then there exists $a_n > 1$, such that $\nu_n^m \in V'$, for all $m < a_n$ and $\nu_n^{a_n} \notin V'$. Therefore, $\nu_n^{a_n} \in V'V' \setminus V' \subset U' \setminus V'$. Let $b_n = k_n - la_n$ if $la_n \le k_n$, otherwise $b_n = 0$. If $b_n = 0$, then $\nu_n^m \in (U')^l$, for all $m \le k_n$. Therefore, for all large n, $\nu_n^m(G(\nu)I(\mu)V) \ge (1-\delta)^l$, and hence $\nu_n^m(G(\nu)I(\mu)U) > (1-\epsilon)^l$ for all $m \le k_n$, as $V \subset U$ and $\delta < \epsilon$.

We now show that $b_n = 0$, for all large n. If $b_n \ne 0$ for infinitely many n, then $\nu_n^{la_n}\nu_n^{b_n}\nu'_n \to \mu$. Since $\{\nu_n^{a_n}\} \subset U' \setminus V'$, by Lemma 2.1 of [S4], $\{\nu_n^{a_n}\}$ is relatively compact and it has a limit point (say) λ, such that $\mu = \lambda^l\lambda'$, for some λ', which is a limit point of $\{\nu_n^{b_n}\nu'_n\}$, i.e. $\lambda \in T_\mu \cap U' \setminus V'$ and $\lambda\lambda' = \lambda'\lambda$. This is a contradiction to the choice of l as above.

Now it is enough to show that A is relatively compact as this would also imply that $\{\nu'_n\}$ is relatively compact. Let $\mu_n = \nu_n^{k_n}\nu'_n$. Then $\mu_n \to \mu$ and for each n, $\nu_n^m \in T_{\mu_n}$ for all $m \le k_n$. Let $F = G(\nu)I(\mu)V$ for V as above. Then F is compact. Let $A' = \{\lambda \in M^1(G) | \lambda(F) \ge (1-\epsilon)^l/2\}$. Then from above, $A \subset A'$. Since $\mu_n \to \mu$, for every $\delta > 0$ such that $\delta < (1-\epsilon)^l/2$, there exists a compact set K_δ such that $\mu_n(K_\delta) > 1-\delta$ (cf. [H], Properties 1.2.20(2)). Therefore, for every n, m as above, there exists $x_{n,m}$ such that $\nu_n^m(K_\delta x_{n,m}) > 1-\delta$. Now since $A \subset A'$, the above implies that $x_{n,m} \in K_\delta^{-1}F$ and hence $\nu_n^m(K'_\delta) > 1-\delta$, where $K'_\delta = K_\delta K_\delta^{-1}F$ which is a compact set. In particular A is relatively compact (cf. [H], 1.2.20). This completes the proof.

We now prove several results which will be needed to prove Theorem 2.1.

Lemma 2.3. *Let G be a locally compact first countable group and let $\{\mu_n\}$, $\{\lambda_n\}$ and $\{\nu_n\}$ be sequences in $M^1(G)$ such that $\lambda_n\nu_n = \nu_n\lambda_n = \mu_n \to \mu$ for some $\mu \in M^1(G)$. Then there exists a sequence $\{x_n\}$ such that $x_n \in N(\mu_n)$ for each n and $\{\lambda_n x_n\}$ and $\{x_n\lambda_n\}$ are relatively compact and all its limit points are supported on* supp μ.

The proof is quite similar to Proposition 1.2 in [DM] and Theorem 2.2 in ch. III of [P].

Proof. For any integer $r > 0$ there exists a compact set $K_r \subset \operatorname{supp}\mu$ such that $\mu(K_r) > 1 - 4^{-(r+1)}$. Without loss of generality, we may assume that $K_r \subset K_{r+1}$ for all r. Let $\{U_r\}$ be a neighbourhood basis of e in G such that each U_r is relatively compact,

$U_{r+1} \subset U_r$ for all r and $\cap_r U_r = \{e\}$. Since $\mu_n \to \mu$, there exists $n_r \in \mathbf{N}$, such that $\mu_n(K_r U_r) > 1 - 2^{-r}$ and $\mu_n(U_r K_r) > 1 - 2^{-r}$, for all $n \geq n_r$. Let $E_n^r = \{x \in G \mid \lambda_n (K_r U_r x^{-1}) > 1 - 2^{-r}\}$ and let $F_n = \cap_{\{r | n_r \leq n\}} E_n^r$.

A simple calculation as in Theorem 2.2 of ch. III in [P] shows that for $n \geq n_r$, $\nu_n(G \setminus E_n^r) \geq 2^{-(r+2)}$ and hence $\nu_n(G \setminus F_n) \geq 1/4$. Similarly, we define $B_n^r = \{x \in G \mid \lambda_n(x^{-1} U_r K_r) > 1 - 2^{-r}\}$ for any r and $C_n = \cap_{\{r|n_r \leq n\}} B_n^r$. Then $\nu_n(G \setminus C_n) \geq 1/4$.

Therefore $\nu_n(F_n \cap C_n) \geq 1/2$. For each n, we pick $x_n \in F_n \cap C_n \cap \operatorname{supp} \nu_n$ as it is nonempty, $x_n \in \operatorname{supp} \nu_n \subset N(\mu_n)$. Then for any $r > 0$, $\lambda_n x_n(K_r U_r) > 1 - 2^{-r}$ and $x_n \lambda_n (U_r K_r) > 1 - 2^{-r}$ for all $n \geq n_r$ and hence by tightness criterion, $\{\lambda_n x_n\}$ and $\{x_n \lambda_n\}$ are tight. Also, since $K_r \subset \operatorname{supp} \mu$ for all r, $\lambda_n x_n((\operatorname{supp} \mu) U_r) > 1 - 2^{-r}$, for all $n \geq n_r$. Since $\cap_r U_r = \{e\}$, it easily follows that for any limit point λ of $\{\lambda_n x_n\}$, $\operatorname{supp} \lambda \subset \operatorname{supp} \mu$. Similarly, the limit points of $\{x_n \lambda_n\}$ are also supported on $\operatorname{supp} \mu$.

Lemma 2.4. *Let G be a locally compact group and let $\mu_n \to \mu$ in $M^1(G)$. Let B be a subgroup which centralizes an open subgroup H containing* $\operatorname{supp} \mu$. *Then the following hold:*

1. *For any sequence $\{x_n\}$ in B, $\{x_n^{-1} \mu_n x_n\}$ is relatively compact and it converges to μ.*
2. *Let $\mu_n = \lambda_n \nu_n = \nu_n \lambda_n$, for all n. If for sequences $\{x_n\}$ and $\{y_n\}$ in B, $\{x_n \lambda_n y_n\}$ is relatively compact then its limit points belong to T_μ; in particular if $\lambda_n a_n \to \lambda$ for some $\{a_n\} \subset B$, then $\lambda \in T_\mu$ and the limit points of $\{x_n \lambda_n y_n\}$ are of the form $z\lambda = \lambda z'$, for some $z, z' \in Z(\mu)$.*

Proof. Let U be any open set contained in H and let $K \subset \operatorname{supp} \mu$ be any compact set such that $\mu(K) > 0$. Then given $0 < \epsilon < \mu(K)$, there exists N such that $\mu_n(KU) > \mu(K) - \epsilon$ for all $n > N$. Since x_n centralizes KU, $x_n^{-1} \mu_n x_n(KU) = \mu_n(KU) > \mu(K) - \epsilon$. Since this is true for all K and U as above, $\{x_n^{-1} \mu_n x_n\}$ is relatively compact and it converges to μ. Let λ' be a limit point of a relatively compact sequence $\{\lambda'_n = x_n \lambda_n y_n\}$, where $x_n, y_n \in B$. Since $\{x_n \mu_n x_n^{-1}\}$ converges to μ, $\{y_n^{-1} \nu_n x_n^{-1}\}$ is relatively compact and there exists a limit point ν' of it such that $\lambda' \nu' = \mu$. Also, $\nu' \lambda'$ is a limit point of $\{y_n^{-1} \mu_n y_n\}$, which converges to μ. Therefore, $\nu' \lambda' = \mu$ and hence $\lambda' \in T_\mu$. Now suppose $\lambda_n a_n \to \lambda$, $\{a_n\} \subset B$, then from above $\lambda \in T_\mu$. Therefore, $\lambda = x\beta$, for some $x \in N(\mu)$ and β supported on $G(\mu) \subset H$. Then $x^{-1} \lambda_n a_n \to \beta$. Let K' be any compact subset in H such that $\beta(K') > 0$. Then for any open subset U contained in H, $\lambda_n a_n(xK'U) = z_n \lambda'_n(xK'U)$, where $z_n = x y_n^{-1} a_n x^{-1} x_n^{-1} \in Z(\mu)$, as $B \subset Z(\mu)$ and $x \in N(\mu)$ which normalizes $Z(\mu)$. Since this is true for all n and all compact subsets K' of $\operatorname{supp} \beta$ it implies that $\{z_n\}$ is relatively compact in $Z(\mu)$. Therefore, $\lambda' = z\lambda$, for λ' as above, where z is a limit point of $\{z_n^{-1}\}$. Now since $\lambda \in T_\mu$ and $z \in Z(\mu)$, $z\lambda = zx\beta = xz'\beta = x\beta z' = \lambda z'$, where $z' = x^{-1} z x \in Z(\mu)$.

