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Variational formulae for Fuchsian groups over families of algebraic curves

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Abstract. We study the problem of understanding the uniformizing Fuchsian groups for a family of plane algebraic curves by determining explicit first variational formulae for the generators.

Keywords. Riemann surfaces; Fuchsian groups; Ahlfors-Bers variational formulae.

1. Introduction

In this paper we make a contribution to the problem of understanding the uniformizing Fuchsian groups for a family of plane algebraic curves by determining explicit first variational formulae for the generators of the Fuchsian groups, say G_t, associated to a t-parameter family of compact Riemann surfaces X_t, where the X_t are the Riemann surfaces for the complex algebraic curves arising from a t-parameter family of irreducible polynomials. The main idea of our work is to utilize explicit quasiconformal mappings between algebraic curves, calculate the Beltrami coefficients, and hence utilize the Ahlfors-Bers variational formulae when applied to quasiconformal conjugates of Fuchsian groups.

We start with a compact Riemann surface X_0, corresponding to the plane algebraic curve $P(x,y) = \sum\sum a_{ij}x^iy^j = 0$, having genus say $g > 1$. Let us assume also that $X_0 = U/G_0$ where G_0 (i.e. the holomorphic deck-transformation group) is known. Then we consider the parametrized family of compact Riemann surfaces X_t corresponding to the polynomial equation $P_t(x,y) = 0$ where $P_t(x,y) = \sum\sum a_{ij}(t)x^iy^j$ such that $a_{ij}(t)$ are holomorphic functions of t (t in a small disk around the origin) with additional restriction that $a_{ij}(0) = a_{ij}$. For such X_t we determine first variational formula for $\gamma_t \in G_t$ where $X_t \equiv U/G_t$ (G_t is the uniformizing Fuchsian group corresponding to X_t)

$$\gamma_t = \gamma + t\dot{\gamma} + \bar{t}\dot{\gamma}^* + o(t), \tag{1}$$

where γ is an element of G_0 (and $\dot{\gamma}$, $\dot{\gamma}^*$ are as in eq. (16)).

Remark. Although we have dealt with compact Riemann surfaces and the torsion-free parabolic-free Fuchsian uniformizing group in the introduction above, the theory of Teichmüller spaces works exactly the same for Riemann surfaces of finite conformal type

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– namely we can allow distinguished points or punctures on the compact Riemann surfaces and correspondingly allow elliptic or parabolic elements in the Fuchsian groups under scrutiny. Those results are exactly parallel and nothing new needs to be said.

2. Invariance of sheet monodromy over families of curves

Monodromy Invariance Lemma. To solve our problems we have to find a correspondence between the ramification (branch) points of $P_t(x,y)=0$ lying on the x-sphere for different values of t. Also we will need to make a correspondence between the algebraic functions $y_t(x)=y(x,t)$ satisfying $P_t(x,y(x,t))=0$ for different values of t, so that the monodromy remains invariant at the corresponding branch points. That will guarantee that the topological structure of the branched covering is kept invariant as t changes.

In order to do this we assume certain restrictions on $P_t(x,y)$:

Assume $\deg P(x,y,t)=D$ for all t. Assume also that there exists r, s such that $r+s=D$ where $0\le r\le m$, $0\le s\le N$ and $a_{rs}(0)\ne 0$ i.e degree $P_0(x,y)=D$.

Assume

(1) $P_0(x,y)$ is irreducible in the polynomial ring $\mathbf{C}[x,y]$.
(2) If degree $P_t(x,y)=D$, then degree $P_0(x,y)=D$; that is if we substitute $t=0$ in $P_t(x,y)$ degree of the polynomial remains the same.
(3) Suppose P_t is of degree N in the y variable for all small t:

$$P_t(x,y)=P_N(x,t)y^N+P_{N-1}(x,t)y^{N-1}+\cdots+P_0(x,t),$$

where

$$P_N(x,t)=a_k(t)x^k+\cdots+a_0(t).$$

Let $D(t)$ denote the discriminant of $P_N(x,t)$. Then assume that $D(0)\ne 0$ and $a_k(0)\ne 0$.
(4) Let $D(x,t)$ be the discriminant of $P_t(x,y)=0$. Then $D(x,t)=P_N(x,t)Q(x,t)$ where

$$Q(x,t)=Q_0(t)x^r+\ldots+Q_r(t).$$

We assume that $Q_0(0)\ne 0$ and $\tilde{D}(0)\ne 0$, where $\tilde{D}(t)=$ discriminant of $Q(x,t)$.
(5) The resultant of $Q(x,t)$ and $P_N(x,t)$ does not vanish at $t=0$.

Assume

$$P(x,y,0)=P_0(x,y)$$

is an irreducible polynomial such that $x=0$ and $x=\infty$ are ordinary points, and the set of ramification points on the x-plane are say located at:

$$\{\zeta_1^0,\ldots,\zeta_k^0\}.$$

Then it is not hard to demonstrate that:

(i) For all t sufficiently close to 0, the polynomial $P_t(x,y)$ is irreducible and $0,\infty$ are ordinary points.
(ii) The ramification points on the x-sphere for $P_t(x,y)$ are holomorphically dependent on t and are given by k holomorphic functions: $\{\zeta_1(t),\ldots,\zeta_k(t)\}$ such that $\zeta_j(0)=\zeta_j^0$ for $0\le j\le k$ and $\zeta_i(t)\ne\zeta_j(t)$ for $i\ne j$ and all t small enough.

(iii) Assume N is the degree of P_t in the y variable (this follows from the stability conditions mentioned above.) Then there exists holomorphic function germs $\{y_1(x,t),\ldots,y_N(x,t)\}$ around $(x,t)=(0,0)\in \mathbf{C}^2$ such that

$$P_t(x,y_j(x,t))=0$$

for all (x,t) sufficiently close to (0,0) and such that N roots of the y equation $P(x,y,t)=0$ are given by $y_j(x,t)$.