PROPOSITION 2.5

Let G be a locally compact group and let C be a closed normal (*real*) *vector subgroup of G. Suppose that $\{\mu_n\} \subset M^1(G)$ be a sequence such that $\mu_n \to \mu$, the closed subgroup (say) H, generated by the centralizer $Z(C)$ of C and* $\operatorname{supp} \mu$, *is open in G. Suppose that there exists a sequence $\{x_n\}$ in C such that $\{x_n^{-1} \mu_n x_n\}$ is relatively compact. Then $\{x_n\}/(Z(\mu) \cap C)$ is relatively compact. In particular $I_\mu \cap C = Z(\mu) \cap C$.*

Proof. Suppose $C \subset Z(\mu)$ then there is nothing to prove. Now let $V = Z(\mu) \cap C$, which is a proper closed subgroup of C. Since C is normal in G, for any $x \in G$, $i_x : C \to C$,

$i_x(c) = xcx^{-1}$ for all $c \in C$, is a continuous homomorphism of C and hence it is a linear operator in $M(d, \mathbf{R})$, where d is such that C is isomorphic to $\mathbf{R}^d$. Now $V = Z(\mu) \cap C = \cap_{x \in \operatorname{supp}\mu} \ker(i_x)$ and hence V is a (possibly trivial) vector subspace and $C = V \times W$, a direct product. Now for each n, $x_n = z_n + y_n$, where $z_n \in V$ and $y_n \in W$. Let $\mu'_n = z_n^{-1}\mu_n z_n$, for each n. Since V centralizes $G(\mu)$ and hence H which is open, by Lemma 2.4, $\mu'_n \to \mu$.

Now it is enough to show that $\{y_n\}$ is relatively compact. If possible, suppose it has a subsequence, denote it by $\{y_n\}$ again, which is divergent, i.e. it has no convergent subsequence. We know that $\{y_n^{-1}\mu'_n y_n = x_n^{-1}\mu_n x_n\}$ is relatively compact. Passing to a subsequence if necessary, we get that $y_n/||y_n|| \to y$ in W, where $|| \; ||$ denotes the usual norm in the vector space C. Since $\mu'_n \to \mu$, arguing as in Proposition 9 in [M1], we get that $G(\mu) \subset Z(y)$, the centralizer of y in G, a contradiction as $y \notin Z(\mu) \cap C = V$, for $y \in W$ and $||y|| = 1$. Therefore, $\{y_n\}$ is relatively compact. If $x \in I_\mu$ then $x\mu x^{-1} = \mu$ therefore, $(I_\mu \cap C)/(Z(\mu) \cap C)$ is a compact group, but since C and $Z(\mu) \cap C$ are both vector groups so is $C/(Z(\mu) \cap C)$ and hence has no nontrivial compact subgroups. Therefore, $I_\mu \cap C = Z(\mu) \cap C$.

PROPOSITION 2.6

Let G and C be as above. Let $\{\nu_n\}$ be a relatively compact sequence in $M^1(G)$ such that $\nu_n^{k_n} \to \mu$ and the closed subgroup (say) H, generated by the centralizer $Z(C)$ of C and $\operatorname{supp}\mu$, *is open in G. Let $A = \{\nu_n^m \mid m \leq k_n\}$. If A/C is relatively compact then so is A.*

Proof. Let A/C be relatively compact. If possible, suppose that A is not relatively compact. That is, there exists a subsequence of $\{\nu_n\}$, denote it by same notation, such that $\{\nu_n^{l(n)}\}$ is divergent, where $l(n) < k_n$ for all n. Passing to a subsequence if necessary, we get that $\nu_n \to \nu$ (say). Let $\pi : G \to G/G^0$ be the natural projection. Since $\{\pi(\nu)^n \mid n \in \mathbf{N}\} \subset \overline{\pi(A)}$ which is compact, $G(\pi(\nu))$ is compact. Also, since $\{\nu^n \mid n \in \mathbf{N}\} \subset T_\mu$, by Theorem 2.4 of [S4], $\operatorname{supp}\nu \subset xI(\mu) = I(\mu)x$, for any $x \in \operatorname{supp}\nu$. Since A/C is relatively compact, there exists a sequence $\{x_{n,m}\}$ in C such that $\{\nu_n^m x_{n,m}\}$ is relatively compact and $\{x_{n,l(n)}\}$ is divergent. Also since $\nu_n^{k_n} \to \mu$ (resp. $\nu_n^{k_n+1} \to \mu\nu = \nu\mu$) the above implies $\{x_{n,m}^{-1}\nu_n^{k_n-m}\}$ (resp. $\{x_{n,m}^{-1}\nu_n^{k_n+1-m}\}$) and hence $\{x_{n,m}^{-1}\nu_n^{k_n}x_{n,m}\}$ (resp. $\{x_{n,m}^{-1}\nu_n^{k_n+1}x_{n,m}\}$) is relatively compact. Now by Proposition 2.5, $\{x_{n,m}\}/(Z(\mu) \cap C)$ (resp. $\{x_{n,m}\}/(Z(\mu\nu) \cap C)$) is relatively compact. As $Z(\mu) \cap Z(\mu\nu) = Z(\mu) \cap Z(\nu)$, the above implies that $\{x_{n,m}\}/(Z(\mu) \cap Z(\nu) \cap C)$ is relatively compact. Without loss of generality we may assume that $\{x_{n,m}\} \subset C' = Z(\mu) \cap Z(\nu) \cap C$, which is a vector group centralizing $G(\nu)$ and H. Therefore, $H' = Z(C')$ contains H and hence it is an open subgroup in G containing $\operatorname{supp}\mu$ and $\operatorname{supp}\nu$. We may also assume that $x_{n,1} = x_{n,k_n} = e$ for every n as $\{\nu_n\}$ and $\{\nu_n^{k_n}\}$ are relatively compact.

Let $n \in \mathbf{N}$ and let $1 \leq m \leq k_n$. From Theorem 2.2, $\nu_n^m(H') > \delta > 0$ and hence $\nu_n^m x_{n,m}(H') > \delta$. Since $\{\nu_n^m x_{n,m}\}$ is relatively compact, there exists a compact set $L \subset H'$, such that $\nu_n^m x_{n,m}(L) > \delta/2$. Let $0 < \epsilon < \min\{\delta/2, 1/4\}$. There exists a compact set $K \subset \operatorname{supp}\mu$ such that $\mu(K) > 1 - \epsilon$. Let $U \subset H'$ be such that U is open in G. Then there exists N, such that for all $n \geq N$, $\nu_n^{k_n}(KU) > 1 - \epsilon$. Let $n \geq N$ and let $1 \leq m \leq k_n$. Then there exists $\{y_{n,m}\} \subset G$, such that $\nu_n^m y_{n,m}(KU) > 1 - \epsilon$. Since $\epsilon < \delta/2$, $KUy_{n,m}^{-1} \cap Lx_{n,m}^{-1} \neq \emptyset$. That is, $y_{n,m}^{-1} \in K'x_{n,m}^{-1}$, where $K' = (KU)^{-1}L \subset H'$ and hence $\nu_n^m x_{n,m}(K_1) > 1 - \epsilon$ and each $x_{n,m}$ commutes with all the elements of $K_1 = KUK' \subset H'$. Now for $m, l < k_n$ such that $m + l \leq k_n$, we get that $\nu_n^{m+l}(K_1 x_{n,m}^{-1} K_1 x_{n,l}^{-1}) \geq (1 - \epsilon)^2$. Since

$\nu_n^{m+l}(K_1 x_{n,m+l}^{-1}) > 1-\epsilon$ and $\epsilon < 1/4$, we get that $K_1 x_{n,m+l}^{-1} \cap K_1 x_{n,m}^{-1} K_1 x_{n,l}^{-1} \neq \emptyset$. Therefore $x_{n,m} x_{n,l} x_{n,m+l}^{-1} \in K_1^2 K_1^{-1} \cap C'$. Since C' is a vector group, C' is strongly root compact by 3.1.12 of [H] and hence by the definition of strong root compactness (see 3.1.10 of [H]), there exists a compact subset K'' such that $x_{n,m} \in K''$, for all m, n. This is a contradiction to the fact that $\{x_{n,l(n)}\}$ is divergent. Therefore A is relatively compact. This completes the proof.