(iv) Analytic continuation of $y_1(x,t)$ for every fixed t, $\mid t\mid\leq \epsilon$ in the x-sphere along the same route (avoiding the branch points) produces the *same permutation* of $\{y_1(x,t),\ldots,y_N(x,t)\}$ – i.e., the monodromy permutations are independent of t.

Idea of the proof for (iv): Follow the construction, as in Siegel [S], for each $\zeta_i(0)$ we consider a circle C_i with center at $\zeta_i(0)$ such that any two of them does not intersect and we join the origin to $\zeta_i(0)$ by a simple curve l_i so that if we cut $\mathbf{CP}^1$ along these curves it remains simply connected. Since ζ_i's are holomorphic function of t we can find a neighborhood of $t=0$ say, $N=\{t:\mid t\mid<\epsilon\}$ such that $\zeta_1(N),\ldots,\zeta_k(N)$ lies inside $C_1,\ldots,C_k$ respectively and each $\zeta_i(N)$ is an open connected subset lying in the interior of C_i $1\leq i\leq n$. Now for each point x_0 on C_i, $1\leq i\leq n$ we can find mutually disjoint neighborhood $W_1(x_0),\ldots,W_N(x_0)$ of $\phi_i(x_0,0)$, $1\leq i\leq N$ (where $P(x_0,\phi_i(x_0,0),0)=0$ and $\phi_i(x,0)$ is an analytic function of x $1\leq i\leq N$) and an open disc $U(x_0)$ of x_0 and an open disc $V(x_0)$ of $t=0$ such that $\forall x\in U(x_0)$, $\forall t\in V(x_0)$, $\phi_i(x,t)\in W(x_0)$ and the function germs are analytic on $U(x_0)$ and $U(x_0)\cap\zeta_i(N)=\varphi$ for all i. Again since the points on C_i $1\leq i\leq n$ form a compact set $D=\cup_{i=1}^k C_i$, the open cover $\{U(x):x\in D\}$ has a finite subcover where $D\subset\cup_{i=1}^n U(x_i)$. Set $V=\cap_{i=1}^n V(x_i)\cap N$. Note that $\phi_i(x,0)=y_j(x_0,0)$ for some j, $1\leq j\leq N$. Let us consider the monodromy permutation around $\zeta_1(0)$. For simplicity let $y_1(x,0)\to y_2(x,0)\to y_3(x,0)\to y_1(x,0)$. We shall prove that for each $t\in V$ $y_1(x,t)\to y_2(x,t)\to y_3(x,t)\to y_1(x,t)$.

Let $U(x_0)$ is a neighborhood of x_0 such that $U(x_0)=U_1(x_0)\cup U_2(x_0)$. Then

$$\begin{aligned}
&\forall x\in U_1(x_0),\quad \forall t\in V,\quad y_1(x,t)\in W_1(x_0)\\
&\forall x\in U_2(x_0),\quad \forall t\in V,\quad y_3(x,t)\in W_1(x_0)\\
&\text{as}\quad y_3(x,0)\longrightarrow y_1(x,0)\ \text{ in the neighborhood of } x=x_0,\\
&\forall x\in U_1(x_0),\quad \forall t\in V,\quad y_2(x,t)\in W_2(x_0)\\
&\forall x\in U_2(x_0),\quad \forall t\in V,\quad y_1(x,t)\in W_2(x_0)\\
&\text{as}\, y_1(x,0)\longrightarrow y_2(x,0),
\end{aligned}$$

and

$$\begin{aligned}
&\forall x\in U_1(x_0),\quad \forall t\in V,\quad y_3(x,t)\in W_3(x_0)\\
&\forall x\in U_2(x_0),\quad \forall t\in V,\quad y_2(x,t)\in W_3(x_0)\\
&\text{as}\, y_2(x,0)\longrightarrow y_3(x,0).
\end{aligned}$$

By construction we can find finite number of points $x_0,\ldots,x_k$ on C_1 and their neighborhood $U(x_0),\ldots,U(x_k)$ and disjoint open set $W_1(x_i),\ldots,W_N(x_i)$ for each fixed i, $0\leq i\leq k$ around $y_j(x_i,0)$, $1\leq j\leq N$ such that $\forall x\in U(x_i)$, $t\in V$, $y_j(x,t)\in W_j(x_i)$ $1\leq j\leq N$. Since $y_1(x,0)$ analytically continues to $y_2(x,0)$, $W_1(x_k)$ (i.e the neighborhood of $y_1(x_k,0)$) intersects $W_2(x_0)$ (which is the neighborhood of $y_2(x_0,0)$).

$$\forall x\in U_2(x_0),\quad y_1(x,0)\in W_2(x_0).$$

Choose

$$\begin{aligned}
&\tilde{x} \in U(x_k) \cap U_2(x_0)\\
&\Longrightarrow y_1(\tilde{x}, 0) \in W_2(x_0)\\
&\Longrightarrow y_1(\tilde{x}, t) \in W_2(x_0) \text{ for } t \text{ small (by continuity of } y_1 \text{ in } t)\\
&\text{as only } \phi_2(\tilde{x}, t) \in W_2(x_0)\ \forall t \in V\\
&\Longrightarrow \phi_2(\tilde{x}, t) = y_1(\tilde{x}, t) \text{ for } t \text{ small}\\
&\Longrightarrow \phi_2(\tilde{x}, t) = y_1(\tilde{x}, t) \quad \forall t \in V \quad (\text{as } y_1 \text{ and } \phi_2 \text{ are analytic function of } t)\\
&\Longrightarrow y_1(\tilde{x}, t) \in W_2(x_0) \quad \forall t \in V,\ \forall \tilde{x} \in U_2(x_0) \cap U(x_k)\\
&\Longrightarrow y_1(x, t) \in W_2(x_0) \quad \forall t \in V,\ x \in U_2(x_0)\\
&(\text{as for } t \text{ fixed } y_1(\tilde{x}, t) = \phi_2(\tilde{x}, t) \quad \forall x \in U_2(x_0) \cap U(x_k)\\
&\Longrightarrow y_1(x, t) = \phi_2(x, t) \quad \forall x \in U_2(x_0) \quad \text{by analyticity in } x).
\end{aligned}$$

So if we continue $y_1(x,t)$ along l_1 we get $\phi_2(x,t)$. Again only $y_2(x,t) \in W_2(x_0)$ $\forall x \in U_1(x_0)$. Let us fix $t \in V$. If we continue $y_1(x,t)$ across l_1 the function we get say $\tilde{y}(x,t)$ which is a solution of $P(x,y,t) = 0$ (for fixed t) and hence belong to either $W_1(x_0)$ or $W_2(x_0)$ or $W_3(x_0)$.

Since

$$y_1(x,t) \in W_2(x_0) \quad \forall x \in U_2(x_0)$$

and

$$W_2(x_0) \cap W_1(x_0) = \varphi, \quad W_2(x_0) \cap W_3(x_0) = \varphi.$$

So

$$\begin{aligned}
&\tilde{y}(x,t) \in W_2(x_0) \quad \forall x \in U_1(x_0)\\
&\Longrightarrow \tilde{y}(x,t) = y_2(x,t) \quad \forall x \in U_1(x_0)\\
&\text{as only } y_2(x,t) \in W_2(x_0) \quad \forall x \in U_1(x_0) \quad \forall t \in V.
\end{aligned}$$

Since $t \in V$ is arbitrary $y_1(x,t)$ continues to $y_2(x,t)$ and thus monodromy remains invariant. □

3. Construction of quasiconformal marking maps

3.1 *Construction of a piecewise-affine mapping* ϕ_t*:* $\mathbf{CP}^1 \to \mathbf{CP}^1$ *which carries ramification points of* $P_0(x,y)$ *to the ramification points of* $P_t(x,y)$

Recall that the ramification points on the Riemann sphere for the covering surface X_t, (i.e., the critical value set for the branched covering map x_t on X_t), are assumed to be located at precisely K points (for each t):

$$(\zeta_1(t), \ldots, \zeta_K(t)).$$

Let g denote the genus of each of the Riemann surfaces X_t.

The aim now is to consider X_0 as the base point for the Teichmüller space $T(X_0) = T_g$, and consequently realise each X_t as a point of the Teichmüller space by constructing an explicit quasiconformal (q.c) marking homeomorphism from X_0 onto X_t:

$$\tilde{\phi}_t : X_0 \longrightarrow X_t.$$

We shall have ϕ_0 as the identity mapping. For these see Nag [N].

Thus the equivalence class of the triple $[X_0, \tilde{\phi}_t, X_t]$ is a point of the Teichmüller space $T(X_0)$. In fact we shall construct a holomorphic 'classifying map' (as the coefficients of P_t vary holomorphically with t):

$$\eta : t \mapsto [X_0, \tilde{\phi}_t, X_t]$$

mapping the t disc $\{|t| < \epsilon\}$ into T_g.

Using the Bers projection

$$\beta : Bel(X_0) \to T(X_0)$$

we will have a lifting of the 'classifying map' η to a map

$$\tilde{\eta} : \{|t| < \epsilon\} \longrightarrow Bel(X_0).$$

The marking homeomorphism between the compact Riemann surfaces X_0 and X_t will be obtained by lifting a mapping ϕ_t between the Riemann spheres that carries corresponding ramification points to ramification points. Construction of $\phi_t : \mathbf{P}^1 \to \mathbf{P}^1$ is detailed below.

Recall that ∞ was set up as an ordinary point for the meromorphic function x on each X_t. Hence all the ramification points, $\zeta_i(t)$ $1 \leq i \leq k$ lie in the finite x-plane. Restrict the parameter t in a relatively compact sub-disc around $t = 0$: $t \in \Delta_\epsilon = \{t :| t |\leq \epsilon\}$. (To save on notation we still call the radius of the sub-disc as ϵ.)

Since the functions ζ_i are analytic in t, we can find a rectangle R containing in its interior all of the points $S = \{\zeta_i(t) : 1 \leq i \leq K, t \in \triangle_\epsilon\}$. Outside R we will define ϕ_t to be the identity mapping.

To define ϕ_t inside R we take the first (domain) copy of $\mathbf{CP}^1$ and triangulate R as follows: we divide R into non-degenerate triangular regions such that each of the points $\zeta_i(0)$ are used as vertices. Thus the triangulation utilizes a set of vertices containing all the K points $\zeta_i(0)$, as well as some extra points ζ_s for some index set $s = K + 1, \ldots, K + L$. (The four vertices of the rectangle R are certainly included amongst these last L vertices. Also note that each triangle utilized is, by requirement, non-degenerate – namely the vertices are always three non-collinear points.)

Now consider another copy of $\mathbf{CP}^1$ (which will serve as the range of the map ϕ_t) and divide the region inside the rectangle R in this second copy into triangular regions in the natural 'corresponding' fashion, as detailed next: namely the vertices of the triangles of this second copy of R consist of the new ramification points $\zeta_i(t)$'s in place of the $\zeta_i(0)$, $1 \leq i \leq K$, – together with the same extra set of points ζ_s (for index set $s = K + 1, \ldots, K + L$) that were used before. Note: these last L vertices are left undisturbed. Of course, the edges of the two triangulations correspond exactly since the vertices have the above correspondence. That is, if $(\zeta_i(0), \zeta_j(0), \zeta_k(0))$ form vertices of a triangle in the first copy then $(\zeta_i(t), \zeta_j(t), \zeta_k(t))$ form vertices of the corresponding triangle in the second copy; similarly, if $(\zeta_i(0), \zeta_p, \zeta_q)$ are vertices of a triangle in the first copy then $(\zeta_i(t), \zeta_p, \zeta_q)$ will be the vertices of the corresponding in the second copy, etc.