Let $\lambda \in M^1(G)$. For some $\alpha = (r_1, l_1, \ldots, r_m, l_m)$, where $m \in \mathbf{N}$, and $r_i, l_i \in \mathbf{N} \cup \{0\}$ fixed, let $\alpha(\lambda) = \lambda^{r_1} \tilde{\lambda}^{l_1} \ldots \lambda^{r_m} \tilde{\lambda}^{l_m}$, where $\lambda^0 = \tilde{\lambda}^0 = \delta_e$. For any such α, the map $\lambda \mapsto \alpha(\lambda)$ on $M^1(G)$ is continuous. Also, $G(\lambda) = \overline{\cup_\alpha \operatorname{supp} \alpha(\lambda)}$ (over all possible choices of α as above).

Proof of theorem 2.1. Without loss of generality we may assume that $\{\nu_n\}$ is convergent, that is $\nu_n \to \nu$ (say). From the hypothesis, $G(\pi(\nu))$ is compact, and hence by Theorem 2.2, $\pi(A)$ is relatively compact. It is enough to show that A is relatively compact as by Theorem 3.6 of [S1], there exists x such that $x\mu = \mu x$ is embeddable.

Step 1. Let K be the maximal compact normal subgroup of G^0, then K is characteristic in G^0 and hence normal in G. Since A is relatively compact if and only if its image on G/K is relatively compact, without loss of generality we may assume that G^0 has no nontrivial compact normal subgroups. In particular, G^0 is a Lie group. Let L be any open Lie projective subgroup of G. Let M be any compact normal subgroup of L such that L/M is a Lie group, then $G^0 M = G^0 \times M$, a direct product, as both G^0 and M are normal in L and $G^0 \cap M = \{e\}$. Moreover $H = G^0 M$ is an open subgroup in G. Since $I(\mu)$ is compact, without loss of generality, we may assume that $I(\mu)$ normalizes H.

Step 2. Now we prove the assertion by induction on the dimension of the Lie group G^0. Let $\dim G^0 = 0$. Then G is totally disconnected and the assertion follows from above. Now suppose that for any $k > 1$, the assertion holds for G such that $\dim G^0 < k$. Now let $\dim G^0 = k$.

Step 3. Suppose that there exists a subsequence of $\{\nu_n\}$, denote it by $\{\nu_n\}$ again, such that $\{\nu_n^{l_n}\}$ is divergent. By Theorem 1.2.21 of [H], there exists a sequence $\{x_n\} \subset G$, such that $\{\nu_n^{l_n} x_n\}$ and hence $\{x_n^{-1} \nu_n^{k_n - l_n}\}$ and $\{x_n^{-1} \nu_n^{k_n} x_n\}$ are relatively compact and we may assume that $\{x_n\}$ is divergent. Since $\pi(A)$ is relatively compact, $\{\pi(x_n)\}$ is relatively compact in G/G^0 and hence we may choose $\{x_n\}$ to be in G^0.

Without loss of generality we may assume that the subgroup N, as in the hypothesis, is the nilradical. Suppose that N is trivial. Then G^0 is a connected semisimple group. Suppose that the center of G is trivial. Then G^0 is an almost algebraic subgroup of $GL_n(\mathbf{R})$. By Propositions 4–6 of [M1], there exists a proper closed subgroup G' of G^0 such that given any relatively compact sequence $\{z_n\} \subset G^0$, the limit points of $\{x_n z_n x_n^{-1}\}$ are contained in G'. Now since $G^0 \subset G(\mu)$, there exists an $x \in G(\mu) \cap (G^0 \setminus G')$. Since $G^0 \setminus G'$ is open in G^0, there exists a set U which is open in G^0 such that $x \in U$, $\overline{U} \subset G^0 \setminus G'$ and $\overline{U}$ is compact. Then for some $\alpha = (r_1, l_1, \ldots, r_m, l_m)$, we have that $\alpha(\mu)(UM) = \delta > 0$, as $UM = U \times M$ is open in G, for a compact group M as above. Since $\alpha(\nu_n^{k_n}) \to \alpha(\mu)$, $\alpha(\nu_n^{k_n})(UM) > \delta/2$ for all large n. Now since $\{x_n^{-1} \nu_n^{k_n} x_n\}$ is relatively compact, so is $\{x_n^{-1} \alpha(\nu_n^{k_n}) x_n\}$. Therefore, there exists a compact set K such that $(x_n^{-1} \alpha(\nu_n^{k_n}) x_n)(K) = \alpha(\nu_n^{k_n})(x_n K x_n^{-1}) > 1 - \delta/4$ for all n. From the above equation $UM \cap x_n K x_n^{-1} \neq \emptyset$, for all large n. Therefore, there exists a sequence $\{a_n\} \subset K$, such that for all large n, $x_n a_n x_n^{-1} = u_n v_n$, where $u_n \in U$ and $v_n \in M$ and hence $x_n a_n v_n^{-1} x_n^{-1} = u_n$. For each n, put $z_n = a_n v_n^{-1}$, then since $x_n, u_n \in G^0$, $z_n \in G^0$. Also $\{z_n\} \subset KM$ is relatively compact. Therefore the limit points of $\{x_n z_n x_n^{-1} = u_n\}$ belong to G'. But $\{u_n\} \subset U$ and $\overline{U} \subset G^0 \setminus G'$, a contradiction. Therefore, A is relatively compact.

Step 4. Now suppose G^0 is a semisimple group with nontrivial center Z. Then Z is a discrete group normal in G and $Z = \mathbf{Z}^n$, for some n, as we have assumed that G^0 has no nontrivial compact subgroups normal in G. The action of G on $\mathbf{Z}^n$ extends to the action of G on $\mathbf{R}^n$. Therefore, we can form a semidirect product $G_1 = G \cdot \mathbf{R}^n$. Let $D = \{(z, z) \mid z \in \mathbf{Z}^n\}$. Then D is normal in G_1. Now G can be embedded as a closed subgroup in $G_2 = G_1/D$ and $G_2^0 = (G^0 \times \mathbf{R}^n)/D$. It is easy to see that the center C of G_2^0 is isomorphic to $\mathbf{R}^n$. Also, C is normal in G_2 and G_2^0/C is a semisimple group with trivial center. Let $\psi : G_2 \to G_2/C$ be the natural projection. It is easy to see that $G(\psi(\mu))$ contains G_2^0/C, the connected component in G_2/C, and hence by the above argument, $\psi(A)$ is relatively compact. Since H centralizes G^0 in G, $H' = H \times \mathbf{R}^n = G^0 \times M \times \mathbf{R}^n$ is open in G_1 and hence H'/D is an open subgroup in G_2 which centralizes C. Now the assertion in this case follows from Proposition 2.6.

Step 5. Now suppose the nilradical N of G is nontrivial. Let C be the center of N. Since G^0 does not contain any compact subgroups normal in G, C is a vector group, i.e. C is isomorphic to $\mathbf{R}^n$, for some n. Since N is normal in G, so is C. Let $\psi : G \to G/C$ be the natural projection. Then since $\dim G^0/C < k$, we have that $\psi(A)$ is relatively compact. Now since C centralizes $N \times M$, M as above, and $\operatorname{supp} \mu$ and N generate a subgroup containing G^0, the assertion follows from Proposition 2.6.

Remark. Theorem 2.1 continues to hold if the conditions in it are replaced by the following: $\nu_n^{k_n} \to \mu$, the closed subgroup generated by $\operatorname{supp} \mu$ and N is whole of G (where N is as in the hypothesis of the theorem), $\{\nu_n\}/G^0$ is relatively compact and for any limit point ν of it, $G(\nu)$ is compact in G/G^0. For the proof, A/G^0 is relatively compact by Theorem 2.2 and the first three steps of the proof of the above theorem will apply word for word. Also, for a normal subgroup C in steps 4 and 5 above, $Z(\mu) \cap C$ is a central vector group in G by the above condition and hence by Proposition 2.5, the relative compactness of A/C implies that of $A/(Z(\mu) \cap C)$. Therefore A is relatively compact by Lemma 3.2 of [S1]. The above variation of Theorem 2.1 generalizes Theorem 3.1 of [S1].

3. Limit theorems on discrete linear groups over R

Theorem 3.1. *Let G be a discrete linear group over $\mathbf{R}$. Let $\{\nu_n\}$ be a sequence in $M^1(G)$ such that $\nu_n^{k_n} \to \mu$, for some $\mu \in M^1(G)$ and some unbounded sequence $\{k_n\}$ in $\mathbf{N}$. Then there exists $x \in I_\mu$, such that $x\mu$ is embeddable.*

Remark. So far, in the limit theorems on discrete groups, one had either the support condition or the infinitesimality condition imposed (see [S4] and Theorem 2.2 above). The above theorem gives a generalization of Theorems 1.5, 1.7(1) of [S4] for this special class of discrete groups. It also generalizes Theorem 1.2 of [DM3]. One cannot get an embedding of μ itself or an element x as above to be infinitely divisible as in $G = GL(1, \mathbf{Z}) = \{-1, 1\}$, for $x = -1$, $\delta_x = \delta_x^{2n+1}$, for all n, but δ_x is clearly not infinitely divisible and hence not embeddable.