Remark. Since the initial triangulation is non-degenerate, namely the vertices of any triangle that was utilized were non-collinear, then, by continuity of the functions $\zeta_j(t)$, that non-degeneracy of the corresponding triangulation (on the range copy) remains valid for all small values of t near $t = 0$.

Affine mapping of one triangle onto another: If (z_1, z_2, z_3) are any three non-collinear points in the plane, then recall that their *closed convex hull*, (smallest closed convex set in the plane containing these points), is precisely the triangle T (includes the interior and the edges) with the given points as vertices. From elementary linear geometry one knows that every point of T has a unique representation as a convex combination of the vertex vectors; namely, each point of T is representable as $\lambda z_1 + \mu z_2 + \nu z_3$, where λ, μ and ν are real numbers in the closed unit interval $[0, 1]$ such that $\lambda + \mu + \nu = 1$.

Clearly then, given any other set of three non-collinear vertices (w_1, w_2, w_3) for a second triangle T', there is a natural *affine mapping* of the first triangle onto the second which simply sends the point $\lambda z_1 + \mu z_2 + \nu z_3$ of T to the point $\lambda w_1 + \mu w_2 + \nu w_3$ of T'.

DEFINITION OF ϕ_t

We therefore define the desired homeomorphism ϕ_t inside the rectangle R by taking the triangles of the first triangulation, by the above affine mappings, onto the corresponding triangles of the second triangulation. Notice that if two triangles share a common edge, then the affine mappings defined on the two abutting triangles will coincide in their definition along the common edge. That is crucial. Consequently we clearly get a well defined homeomorphism ϕ_t of the rectangle R on itself, and outside R we simply extend ϕ_t by the identity map to the whole Riemann sphere.

It is clear that ϕ_t is a C^∞-diffeomorphism when restricted to the interiors of the triangles used in triangulating R, and also, of course, on the exterior of R.

Lemma. *ϕ_t is quasiconformal for each t in the ϵ disc. The Beltrami coefficient of ϕ_t, is a complex constant (of modulus less than unity) when restricted to the interior of each triangle in the initial triangulation of the rectangle R. Of course, the Beltrami coefficient is identically zero in the exterior of R.*

3.2 *Lifting of* $\phi_t : \mathbf{CP}^1 \longrightarrow \mathbf{CP}^1$ *to* $\tilde{\phi}_t : X_0 \longrightarrow X_t$

Consider the following diagram of Riemann surfaces with the vertical arrows being, as we know, holomorphic branched coverings:

$$\begin{array}{ccc} X_0 & \xrightarrow{\tilde{\phi}_t} & X_t \\ \Big\downarrow x & & \Big\downarrow x_t \\ \mathbf{CP}^1 & \xrightarrow{\phi_t} & \mathbf{CP}^1 \end{array}$$

PROPOSITION

There exists a quasiconformal, orientation preserving homeomorphism:

$$\tilde{\phi}_t : X_0 \longrightarrow X_t$$

lifting the map $\phi_t : \mathbf{CP}^1 \to \mathbf{CP}^1$ and making the above diagram commute. (Note that $\tilde{\phi}_0$ is the identity.)

Proof. In fact, in order to deal with unbranched covering spaces, we define the following punctured Riemann surfaces:

$$X_0' = x^{-1}\{\mathbf{CP}^1 - \text{all critical values of } x\}$$

and

$$X_t' = x_t^{-1}\{\mathbf{CP}^1 - \text{all critical values of } \; x_t\}.$$

Restricted to X_0' and X_t', the vertical mappings are now smooth (=unbranched) covering projections. Observe that the ϕ_t was designed so as to map the critical values of x onto those of x_t. Now we can apply the standard lifting criterion for maps from the theory of covering spaces to demonstrate that ϕ_t lifts. Consequently, at the level of fundamental groups we need to look at the image of the action on π_1 of $(\phi_t \circ x)$ as compared with that of x_t. (See, for instance, Theorem 5.1, p. 128, of Massey [M] for the statement of the usual lifting criterion.)

Since the monodromy permutation at any critical point say $\zeta_m(0)$ is the same as that around the perturbed critical point $\zeta_m(t)$, and since $\phi_t(\zeta_m(0)) = \zeta_m(t)$, we see that:

$$\pi_1(\phi_t \circ x)\pi_1(X_0', w_0) = \pi_1(x_t)\pi_1(X_t', \beta_0),$$

(where $w_0 \in X_0'$ and $x(w_0) = z_0$ and $\beta_0 \in X_t'$ such that $x_t(\beta_0) = \phi_t(z_0)$).

Clearly then the lifting criterion is satisfied, and hence the homeomorphism ϕ_t lifts to a homeomorphism $\tilde{\phi}_t$, as desired. Certainly the lift is quasiconformal since the vertical mappings are holomorphic. This completes the proof of the proposition. In this connection recall the following result.

Theorem. *If U and V are open subsets of compact surfaces X and Y respectively with finite complements, then any homeomorphism from of U onto V extends uniquely to one of X onto Y.* □

Finally then, for our applications to the variation of Fuchsian groups we may *lift all the way to the universal covering upper half-planes* and obtain the quasiconformal homeomorphism $\Phi_t(z) = \Phi(z,t)$ from U to U, obtained by lifting the mapping to $\tilde{\phi}_t : X_0 \longrightarrow X_t$.