To prove the theorem, we need preliminary results.

Lemma 3.2. *Let V be a finite dimensional vector space over* $\mathbf{R}$. *Let* $\{\tau_n\}$ *be a divergent sequence in* $GL(V)$ *such that for some* $b > 0$, $|\det(\tau_n)| \geq b$ *for all n. Then there exists a proper subspace W of V such that the following holds: if* $\{\mu_n\} \subset M^1(V)$ *is such that* $\mu_n \to \mu$ *and* $\{\tau_n(\mu_n)\}$ *is relatively compact, then* $\operatorname{supp}\mu \subset W$.

The proof of the Lemma is exactly same as the proof of Proposition 3.2 in [S2] using Proposition 1.4 in [DM1]. We will not repeat it here.

PROPOSITION 3.3

Let G be a discrete linear group over $\mathbf{R}$ *and let* $\{\mu_n\}$ *be a sequence converging to* μ *in* $M^1(G)$. *Let* $\lambda_n \in T_{\mu_n}$ *for each n. Then there exist sequences* $\{z_n\}$ *and* $\{z'_n\}$ *in* $Z(\mu)$ *such that* $\{\lambda_n z_n\}$ *and* $\{z'_n\lambda_n\}$ *are relatively compact and all their limit points belong to* T_μ.

Proof. There exists a sequence $\{\lambda'_n\}$ in $M^1(G)$, such that $\lambda_n\lambda'_n = \lambda'_n\lambda_n = \mu_n \to \mu$. By Lemma 2.3, there exists a sequence $\{x_n\}$ in G such that $\{\lambda_n x_n\}$ and $\{x_n\lambda_n\}$ are relatively compact and all its limit points are supported on $\operatorname{supp}\mu$. Therefore, by Theorem 1.2.21 of [H], $\{x_n^{-1}\lambda'_n\}$, $\{\lambda'_n x_n^{-1}\}$ and hence $\{x_n^{-1}\mu_n x_n\}$ and $\{x_n\mu_n x_n^{-1}\}$ are all relatively compact. If ν is a limit point of $\{x_n^{-1}\lambda'_n\}$ then there exists a limit point λ of $\{\lambda_n x_n\}$ such that $\lambda\nu = \mu$. Since $\operatorname{supp}\lambda \subset \operatorname{supp}\mu = \overline{\operatorname{supp}\lambda\ \operatorname{supp}\nu}$, $\operatorname{supp}\nu \subset G(\mu)$. Therefore all the limit points of $\{x_n^{-1}\lambda'_n\}$ and also of $\{x_n^{-1}\mu_n x_n\}$ are supported on $G(\mu)$.

Similarly, the limit points of $\{x_n\mu_n x_n^{-1}\}$ are also supported on $G(\mu)$, and $\{x_n^{-1}\alpha(\mu_n)x_n\}$ and $\{x_n\alpha(\mu_n)x_n^{-1}\}$ are relatively compact and their limit points are supported on $G(\mu)$, for any α (where α and $\alpha(\mu_n)$ are defined as in § 2). Also, for any $\epsilon > 0$, there exists a compact set K such that $(x_n^{-1}\mu_n x_n)(K) > 1 - \epsilon$ for all n. Now for any limit point γ of $\{x_n^{-1}\mu_n x_n\}$, $\gamma(K \cap G(\mu)) > 1 - \epsilon$. Therefore it is easy to see that $(x_n^{-1}\mu_n x_n)(K') > 1 - \epsilon$, for all large n, where $K' = K \cap G(\mu)$.

We know that $G \subset GL(n, \mathbf{R}) \subset M(n, \mathbf{R})$. Let V_μ be the vector space generated by $G(\mu)$ in $M(n, \mathbf{R})$. There exists a finite set $\{y_1, \ldots, y_m\} \subset G(\mu)$ such that $\{y_1, \ldots, y_m\}$ generates V_μ. Since $G(\mu) = \cup_\alpha \operatorname{supp}\alpha(\mu)$, where α and $\alpha(\mu)$ are as defined in § 2, there exist $\alpha_1, \ldots, \alpha_m$ such that $y_i \in \operatorname{supp}\alpha_i(\mu)$, for each i. Therefore, as G is discrete, for some $\delta > 0$, $\alpha_i(\mu)\{y_i\} > \delta$ for all i. Since $\alpha_i(\mu_n) \to \alpha_i(\mu)$, there exists N such that $\alpha_i(\mu_n)\{y_i\} > \delta/2$, for all $n > N$, for all i.

Now since $\{x_n^{-1}\alpha_i(\mu_n)x_n\}$ is relatively compact and all its limit points are supported on $G(\mu)$, arguing as above we can get a compact set $K_1 \subset G(\mu)$, such that $(x_n^{-1}\alpha_i(\mu_n)x_n)(K_1) > 1 - \delta/2$ for all i, for all large n. That is, $\alpha_i(\mu_n)(x_n K_1 x_n^{-1}) > 1 - \delta/2$ for all i, for all large n. Therefore, $y_i \in x_n K_1 x_n^{-1}$, or $x_n^{-1}y_i x_n \in K_1 \subset G(\mu) \subset V_\mu$, for all large n. Since V_μ is generated by $\{y_1, \ldots, y_m\}$, the above implies that $x_n^{-1}V_\mu x_n = V_\mu$, for all large n.

Let $\tilde{G}$ be the Zariski closure of G in $GL(d, \mathbf{R})$ and let $N(V_\mu)$ (resp. $Z(V_\mu)$) be the normaliser (resp. centraliser) of V_μ in $\tilde{G}$. Then $Z(V_\mu)$ and $N(V_\mu)$ are algebraic subgroups of $\tilde{G}$ and $Z(V_\mu)$ is normal in $N(V_\mu)$. Now $N(V_\mu)$ acts on V_μ linearly and the map $\rho : N(V_\mu) \to GL(V_\mu)$ is a rational morphism, as in the proof of Theorem 3.2 in [DM2]. Therefore, the image of ρ, $\operatorname{Im}(\rho)$ is closed in $GL(V_\mu)$ and since $\ker\rho = Z(V_\mu)$, $\rho' : N(V_\mu)/Z(V_\mu) \to \operatorname{Im}\rho$ is a topological isomorphism.

We know that $\{x_n\} \subset N(V_\mu)$. Now if possible, suppose that $\{x_n\}/Z(V_\mu)$ is not relatively compact. Going to a subsequence if necessary, without loss of generality, we may assume that $\{x_n\}/Z(V_\mu)$ is divergent; i.e. it has no convergent subsequence, and for some $\delta > 0$, either $|\det\rho'(x_n Z(V_\mu))| = |\det\rho(x_n)| > \delta$ or $|\det\rho'(x_n^{-1}Z(V_\mu))| > \delta$.

Suppose $|\det\rho'(x_n Z(V_\mu))| = |\det\rho(x_n)| > \delta$ for all n. By Lemma 3.2, there exists a proper subspace W of V_μ such that $\operatorname{supp}\alpha(\mu) \subset W$ for all α, as $\alpha(\mu_n) \to \alpha(\mu)$ and $\{(\rho'(x_n Z(V_\mu)))(\alpha(\mu_n)) = x_n\alpha(\mu_n)x_n^{-1}\}$ is relatively compact. This implies that $G(\mu) = \cup_\alpha \operatorname{supp}\alpha(\mu) \subset W$, a contradiction as $G(\mu)$ generates V_μ and W is a proper subspace.

Now suppose $|\det\rho'(x_n^{-1}Z(V_\mu))| > \delta$. Now using the fact that for every α, $\{(\rho'(x_n^{-1}Z(V_\mu)))(\alpha(\mu_n)) = x_n^{-1}\alpha(\mu_n)x_n\}$ is relatively compact and replacing $\{x_n\}$ by $\{x_n^{-1}\}$ in the above argument we arrive at a contradiction. Therefore, $\{x_n\}/Z(V_\mu)$ is relatively compact.