Thus we have determined $\Phi_t(z)$ so that the following diagram commutes:

$$\begin{array}{ccc}
z \in U & \xrightarrow{\Phi_t(z) = \Phi(z,t)} & U \\
\downarrow \pi & & \downarrow \pi_t \\
U/G_0 \equiv X_0 & \xrightarrow{\tilde{\phi}_t} & X_t \equiv U/G_t \\
\downarrow x_0 & & \downarrow x_t \\
x \in \mathbf{CP}^1 & \xrightarrow[= \phi(z,t)]{\phi_t(z)} & \mathbf{CP}^1
\end{array} .$$

4. Variational formulae for the Fuchsian groups of varying curve

4.1 *The fundamental variational term*

Let $\mu_t(z)$ denote a one-parameter family of Beltrami coefficients on the upper half-plane depending real or complex analytically on the (real or complex) parameter t near $t = 0$. Suppose also that $\mu_0(z) \equiv 0$. We come now to the main formula that we shall apply. If $\mu_0 \equiv 0$, and if for small t the Beltrami coefficient is given by

$$\mu_t(z) = t\hat{\nu}(z) + o(t), \text{ where } \hat{\nu} \in L^\infty(U), \tag{2}$$

then one has an important integral formula expressing the solutions of the family of Beltrami equations, as a perturbation of the identity homeomorphism

$$w_{\mu_t}(z) = z + tw_1(z) + o(t), z \in U.$$

Indeed, the crucial first variation term, $w_1 = \dot{w}$, for real t is given by

$$w_1(z) = -\frac{1}{\pi}\int\int_U [\hat{\nu}(\zeta)R(\zeta, z) + \overline{\hat{\nu}(\zeta)}R(\bar{\zeta}, z)]\mathrm{d}\xi\mathrm{d}\eta,$$

$$R(\zeta, z) = \frac{z(z-1)}{\zeta(\zeta-1)(\zeta-z)}, \text{ and } \zeta = \xi + i\eta.$$

This perturbation formula (see Ahlfors [A], or section 1.2.13, 1.2.14, as well as page 175, eq. (1.21) of Nag [N]), will be fundamental for us. We shall apply it to the family of quasiconformal mappings Φ_t (§3) standing for the family w_{μ_t}.

Since in our set up t is a *complex* parameter we may as well deduce the form of the variational terms for general t complex – which follows by simply applying the real t formula above appropriately. We show this:

If t is complex, write in polar form: $t = |t|\mathrm{e}^{i\alpha}$ then put $\tau = \mathrm{e}^{-i\alpha}t = |t|$. Then it is straight forward to see that

$$w_1(z) = -\frac{1}{\pi}\int\int_U [\hat{\nu}(\zeta)R(\zeta, z) + \mathrm{e}^{-2i\alpha}\,\overline{\hat{\nu}(\zeta)}R(\bar{\zeta}, z)]\mathrm{d}\xi\mathrm{d}\eta,$$

where $\alpha = \arg(t)$. But $t\mathrm{e}^{-2i\alpha}$ is the conjugate of t. Therefore, this last formula says that for complex t we have the final important formulae:

$$w_{\mu_t}(z) = z + tw_1(z) + \bar{t}w_1^*(z) + o(t), \quad z \in U \tag{3}$$

where

$$(w_1(z), w_1^*(z)) = \left(-\frac{1}{\pi}\int\int_U [\hat{\nu}(\zeta)R(\zeta, z)]\mathrm{d}\xi\mathrm{d}\eta, -\frac{1}{\pi}\int\int_U [\overline{\hat{\nu}(\zeta)}R(\bar{\zeta}, z)]\mathrm{d}\xi\mathrm{d}\eta\right). \tag{4}$$

Equation (4) will be manipulated to produce the chief formulae of §4.

Let $\Gamma \equiv G_0 \subset PSL(2, \mathbf{R})$ denote the uniformizing Fuchsian group acting as deck transformations for the covering π. Then there is a biholomorphic equivalence:

$$X_0 = U/G_0. \tag{5}$$

It follows from the standard Ahlfors-Bers deformation theory of Fuchsian groups (see Nag [N]) that the quasiconformal homeomorphism Φ_t is *compatible with the Fuchsian group* G_0, in the sense that $g_t = \Phi_t \circ g \circ \Phi_t^{-1}$ is again a Möbius transformation in

$PSL(2, \mathbf{R})$ for every $g \in G_0$, and the new Fuchsian group (which evidently remains abstractly isomorphic to G_0) is the Fuchsian group:

$$G_t = \Phi_t \circ G_0 \circ {\Phi_t}^{-1}. \tag{6}$$

This is the group of deck transformations for the covering π_t, so that X_t is biholomorphically equivalent to U/G_t. We shall write

$$g_t = \Phi_t \circ g \circ {\Phi_t}^{-1} \in G_t \tag{7}$$

for any fixed $g \in G_0 \equiv \Gamma$.

In this notation, the central problem of our work is to determine explicit and applicable formulae for the variation of g_t – or, equivalently, to compute the t-derivative: $\dot{g}_t$ at $t = 0$. As g varies over any generating set of elements for the group G_0, we shall then obtain, up to first order approximation, a corresponding set of generating elements for the deformed groups G_t.

4.2 *The Beltrami coefficient μ_t of Φ_t*

Notational set up. Let us, for notational convenience, denote as x_* the meromorphic function on U given by $x \circ \pi$, (this is, of course, a holomorphic branched covering of the Riemann sphere by the upper half plane). Clearly, x_* is automorphic with respect to the Fuchsian group Γ, since x_* descends onto the surface X_0 as the meromorphic function x thereon. In particular, let us note the well-known fact that this function, x_*, can be expressed in terms of the standard Poincaré theta-series on U with respect to the group Γ.

Now recall from the previous section that the mapping ϕ_t was, by our very definition, a piecewise affine quasiconformal mapping. So the Beltrami coefficient of ϕ_t was a complex constant on each triangle of the triangulation of the domain rectangle R. (The Beltrami coefficient need only be specified almost everywhere – therefore we will ignore it on the edges and vertices of the triangulation.)

Moreover we know that the vertices of the triangulation (in the image plane) depend holomorphically on t – since the ramification points $\zeta_j(t)$ were holomorphic functions of t. Here is the main proposition we require.