Clearly, $N(V_\mu)\cap G$ normalizes $Z(V_\mu)$. Let $H = (N(V_\mu)\cap G)Z(V_\mu)$ and let $x \in H$. Then $x(V_\mu \cap G)x^{-1} = V_\mu \cap G$. Let G_μ be the closed subgroup generated by $V_\mu \cap G$ in G. Then $G(\mu) \subset G_\mu$ and x normalizes G_μ. Therefore $\overline{H}$ is a closed subgroup (in $\tilde{G}$) normalizing G_μ. Since G_μ is discrete, the connected component $\overline{H}^0$ of $\overline{H}$, centralizes G_μ and hence $\overline{H}^0 \subset Z(V_\mu) \subset H$ as V_μ is generated by $G(\mu)$ and $G(\mu) \subset G_\mu$. Since $\overline{H}^0$ is open in $\overline{H}$, it follows that H is open in $\overline{H}$. That, is $\overline{H} = H$ and H is a closed subgroup. This implies that $((N(V_\mu)\cap G)Z(V_\mu))/Z(V_\mu)$ is isomorphic to $(N(V_\mu)\cap G)/(Z(V_\mu)\cap G)$. Therefore $\{x_n\}/Z(\mu)$ is relatively compact as $Z(V_\mu)\cap G = Z(\mu)$. Therefore $x_n = z_n a_n = a_n z_n'$, for some relatively compact sequence $\{a_n\}$ in G and some sequences $\{z_n\}$ and $\{z_n'\}$ in $Z(\mu)$. Also, since $\{\lambda_n x_n\}$ and $\{x_n\lambda_n\}$ are relatively compact, so are $\{\lambda_n z_n\}$ and $\{z_n'\lambda_n\}$, and all their limit points belong to T_μ by Lemma 2.4.

Proof of Theorem 3.1. Since $\nu_n^{k_n} \to \mu$, by Proposition 3.3, for any m, there exists a sequence $\{z_{m,n}\} \subset Z(\mu)$ such that $\{\nu_n^m z_{m,n}\}$ is relatively compact. Passing to a subsequence if necessary, without loss of generality, we may assume that $\{\nu_n z_{1,n}\}$ is convergent, with the limit ν. Then $\nu \in T_\mu$ by Lemma 2.4. Also, for any m, $\{\nu_{m,n} = z_{m,n}^{-1}\nu_n z_{m+1,n}\}$ is relatively compact and its limit points are of the form $z\nu = \nu z'$, for some $z, z' \in Z(\mu)$ (cf. Lemma 2.4).

Suppose for any fixed m, the limit points of $\{\nu_n^m z_{m,n}\}$ are of the form $\nu^m z_m$ for some $z_m \in Z(\mu)$. Then combining the above two statements, we get that the limit points of $\{\nu_n^{m+1} z_{m+1,n}\}$ have the form $\nu^m z_m z\nu = \nu^{m+1} z_{m+1}$, for some $z_{m+1} \in Z(\mu)$. By induction, for any m, the limit points of $\{\nu_n^m z_{m,n}\}$ are of the form $\nu^m z_m$, for some $z_m \in Z(\mu)$. Moreover, by Lemma 2.4, $\nu^m \in T_\mu$, as it is a limit point of $\{\nu_n^m z_{m,n} z_m^{-1}\}$, for each m. Also $\operatorname{supp}\nu \subset N(\mu)$.

Now by Proposition 3.3, $\{\nu^n\}/Z(\mu)$ is relatively compact. Therefore $G(\nu)Z(\mu)/Z(\mu)$ is compact and hence finite of order (say) s, as G is discrete. Let $x \in \operatorname{supp}\nu$, then $x^s \in Z(\mu)$. Let $\beta = \nu^s z = z\nu^s$ for $z = x^{-s} \in Z(\mu)$. Then $e \in \operatorname{supp}\beta$ and $\beta^n \in T_\mu$ for all n. Therefore by Theorem 2.4 of [S4], $\operatorname{supp}\beta \subset I(\mu)$ and, furthermore, $\beta^n \to \omega_H$, where $H = G(\beta) \subset I(\mu)$. Hence $\operatorname{supp}\nu \subset xH \cap Hx$. Therefore $x\mu = \nu\mu = \mu\nu = \mu x$, and hence $x \in I_\mu$, for all $x \in \operatorname{supp}\nu$.

Now we show that μ has a shift which is infinitely divisible. Let $l \in \mathbf{N}$ be fixed. Let $a_n = [k_n/l]$ and $b_n = k_n - la_n$. Then for any, $m \le l$, $\nu_n^{ma_n}\nu_n^{k_n - ma_n} \to \mu$ and hence there exist sequences $\{z_{m,n}'\}$ in $Z(\mu)$ such that $\{\nu_n^{ma_n} z_{m,n}'\}$ are relatively compact. Arguing as above, we get that the limit points of $\{\nu_n^{la_n} z_{l,n}'\}$ are of the form $\lambda_l^l z$, for some $z \in Z(\mu)$ and some limit point λ_l of $\{\nu_n^{a_n} z_{1,n}'\}$. Let $r \in \mathbf{N}$ be fixed. Since $a_n \to \infty$, for large n such that $a_n > r$, $\nu_n^{a_n} z_{1,n}' = \nu_n^r z_{r,n}\gamma_n$, where $\{\gamma_n = z_{r,n}^{-1}\nu_n^{a_n - r} z_{1,n}'\}$ which is relatively compact and hence $\lambda_l = \nu^r\gamma$ for some γ. Also $\nu_n^{a_n} z_{1,n}' = \nu_n^{a_n - r}\nu_n^r z_{1,n}'$. By Proposition 3.3, there exists $\{y_n\}$ in $Z(\mu)$ such that $\{\nu_n^{a_n - r} y_n\}$ is relatively compact and hence so is $\{y_n^{-1}\nu_n^r z_{1,n}'\}$

and all its limit points are of the form $z'\nu^r$ for some $z' \in Z(\mu)$ (cf. Lemma 2.4). That is, $\lambda_l = \gamma'\nu^r$, for some γ' and hence for $\beta = \nu^s z = z\nu^s$ defined as above, $\lambda_l = \beta^r\beta' = \beta''\beta^r$ for some β', β''. Since this is true for all r, $\omega_H \in T_{\lambda_l}$. That is, $\lambda_l\omega_H = \omega_H\lambda_l = \lambda_l$ for all l.

For each n, let $z_n = (z'_{l,n})^{-1}$. Then the sequence $\{z_n\nu_n^{b_n}\}$ is relatively compact. Clearly, $b_n < l$ for all n. Let $r < l$ be such that $r = b_{n_k}$ for infinitely many n_k. Then clearly the limit points of $\{z_n\nu_n^{b_n}\}$ are contained in $\{g\nu^r \mid r < l, g \in Z(\mu)\}$ and hence if ρ_l is any such limit point then $\operatorname{supp}\rho_l \subset G(\nu)Z(\mu) \subset I_\mu$ and $\rho_l\omega_H = x_l\omega_H$ (resp. $\omega_H\rho_l = \omega_H x_l$), where $x_l^s \in Z(\mu)$, where s is the cardinality of $G(\nu)Z(\mu)/Z(\mu)$.