PROPOSITION

The Beltrami coefficient of Φ_t is

$$\mu_t(z) = t\hat{\nu}(z) + o(t), z \in U, \hat{\nu}(z) \in L^\infty(U),$$

where

$$\hat{\nu}(z) = \nu(w)\frac{\overline{(x \circ \pi)'}(z)}{(x \circ \pi)'(z)}, \quad \textit{where} \quad w = (x \circ \pi)(z) = x_*(z) \in \mathbf{CP}^1. \tag{8}$$

Here the Beltrami coefficient for the piecewise-affine mappings ϕ_t on the Riemann w-sphere has been expanded up to first order in t as below:

$$\frac{\phi_{t,\bar{w}}(w)}{\phi_{t,w}(w)} = t\nu(w) + o(t), \nu \in L^\infty(\mathbf{CP}^1). \tag{9}$$

Further note that $\nu(w)$ is a *constant on each triangle* of the first (domain) triangulation of R, and it is zero for all w outside R.

Note. The Γ invariant Beltrami coefficient $\hat{\nu}$ above, represents the *tangent vector* to the one parameter family of Beltrami coefficients μ_t which arise from the one parameter family of quasiconformal mappings Φ_t.

Proof. From the above commutative diagram for the liftings we have

$$(x_t \circ \pi_t) \circ \Phi_t = \phi_t \circ (x \circ \pi). \tag{10}$$

Taking the ∂ and $\bar{\partial}$ derivatives in (10), and remembering that all the vertical maps are *holomorphic* coverings (possibly branched as we know), we obtain the Beltrami coefficient of Φ_t on U:

$$\mu_t(z) \equiv \frac{\Phi_{t,\bar{z}}}{\Phi_{t,z}} = \frac{\phi_{t,\bar{w}}(w)\overline{(x \circ \pi)'(z)}}{\phi_{t,w}(w)(x \circ \pi)'(z)}, \quad z \in U. \tag{11}$$

Clearly then the statements in the Proposition follow because the $\phi_t(w)$ are a family of piecewise affine quasiconformal homeomorphisms on the w-sphere which vary holomorphically in t. Thus, remembering that ϕ_0 is the identity, we see that the Beltrami coefficients of the family ϕ_t indeed must have an expression as in (9) with $\nu(w)$ being piecewise-constant. □

Remark. Note that the Beltrami coefficient of ϕ_t is a *holomorphic* function of the parameter t in the neighborhood of $t = 0$. The map

$$t \mapsto \frac{\phi_{t,\bar{z}}}{\phi_{t,z}}$$

takes values in the complex Banach space $L^\infty(\mathbf{CP}^1)$, – and the holomorphy is as a map into this Banach space.

Beltrami coefficients automorphic with respect to Γ. We must remember from the general theory (see §1.3.3 of Nag [N]) one further fundamental fact. Since the quasiconformal maps Φ_t are compatible with Γ their Beltrami coefficients are $(-1,1)$ forms on U with respect to Γ. (We called them Γ-invariant Beltrami coefficients.)

Indeed, if μ is the complex dilatation of a quasiconformal mapping that conjugates Γ into any group of Möbius transformations, then

$$(\mu \circ g)(\bar{g'}/g') = \mu, a.e., \quad \text{for all} \quad g \in \Gamma. \tag{12}$$

We denote the Banach space of complex valued L^∞ functions on U that satisfy equation (12) for every $g \in \Gamma$, by the notation: $L^\infty(U,\Gamma)$. See p. 49 of [N]. Thus, μ_t belongs to the open unit ball of this Banach space for all small t , and also therefore $\hat{\nu}$ belongs to this Banach space of automorphic objects.

4.3 *The variational formula for* Φ_t

We come to the chief application of the perturbation formula (eq. (4)) in our specific context of varying algebraic curves.

Let F denote a closed fundamental domain, with boundary of two-dimensional measure zero, for the action of Γ on U; (for instance, we may choose F as any standard Dirichlet fundamental polygon for the Fuchsian group Γ). Thus π maps F onto X_0, and π is one-to-one when restricted to the interior of F.

Recall that x was itself a meromorphic function of degree N on the compact Riemann surface X_0, (see §2, 3). Consequently, when restricted to the interior of F the mapping x_* is a N-to-1 branched holomorphic covering map onto the Riemann sphere – missing only a set of areal measure zero. Since this is a finite covering space situation (aside from a measure zero set of branch points which we may discard to start with), we may choose a decomposition of F into N regions:

$$F = D_1 \cup D_2 \cup \cdots \cup D_N. \tag{13}$$

Here the D_j are mutually disjoint domains (except for boundary contact, as usual in choice of fundamental regions), partitioning F, with the basic property that each D_j maps, via x_*, in a one-to-one fashion onto the entire Riemann sphere (missing atmost a measure zero subset). (Recall that the compact Riemann surface X_0 was described as an N-sheeted branched cover of the sphere – by the degree N meromorphic function x.)

A kernel function associated to Γ. We introduce as an useful matter of notation, the following function of two variables: $z \in U$, $\tau \in \mathbf{C}$ (not lying on the Γ orbit of z):

$$K_\Gamma(z,\tau) = \sum_{g\in\Gamma} \frac{[g'(z)]^2}{g(z)(g(z)-1)(g(z)-\tau)}. \tag{14}$$

We are now in a position to state a main result.

Theorem. *On variation of* Φ_t. *The lifted quasiconformal maps* Φ_t *on* U *satisfy the following first order expansion for small* t,

$$\Phi_t(z) = z + t w_1(z) + \bar{t} w_1^*(z) + o(t), z \in U, \tag{15}$$

where

$$w_1(z) = \frac{z(z-1)}{2\pi\sqrt{-1}} \sum_{k=1}^{N} \int\int_{\mathbf{CP}^1} \left\{ \nu(w) K_\Gamma(x_{*,k}^{-1}(w), z) \left[\frac{\partial x_{*,k}^{-1}}{\partial w}(w) \right]^2 \right\} \mathrm{d}w \wedge \mathrm{d}\bar{w},$$

$$w_1^*(z) = \frac{z(z-1)}{2\pi\sqrt{-1}} \sum_{k=1}^{N} \int\int_{\mathbf{CP}^1} \left\{ \overline{\nu(w)} K_\Gamma(\overline{x_{*,k}^{-1}(w)}, z) \left[\overline{\frac{\partial x_{*,k}^{-1}}{\partial w}(w)} \right]^2 \right\} \mathrm{d}w \wedge \mathrm{d}\bar{w}.$$

Here we have denoted by $x_{*,k}$ *the restriction of the projection* $x_* = x \circ \pi$ (*which is a meromorphic and* Γ*-automorphic function on* U), *to the region* $D_k \subset F$, $k = 1, \ldots, N$. *Here* ν *denotes the function on the* w*-sphere appearing in formula* (9) *of the Proposition in sub-section* 4.2 *above.* (*Recall that* ν *is simply a constant assigned on each triangle in the triangulation of* R, *with* ν *being identically zero outside* R.)

Note furthermore, that since x_* *is a meromorphic function on* U, *we may replace in the above formula the derivative of its inverse by the reciprocal of its own derivative, as shown below:*

$$\frac{\partial x_{*,k}^{-1}}{\partial w}(w) = 1 \Big/ \frac{\mathrm{d}x_{*,k}}{\mathrm{d}z}(z), \quad w = x_*(z), \quad z \in D_k.$$

These derivatives can therefore be calculated from the expression for x_* *which will be available in terms of the standard Poincaré theta series on* U *with respect to* γ. (*Therefore we see that if* $\gamma \in G_0$ *then the variational formula for* $\gamma_t = \Phi_t \circ \gamma \circ \Phi_t^{-1} \in G_t$ *is*

$$\gamma_t = \gamma + t\dot{\gamma} + \bar{t}\dot{\gamma}^* + o(t), \tag{16}$$

where

$$\dot{\gamma} = w_1 \circ \gamma - \gamma' w_1,$$
$$\dot{\gamma}^* = w_1^* \circ \gamma - \gamma' w_1^*.$$

For this, see Nag [*N*].)

During the course of the proof we shall show that all integrals and summations appearing in sight are absolutely convergent. For facts regarding Poincaré theta series and their utilization in expressing meromorphic functions on U/Γ, see [Kra, Kr].

Proof. We shall have to manipulate the variational formula (4) which said:

$$w_1(z) = \frac{1}{2i\pi}\int\int_U [\hat{\nu}(w)R(w,z) + \overline{\hat{\nu}(w)}R(\bar{w},z)]\mathrm{d}w \wedge \mathrm{d}\bar{w}$$

with $R(w,z) = \frac{z(z-1)}{w(w-1)(w-z)}$.

By general theory quoted above, the integrals involved in (4) are necessarily absolutely convergent.

To obtain the final result for w_1 and w_1^*, there are several chief ideas which we first explain in words:

(i) Write each of the two integrals over U as a sum of integrals over all the tiles in the Γ-tessellation of U – obtained by decomposing U as the union of the fundamental domain F and its translates: i.e., $U = \cup_{g\in\Gamma}(g(F))$.

(ii) Utilizing the Γ-automorphic nature of the Beltrami coefficient $\hat{\nu}$ (see eq. (12) above), and making a change of variables by $w = g(z)$, we can transform the integral over $g(F)$ to an integral again over F itself.

(iii) Consequently, the original expression for w_1 becomes simply an integration over F of a certain expression on F, after interchanging summation and integration. (The validity of the interchange is guaranteed by the absolute convergence of the result, together with the dominated convergence theorem. The main details of this critical interchange of sum and integral are spelled out in the remarks attached at the end of the proof.)

(iv) Finally we decompose F itself into the N pieces $D_1, \ldots, D_N$ (as explained with eq. (13) above) – and hence we may eliminate $\hat{\nu}$ by replacing it with occurrences of ν itself, and thus express the final result as integrations over the Riemann sphere $\mathbf{CP}^1$, as desired.

The first three of the above steps are carried out e.g. in [A]. Let us now get down to the main business of showing the exact nature of how these transformations come about in the expression for w_1. First of all note:

$$\int\int_U \frac{\hat{\nu}(w)}{w(w-1)(w-\tau)}\mathrm{d}w \wedge \mathrm{d}\bar{w}$$
$$= \sum_{g\in\Gamma}\int\int_{g(F)} \frac{\hat{\nu}(w)}{w(w-1)(w-\tau)}\mathrm{d}w \wedge \mathrm{d}\bar{w}, \quad F = \text{fundamental}$$

region of Γ in U.

Perform a change of variables on $g(F)$ by $w = u + iv = g(z)$

$$= \sum_{g\in\Gamma} \iint_F \frac{\hat{\nu}(z)\frac{g'(z)}{\overline{g'(z)}} \mid g'(z) \mid^2}{g(z)(g(z)-1)(g(z)-\tau)} \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

$$= \sum_{g\in\Gamma} \iint_F \frac{\hat{\nu}(z)[g'(z)]^2}{g(z)(g(z)-1)(g(z)-\tau)} \mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

For convergence arguments we note that since

$$\sum_{g\in\Gamma} \iint_F \frac{\mid \hat{\nu}(z) \mid \frac{|g'(z)|}{|\overline{g'(z)}|} \mid g'(z) \mid^2}{\mid g(z) \mid\mid g(z)-1 \mid\mid g(z)-\tau \mid} \mathrm{d}x\mathrm{d}y$$

$$= \sum_{g\in\Gamma} \iint_{g(F)} \frac{\mid \hat{\nu}(w) \mid}{\mid w \mid\mid w-1 \mid\mid w-\tau \mid} \mathrm{d}u\mathrm{d}v < \infty.$$