Combining the above we get that $\mu = \lambda_l^l\rho_l = \lambda_l^l\omega_H\rho_l = \lambda_l^l x_l (= x_l\lambda_l^l)$ for some $x_l \in \operatorname{supp}\rho_l \subset I_\mu$, for each l. That is, μ is weakly infinitely divisible. As $\lambda_l \in T_\mu$, $\operatorname{supp}\lambda_l \subset y_l G(\mu)$ for some $y_l \in \operatorname{supp}\lambda_l \subset N(\mu)$. Since for each l, $\mu = \lambda_l^l x_l$ and $x_l \in G(\nu)Z(\mu)$, we get that $y_l^l \in G(\nu)Z(\mu)G(\mu)$. Hence $(y_l)^{ls} \in G(\mu)Z(\mu)$, as $G(\nu)Z(\mu)/Z(\mu)$ is a finite group of order s. Since $T_\mu/Z(\mu)$, is relatively compact, arguing as in Theorem 3.1 of [DM3], we get that $F = T_\mu/G(\mu)Z(\mu)$ is finite and it obviously consists of dirac measures. Also, the above implies that the image of λ_l on $G' = N(\mu)/G(\mu)Z(\mu)$ is $\delta_{\bar{y}_l}$, where $\bar{y}_l = y_l G(\mu)Z(\mu)$ in G' and $\bar{y}_l^{\,ls} = \bar{e}$, the identity in G'. Let $B = \{\gamma \in F \mid \gamma^r = \delta_{\bar{e}}$ for some $r \in \mathbf{N}\}$. Since F is finite, so is B and there exists an element of maximal order in B; let i be the maximal order. Then $\gamma^{i!} = \delta_{\bar{e}}$ for all $\gamma \in B$. Since the image of λ_l on $N(\mu)/G(\mu)Z(\mu)$ belongs to B, we have that $\operatorname{supp}\lambda_l^{i!} \subset G(\mu)Z(\mu)$, for all l. Now for each m, let $\beta_m = \lambda_{i!m}^{i!}$, where $\mu = \lambda_{i!m}^{i!m}x$, for some $x \in I_\mu$. Then $\mu = \beta_m^m x$ and $\operatorname{supp}\beta_m \subset G(\mu)Z(\mu)$. Also, since $\operatorname{supp}\beta_m \subset yG(\mu)$ for some $y \in \operatorname{supp}\beta_m$, $y = zy' = y'z$, for some $y' \in G(\mu), z \in Z(\mu)$. Then $\beta'_m = z^{-1}\beta_m = \beta_m z^{-1}$ is supported on $G(\mu)$. Also, $\mu = \beta_m^m x = (\beta'_m)^m z^m x = (\beta'_m)^m x'$, where $x' = z^m x \in I_\mu \cap G(\mu)$ as $\operatorname{supp}\beta'_m \subset G(\mu)$. That is, μ is weakly infinitely divisible on $G(\mu)$. Moreover, from the above equation, we have that $\{\beta'_m\}/Z_\mu$ is relatively compact, where $Z_\mu = G(\mu) \cap Z(\mu)$ is the center of $G(\mu)$ (cf. [DM3], Theorem 2.1). In fact, $\{\beta'_m z_m\}$ is relatively compact for some sequence $\{z_m\}$ in Z_μ. Let $\gamma'_m = \beta'_m z_m$. Then $(\gamma'_m)^m = (\beta'_m)^m z_m^m$ and hence $\mu = (\gamma'_m)^m x_m$ for some $x_m \in I_\mu \cap G(\mu)$, for all m. Now if γ' is a limit point of $\{\gamma'_m\}$ then $(\gamma')^n \in T_\mu$ for all n and hence, as earlier, $\operatorname{supp}\gamma' \subset xI(\mu) = I(\mu)x$, for some $x \in I_\mu \cap G(\mu)$. Since $(I_\mu \cap G(\mu))/Z_\mu$ is finite (cf. [DM3], Theorem 2.1), if a is its cardinality then $\operatorname{supp}(\gamma')^a \subset zI(\mu) = I(\mu)z$ for some $z \in Z_\mu$. Therefore limit points of $\{(\gamma'_m)^a\}$ are supported on $zI(\mu) = I(\mu)z$, $z \in Z_\mu$. Let $\gamma_m = (\gamma'_{am})^a$. Then $\mu = \gamma_m^m x_{am}$, where $x_{am} \in I_\mu \cap G(\mu)$. Let $\{\gamma_{c_m}\}$ be a convergent subsequence of $\{\gamma_{m!}\}$ converging to γ. Then from above, $\operatorname{supp}\gamma \subset zI(\mu) = I(\mu)z$ for some $z \in Z_\mu$. Therefore, for each m, replacing γ_{c_m} by $\gamma_{c_m}z^{-1}$ (and using the same notation), we get that $\mu = \gamma_{c_m}^{c_m} y_m$, $y_m \in I_\mu \cap G(\mu)$ and $\gamma_{c_m} \to \gamma$ and $G(\gamma) \subset I(\mu)$, which is compact. Also $\{y_m\}/Z_\mu$ is finite, and hence passing to a subsequence again, we may assume that $y_m = az'_m = z'_m a$, where $a \in I_\mu \cap G(\mu)$ and $z'_m \in Z_\mu$. Therefore, $\gamma_{c_m}^{c_m} z'_m = a^{-1}\mu = \mu a^{-1}$. Now applying Theorem 2.2, we get that $A = \{\gamma_{c_m}^n \mid n \le c_m\}$ and $\{z'_m\}$ are relatively compact. Now if β is a limit point of $\{\gamma_{c_m}^{c_m}\}$ then $a^{-1}\mu = \beta z'$ for some $z' \in Z_\mu$. Since for all m, $c_m = l_m!$, where $l_m \to \infty$, any n divides c_m for all large m. Also since A is relatively compact, it is easy to see that β has an n-th root in $\overline{A}$, namely, any limit of the sequence $\{\gamma_{c_m}^{c_m/n}\}$. Therefore, $y\mu = \beta$ is infinitely divisible in the compact set $\overline{A}$, where $y = (z')^{-1}a^{-1} \in I_\mu \cap G(\mu)$. Now as in the proof of Theorem 3.1.32 of [H], $y\mu$ is rationally embeddable, i.e. there exists a homomorphism $f : \mathbf{Q}_+^* \to M^1(G)$ such that $f(]0,1[\cap\mathbf{Q}_+^*) \subset \overline{A}$ is relatively compact and $f(1) = \mu$. Now since G is discrete, any compact connected subgroup of G has to be $\{e\}$. Therefore, as in the proof of Theorem 3.5.4 of [H], f extends to $\mathbf{R}_+$ and hence $y\mu$ is embeddable.

4. Infinitesimally divisible measures on algebraic groups

We first recall that an element s, in a Hausdorff semigroup S with identity e, is said to be infinitesimally divisible if for every neighbourhood U of e in S, s has a U-decomposition, i.e. there exist $s_1, \ldots, s_n \in U$ such that s_i's commute and $s = s_1 \cdots s_n$. The following theorem generalizes Theorem 1.2 of [S3] in a certain sense.

Theorem 4.1. *Let G be a real algebraic group and let $\mu \in M^1(G)$ be infinitesimally divisible in $M^1(G)$. Then there exist a closed semigroup $S \subset M^1_H(G)$, with identity ω_H for some compact subgroup H of $I(\mu)$, and an equivalence relation $\sim$, such that $\mu \in S$ and if $\rho : S \to S^* = S/\sim$ is the natural map then $\rho(\mu)$ is bald and infinitesimally divisible in S^*, and $T_{\rho(\mu)}$ is compact and associatefree in S^*. Moreover, if G is connected and nilpotent then μ is embeddable.*

Before proving the above theorem, we define an equivalence relation on a certain kind of subsemigroup of $M^1(G)$, for any locally compact (Hausdorff) group G. For a $\mu \in M^1(G)$, let S_μ be the closed subsemigroup generated by T_μ in $M^1(G)$. Since $T_\mu \subset M^1(N(\mu))$, $S_\mu \subset M^1(N(\mu))$. In fact, for any $\lambda \in T_\mu$, $\operatorname{supp} \lambda \subset xG(\mu)$, for some $x \in \operatorname{supp} \lambda \subset N(\mu)$. Therefore, it easily follows that for any $\beta \in S_\mu$, $\operatorname{supp} \beta \subset xG(\mu)$, for any $x \in \operatorname{supp} \beta \subset N(\mu)$. We also know that $Z(\mu) \subset T_\mu \subset S_\mu$ and $Z(\mu)T_\mu = T_\mu Z(\mu) = T_\mu$. Let us define an equivalence relation '$\approx$' on S_μ as follows: for any

$$\beta, \lambda \in S_\mu, \beta \approx \lambda \text{ if } \quad \beta = z\lambda \text{ for some } \quad z \in Z(\mu).$$

For $\{\beta_n\}, \{\lambda_n\} \subset S_\mu$, suppose $\beta_n \approx \lambda_n$, i.e. $\beta_n = z_n \lambda_n$ for some $z_n \in Z(\mu)$, for each n. Now if $\beta_n \to \beta$ and $\lambda_n \to \lambda$, then we have that $\{z_n\}$ is relatively compact and for any limit point z of it, $z \in Z(\mu)$ and $\beta = z\lambda$. Therefore, $\beta \approx \lambda$.

Now for $\lambda \in S_\mu$, for any fixed $x \in \operatorname{supp} \lambda$, $\operatorname{supp}(\lambda x^{-1}) \subset G(\mu)$. For any $z \in Z(\mu)$, $z' = xzx^{-1} \in Z(\mu)$ as $Z(\mu)$ is normal in $N(\mu)$ and hence

$$\lambda z = (\lambda x^{-1})xz = (\lambda x^{-1})z'x = z'(\lambda x^{-1})x = z'\lambda.$$

Similarly, one can also show that $z\lambda = \lambda z''$, for some $z'' \in Z(\mu)$.

Now for $i \in \{1, 2\}$, $\beta_i, \lambda_i \in S_\mu$, let $\beta_i \approx \lambda_i$, i.e. there exist $z_i \in Z(\mu)$, such that $\beta_i = z_i \lambda_i$, Then from the above equation, $\beta_1\beta_2 = z_1\lambda_1 z_2\lambda_2 = z_1 z_2' \lambda_1\lambda_2$ for some $z_2' \in Z(\mu)$. That is, $\beta_1\beta_2 \approx \lambda_1\lambda_2$. Let $\psi : S_\mu \to S^*_\mu = S_\mu/\approx$ be the natural projection. Then ψ is a continuous open homomorphism and it is also easy to show that S^*_μ is Hausdorff.

In case of a real algebraic group G, we define an analogous equivalence relation $\approx'$ with respect to $Z^0(\mu)$, the connected component of the identity in $Z(\mu)$, i.e. for $\beta, \lambda \in S_\mu$, $\beta \approx' \lambda$ if $\beta = z\lambda$, for some $z \in Z^0(\mu)$. It is easy to verify as above that this is an equivalence relation using the fact that $Z^0(\mu)$ is normal in $N(\mu)$.

Proof of Theorem 4.1. Let G be a real algebraic group and let μ be infinitesimally divisible in $M^1(G)$. Since G is metrizable, so is $M^1(G)$.

Step 1. Let S_μ, $\approx'$, S^*_μ and $\psi : S_\mu \to S^*_\mu$ be as above. Clearly, S_μ and S^*_μ are second countable and $\psi(\mu)$ is infinitesimally divisible in S^*_μ.

Since G is algebraic, by Theorem 3.2 of [DM2], $T_\mu/Z^0(\mu)$ is relatively compact. Clearly, $\psi(T_\mu) \subset T_{\psi(\mu)}$. Now for any $\{\lambda_n\} \subset T_\mu$, there exists a sequence $\{z_n\} \subset Z^0(\mu)$, such that $\{\lambda_n z_n\}$ is relatively compact and hence $\{\psi(\lambda_n)=\psi(\lambda_n z_n)\}$ is also relatively compact. Since $T_\mu Z(\mu)=T_\mu$, $\{\lambda_n z_n\}\subset T_\mu$ and the above implies that $\psi(T_\mu)$ is compact in S^*_μ.

Since μ is infinitesimally divisible so is $\psi(\mu)$ in S^*_μ. We can choose a neighbourhood basis $\{U_i\}_{i\in\mathbf{N}}$ of δ_e in $M^1(G)$. For any i, there exist $\mu_{i1},\ldots,\mu_{in_i}\in U_i\cap T_\mu$, such that μ_{ij}s commute and $\mu=\mu_{i1}\cdots\mu_{in_i}$. Therefore $\psi(\mu)=\psi(\mu_{i1})\cdots\psi(\mu_{in_i})$ is a $\psi(U_i)$-decomposition of $\psi(\mu)$ in $\psi(T_\mu)$. Let $\Delta=(\mu_{ij})^{n_i}_{i\in\mathbf{N},j=1}$ and $\psi(\Delta)=(\psi(\mu_{ij}))^{n_i}_{i\in\mathbf{N},j=1}$. Then Δ (resp. $\psi(\Delta)$) is a commutative infinitesimal triangular system in S_μ (resp. in S^*_μ) converging to μ (resp. $\psi(\mu)$). In fact, $\mu=\Pi^{n_i}_{j=1}\mu_{ij}$ and $\psi(\mu)=\Pi^{n_i}_{j=1}\psi(\mu_{ij})$.

Step 2. Since I^0_μ is open in I_μ, one can choose U and W to be neighbourhoods of $I^0_\mu J_\mu$ such that $U=\{\nu\in M^1(G)\mid \nu(I^0_\mu I(\mu)V)>\delta\}$, for some $\delta>0$, $\overline{U}\cap I_\mu J_\mu=I^0_\mu J_\mu$, for some relatively compact neighbourhood V of e in G^0 and $WW\subset U$. Now let $\lambda\in S_\mu\cap\overline{U}\setminus W$ be such that $\psi(\lambda^n)\in T_{\psi(\mu)}$ in S^*_μ for all n, then $\mu=\lambda^n\nu_n=\nu'_n\lambda^n$, for some ν_n,ν'_n in S_μ for all n. Then the concentration functions of both λ and $\tilde{\lambda}$ do not converge to zero. Since λ commutes with μ, as in the proof of Theorem 2.4 of [S4], $\operatorname{supp}\lambda\subset xI(\mu)=I(\mu)x$, for some $x\in\operatorname{supp}\lambda\subset I_\mu\cap\overline{U}$. i.e. $\lambda\in I^0_\mu J_\mu$, a contradiction as $\lambda\notin W$. Now as in the proof of Lemma 2.5 in [S4], there exists n such that for any $m\geq n$, $\psi(\mu)$ cannot be expressed as $\psi(\mu)=\psi(\lambda_1)\cdots\psi(\lambda_m)\psi(\nu)$, where $\psi(\lambda_j)$s commute with each other and also with $\psi(\nu)$ for any $\lambda_j\in S_\mu\cap\overline{U}\setminus W$, for all j.

Since $I_\mu\subset T_\mu$, $\psi(I_\mu)$ is compact. Let $K=\psi(I^0_\mu J_\mu)$. Then K is a compact semigroup and $\psi(U\cap S_\mu)$ and $\psi(W\cap S_\mu)$ are neighbourhoods of K in S^*_μ.

Since $\psi(\mu)$ is a limit of a triangular system as above, as in Lemma 2.6 of [S4], given any neighbourhood U' of K in S^*_μ, one can choose small neighbourhoods U and W as above such that $\psi(U\cap S_\mu)\subset U'$ and show that there exists a U'-decomposition of $\psi(\mu)$ in $\psi(T_\mu)$, namely, $\psi(\mu)=\psi(\mu_1)\cdots\psi(\mu_n)$, where each $\psi(\mu_i)\in U'$ is a limit of a subsystem of $\psi(\Delta)$.

Step 3. Let $\{U'_n\}$ be a neighbourhood basis of K in S^*_μ such that $U'_{n+1}\subset U'_n$ for all n and $\cap_{n\in\mathbf{N}}U'_n=K$. Now let $\psi(\mu)=\gamma_1\cdots\gamma_n$ be a U'_1-decomposition of $\psi(\mu)$ in $\psi(T_\mu)$ obtained as above. Given any U'_k-decomposition of $\psi(\mu)$ as $\psi(\mu)=\nu_1\cdots\nu_r$, $\nu_l=\Pi_{j\in J_{il}}\psi(\mu_{k(i)j})$, where $\cup_l J_{il}=\{1,\ldots,n_{k(i)}\}$ we get U'_{k+1}-decomposition of each ν_l in such a way that $\nu_l=\nu_{l1}\cdots\nu_{ln_l}$, $\nu_{lm}\in U'_{k+1}$, where $\nu_{lm}\nu_{pq}=\nu_{pq}\nu_{lm}$, for all l,m,p,q, and all the ν_{lm} are limits of a subsystem of $(\psi(\mu_{(k+1)(i)j}))$, where $\{(k+1)(i)\}$ is a subsequence of $\{k(i)\}$. Clearly $\psi(\mu)=\Pi_{l,m}\nu_{lm}$ is a U'_{k+1}-decomposition for $\psi(\mu)$.

For each $k\in\mathbf{N}$, let M_k be the subsemigroup of S^*_μ generated by U'_k-decomposition obtained in above manner. Then each M_k is abelian, $\mu\in M_k$ and $M_k\subset M_{k+1}$. Let $M=\overline{\cup_k M_k}$ and let $K'=K\cap M=\psi(I^0_\mu J_\mu)\cap M$. Then M (resp. K') is a closed (resp. compact) abelian semigroup. Also, given any neighbourhood U' of K' in M, there exists a neighbourhood U'' of K in S^*_μ, such that $U''\cap M\subset U'$. Hence μ has a U'-decomposition in M for every neighbourhood U' of K'.

Step 4. We now show that $T_{\psi(\mu)}$ is compact in M. Let U, W and V be as in Step 2. Let $\nu\in S_\mu$ be such that $\psi(\nu)\in T_{\psi(\mu)}$ in M. Now $\mu=\nu\nu'=\nu''\nu$ for some $\nu',\nu''\in S_\mu$. Arguing as in Step 2, there exists n (which does not depend on the choice of $\psi(\nu)\in T_{\psi(\mu)}$) such that for any $m\geq n$, $\psi(\nu)$ cannot be expressed as $\psi(\nu)=\psi(\lambda_1)\cdots\psi(\lambda_m)\psi(\beta)$ in M for $\lambda_j\in S_\mu\cap\overline{U}\setminus W$, for all j, and $\psi(\lambda_j)$s commute and they also commute with $\psi(\beta)$. Here, $\psi(\nu)$ is a limit of a commutative K'-infinitesimal triangular system in M, i.e.

$\psi(\nu) = \lim_{i\to\infty} \Pi_{j=1}^{n_i}\psi(\nu_{ij})$ for some $\nu_{ij} \in S_\mu$. Again arguing as in Step 2, $\psi(\nu) = \psi(\nu_1)\cdots\psi(\nu_n)$ for $\psi(\nu_i) \in T_{\psi(\mu)} \cap \psi(\overline{U} \setminus W)$. That is, $\psi(\nu) \in (\psi(\overline{U} \setminus W)^n$. Since n does not depend on the choice of $\psi(\nu)$ in $T_{\psi(\mu)}$, $T_{\psi(\mu)} \subset (\psi(\overline{U} \setminus W))^n$. Hence it is easy to show as in the proof of Lemma 2.1 of [S4] that $T_{\psi(\mu)}$ is relatively compact.