This demonstrates that the series

$$\sum_{g\in\Gamma} \iint_F \frac{\hat{\nu}(z)[g'(z)]^2}{g(z)(g(z)-1)(g(z)-\tau)} \mathrm{d}z \wedge \mathrm{d}\bar{z} = \sum_{g\in\Gamma} \iint_F \psi_g(z)\mathrm{d}z \wedge \mathrm{d}\bar{z} \tag{17}$$

is absolutely convergent. Note that, for convenience, we have written ψ_g here for the following frequently recurring expression:

$$\psi_g(z) = \frac{\hat{\nu}(z)[g'(z)]^2}{g(z)(g(z)-1)(g(z)-\tau)}.$$

We shall show by a measure-theoretic lemma in the remarks appended to the bottom of this proof, that we are allowed to change summation and integration in the summation (17). We shall utilize crucially this interchange immediately in what follows. Returning therefore to the actual expression for the variational term w_1, we now obtain:

$$\begin{aligned}
w_1(z) &= -\frac{1}{2i\pi} \iint_U \hat{\nu}(\zeta) R(\zeta, z)\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \\
&= \frac{z(z-1)}{2\pi i} \iint_U \frac{\hat{\nu}(\zeta)}{\zeta(\zeta-1)(\zeta-z)} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \\
&= \frac{z(z-1)}{2\pi i} \sum_{g\in\Gamma} \iint_F \frac{\hat{\nu}(\zeta)[g'(\zeta)]^2}{[g(\zeta)][g(\zeta)-1][g(\zeta)-z]} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \\
&= \frac{z(z-1)}{2\pi i} \iint_F \sum_{g\in\Gamma} \frac{\hat{\nu}(\zeta)[g'(\zeta)]^2}{g(\zeta)(g(\zeta)-1)(g(\zeta)-z)} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \\
&= \frac{z(z-1)}{2\pi i} \iint_F \hat{\nu}(\zeta) K_\Gamma(\zeta, z)\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}.
\end{aligned}$$

Similarly

$$w_1^*(z) = \frac{z(z-1)}{2\pi i} \iint_F \overline{\hat{\nu}(\zeta)} K_\Gamma(\bar{\zeta}, z)\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}.$$

That completes the manipulation of the formula to a point that already has points of interest; we have carried out steps (i), (ii), (iii) – and now we are integrating over F (i.e., over X_0), rather than over U.

The final steps are for carrying out the program outlined in point number (iv) above. This goes as detailed below:

Let

$$(x \circ \pi)(D_i) = \mathbf{CP}^1, \quad \text{and denote } x \circ \pi|_{D_i} = x_{*,i}$$

for each $i = 1, \ldots, N$. Setting $(x \circ \pi)(\zeta) = w$, $\zeta \in U$ and $w \in \mathbf{CP}^1$, and using the relation (eq. (8)) between $\hat{\nu}$ and ν, we will have

$$w_1(z) = \frac{z(z-1)}{2\pi i} \sum_{i=1}^{N} \iint_{\mathbf{CP}^1} \left[\sum_{g\in\Gamma} \left(\frac{\nu(w)([\overline{\mathrm{d}w/\mathrm{d}\zeta}]/[\mathrm{d}w/\mathrm{d}\zeta])[g'(x_{*,i}^{-1}(w))]^2}{g(x_{*,i}^{-1}(w))(g(x_{*,i}^{-1}(w))-1)(g(x_{*,i}^{-1}(w))-z)} \right) \right] \frac{\mathrm{d}w \wedge \mathrm{d}\bar{w}}{|\mathrm{d}w/\mathrm{d}\zeta|^2}$$

$$= \frac{z(z-1)}{2\pi i} \sum_{i=1}^{N} \iint_{\mathbf{CP}^1} \left[\sum_{g\in\Gamma} \frac{\nu(w)[g'(x_{*,i}^{-1}(w))]^2[\partial x_{*,i}^{-1}/\partial w(w)]^2}{g(x_{*,i}^{-1}(w))(g(x_{*,i}^{-1}(w))-1)(g(x_{*,i}^{-1}(w))-z)} \right] \mathrm{d}w \wedge \mathrm{d}\bar{w}$$

$$= \frac{z(z-1)}{2\pi i} \sum_{i=1}^{N} \iint_{\mathbf{CP}^1} \left[\nu(w) K_\Gamma(x_{*,i}^{-1}(w), z) \left[\frac{\partial x_{*,i}^{-1}}{\partial w}(w) \right]^2 \right] \mathrm{d}w \wedge \mathrm{d}\bar{w}.$$

Similarly

$$w_1^*(z) = \frac{z(z-1)}{2\pi i} \sum_{i=1}^{N} \iint \left[\overline{\nu(w)} K_\Gamma(\overline{x_{*,i}^{-1}(w)}, z) \left[\overline{\frac{\partial x_{*,i}^{-1}}{\partial w}(w)} \right]^2 \right] \mathrm{d}w \wedge \mathrm{d}\bar{w}$$

That at last is exactly the expression desired and claimed in the Theorem and we are through. □

The interchange of summation and integration above in the series (17), follows from some straightforward facts of the theory of measure and integration. For instance, our purposes are adequately served by the following result (see Rudin [R]):

Lemma. Suppose $\{f_n\}$ is a sequence of complex measurable functions defined almost everywhere on a complete measure space (X, μ) such that

$$\sum_1^\infty \int_X |\, f_n \,| \, \mathrm{d}\mu < \infty.$$

Then the series $f(x) = \sum_1^\infty f_n(x)$ converges absolutely for almost all x, and $f \in L^1(\mu)$; moreover, the summation and integration can be interchanged, namely:

$$\int_X f \mathrm{d}\mu = \sum_1^\infty \int_X f_n \mathrm{d}\mu.$$

□

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