Step 5. Let $J = \psi(J_\mu) \cap M$. Then J is a compact semigroup and there exists a maximal idempotent h_1 in J. Then $J' = Jh_1$ is a group. Let $H = \{x \in I(\mu) \mid \psi(x)h_1 \in J'\}$. It is easy to check that H is a compact group. Let $h = \omega_H$ and let $h^* = \psi(\omega_H)$. Then $Jh^* = J'h^* = h^*$ and $K'' = K'h^* = (\psi(I_\mu^0) \cap M)h^*$, which is a compact group. Let $M^* = Mh^*$. M^* is a closed abelian semigroup with identity h^* and $K'' \subset M^* \subset M$. Now if U is a neighbourhood of K'' in M^* then there exists a neighbourhood U' of K' in M such that $U'h^* \subset U$, and hence if $\psi(\mu) = \lambda_1\cdots\lambda_n$ is a U'-decomposition of $\psi(\mu)$ in M, then since $\mu = \mu h = \mu h^n$, $\psi(\mu) = \lambda_1\cdots\lambda_n\psi(h^n)$ and hence $\psi(\mu) = \lambda_1 h^*\cdots\lambda_n h^*$ is a U-decomposition of $\psi(\mu)$ in M^*. Now we define an equivalence relation $\sim'$ on M^* as follows: For

$$\lambda, \nu \in M^*, \lambda \sim' \nu \text{ if } \quad \lambda = k\nu \text{ for some } \quad k \in K''.$$

Let $S^* = M^*/\sim'$ and let $\phi : M^* \to S^*$ be the natural projection and let $\rho = \phi\circ\psi$. Then S^* is a Hausdorff abelian semigroup with identity $\phi(h^*)$, $\rho^{-1}(S^*) = S$ is a closed semigroup in $M_H^1(G)$, the relation $\sim$ is defined by ρ on S, each $\rho(\lambda)$ in $T_{\rho(\mu)}$ is inifinitesimally divisible in S^* and by step 4, $T_{\rho(\mu)}$ is compact. Now if $a, b \in T_{\rho(\mu)}$ are associates then $a = a'b$ and $b = b'a$. Let $\beta, \beta' \in S_1$ be such that $\rho(\beta) = b$ and $\rho(\beta') = b'$ and $\rho(\gamma) = a'$, then since $b = b'a'b$, $\psi(\beta) = k\psi(\beta')\psi(\gamma)\psi(\beta)$ for some $k \in K''$ and hence $\psi(\beta')^n \in T_{\psi(\beta)}$ for all n. As in step 2, $\operatorname{supp}\beta' \subset xI(\beta) = I(\beta)x$, for some $x \in I_\mu$, and since $\rho(\beta') = b'$ is infinitesimally divisible, it is easy to show that $x \in I_\mu^0$. Therefore b' is identity in S^* and $b = a$, i.e. $T_{\rho(\mu)}$ is associatefree.

Now if $\beta \in S_\mu$ be such that $\rho(\beta) \in T_{\rho(\mu)}$ is an idempotent then $\psi(\beta)^n \in T_{\psi(\mu)}$ for all n and hence as in step 2, $\operatorname{supp}\beta \subset xI(\mu) = I(\mu)x$ for some $x \in I_\mu$. Since $\rho(\beta)$ is also infinitesimally divisible in S^* one can easily show that $x \in I_\mu^0$ and $\beta = x\omega_{H'} = \omega_{H'}x$ for some $H' \subset I(\mu)$ and hence $\psi(\beta) \in K''$ and hence $\rho(\beta)$ is identity in S^*. Therefore $\rho(\mu)$ is bald.

Step 6. Now let G be connected and nilpotent and let Z be the center of G. Then G/Z is simply connected and hence so are $N(Z(\mu))/Z$ and $N(Z(\mu))/Z(\mu)$, where $N(Z(\mu))$ is the normaliser of $Z(\mu)$, and both of them are connected. Therefore, $I_\mu = Z(\mu)$ as $I_\mu/Z(\mu)$ is compact. Hence, in the above equation $K'' = h^*$ and $\sim'$ is a trivial relation, i.e. $S^* = M^*$ and also $\rho = \psi$.

Now we show that for $s \in T_{\psi(\mu)} \setminus \psi(h)$ in S^*, there exists a continuous s-norm f_s on T_s (in S^*) such that $f_s(s) > 0$, (an s-norm on T_s (in S^*) is a map $f_s : T_s \to \mathbf{R}_+$ which is continous at the identity and it is a partial homoporphism, i.e. $f_s(s_1s_2) = f_s(s_1) + f_s(s_2)$ if $s_1, s_2, s_1s_2 \in T_s$). This would imply the embedding of $\psi(\mu)$ in a continuous real one-parameter semigroup $\{\gamma_t\}_{t\in\mathbf{R}_+}$ in S^* (cf. [S3], Theorem 2.3 or [S4], Theorem 4.1) and in particular, $\mu = \lambda_n^n x_n$ $x_n \in Z(\mu) = Z^0(\mu)$.

Let $\lambda \in S$ be such that $\psi(\lambda) = s$. If λ is not a translate of an idempotent then as in the proof of Theorem 5.1 in [S3], there exists a continuous λ-norm f_λ on S such that $f_\lambda(\lambda) > 0$, (it is easy to see that one does not need the underlying semigroup to be abelian in that proof). Moreover, if $\psi(\nu_1) = \psi(\nu_2)$ then $\nu_1 = \nu_2 x$ for some $x \in Z(\mu)$. Then $\nu_1\tilde{\nu}_1 = \nu_2\tilde{\nu}_2$ and $f_\lambda(\nu_1) = f_\lambda(\nu_2)$ (see the proof of Theorem 5.1 in [S3]). Therefore, we can define a s-norm f_s on T_s in S^* such that $f_s(\psi(\nu)) = f_\lambda(\nu)$. Now if λ is indeed a translate of an idempotent, i.e. $\lambda = x\omega_K = \omega_K x$ for some compact group $K \subset I(\mu) \subset Z$, then clearly $x \in I_\mu = Z(\mu)$ and hence $s = \psi(\lambda)$ is an idempotent. Now since $\psi(\mu)$ is bald $s = \psi(h)$, a contradiction.

The embeddability of $\psi(\mu)$ in particular implies that $\psi(\mu) = \psi(\lambda_n)^n$, and hence $\mu = \lambda_n^n x_n$, $x_n \in Z(\mu)$ for all n. Therefore, $\operatorname{supp} \lambda_n^n \subset G(\mu)Z(\mu)$. Here, $\operatorname{supp} \lambda_n \subset y_n G(\mu)$ for some $y_n \in \operatorname{supp} \lambda_n \subset N(\mu)$. Therefore, $y_n^n \in G(\mu)Z(\mu) \subset \tilde{G}(\mu)Z(\mu)$, where $\tilde{G}(\mu)$ is the Zariski closure of $G(\mu)$. Since $N(\mu)/\tilde{G}(\mu)Z(\mu)$ is simply connected, $y_n \in \tilde{G}(\mu)Z(\mu)$ for all n. That is, for each n, $\operatorname{supp} \lambda_n \subset \tilde{G}(\mu)Z(\mu)$ and hence $\lambda_n = \beta_n z_n$ for some $z_n \in Z(\mu)$ and $\operatorname{supp} \beta_n \subset \tilde{G}(\mu)$ and $\mu = \beta_n^n z_n'$, where $z_n' = z_n^n x_n \in Z(\mu)$. Now we have that $z_n' \in C = \tilde{G}(\mu) \cap Z(\mu)$, which is the center of $\tilde{G}(\mu)$. Therefore, $CZ \subset Z(\mu)$ is an abelian algebraic subgroup containing the center Z of G. Therefore CZ is connected, and hence it is divisible. In particular, each z_n' is infinitely divisible in CZ, and hence μ is infinitely divisible on G which is a connected nilpotent Lie group, therefore μ is embeddable (cf. [BM]).

Remark. As remarked in [S4], Theorem 4.1 also holds for $\mu \in M_H^1(G)$ which is infinitesimally divisible in $M_H^1(G)$.

We now state the following theorem for maximally almost periodic groups without a proof. A locally compact group G is said to be *maximally almost periodic* if its irreducible finite dimensional unitary representations separate points of G.

Theorem 4.2. *Let G be a maximally almost periodic first countable group. Let Δ be a commutative infinitesimal triangular system of probability measures converging to μ in $M^1(G)$. Then there exists an $x \in G^0$ such that $x\mu = \mu x$ is embeddable.*

If G is as above then there exists a normal vector subgroup V, such that G^0/V is compact and V centralises an open subgroup of finite index in G (cf. [RW], Theorems 1, 2]. The above theorem can be proven using the above fact, Proposition 2.5, Lemma 2.4, Proposition 3.3 and Theorem 4.2 of [S4] and the techniques developed above.

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