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A quantum spin system with random interactions I

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Abstract. We study a quantum spin glass as a quantum spin system with random interactions and establish the existence of a family of evolution groups $\{\tau_t(\omega)\}_{\omega\in\Omega}$ of the spin system. The notion of ergodicity of a measure preserving group of automorphisms of the probability space Ω, is used to prove the almost sure independence of the Arveson spectrum $\mathrm{Sp}(\tau(\omega))$ of $\tau_t(\omega)$. As a consequence, for any family of $(\tau(\omega),\beta)$-KMS states $\{\rho(\omega)\}$, the spectrum of the generator of the group of unitaries which implement $\tau(\omega)$ in the GNS representation is also almost surely independent of ω.

Keywords. Spin; system; quasi-local; random; dynamics; evolution; independent; Arveson; KMS.

1. Introduction

Traditionally, spin glasses [1] have been studied as spin systems with random interactions. These models are essentially Ising-type models with random coupling. Extensive investigations on the existence of the thermodynamic limit have been made e.g. van Enter *et al* [5, 6], and the equilibrium statistical mechanics of such systems has been studied. In order to study the dynamics of a quantum spin glass we model it as a quantum spin system on an infinite lattice with random interactions. We introduce a measure preserving group of automorphisms $\{T_a\}_{a\in\mathbb{Z}^\nu}$ with an ergodic action on a complete probability space $(\Omega,\mathcal{S},P)$, and consider the class of interactions Φ which satisfy a compatibility condition involving the measure preserving group of automorphisms and the action of the lattice $\mathbb{Z}^\nu$ on the quasi-local algebra. We establish the existence of a family $\{\tau_t(\omega)\}$, of strongly continuous one-parameter groups of *-automorphisms of the quasi-local algebra $\mathcal{A}$ associated with the infinite system. Some interesting algebraic properties of $\tau_t(\omega)$ as well as those of its generator have been derived. Finally, we show that the Arveson spectrum of the evolution group $\tau_t(\omega)$ is almost surely independent of ω. Also, for any given family of $(\tau(\omega),\beta)$-KMS states $\{\rho(\omega)\}$, we report an ergodic property of the spectrum of the generator of the group of unitaries which implement $\tau(\omega)$ in the GNS representation. Results pertaining to the ergodic properties of the spectra of random self adjoint operators have also been reported in chapter 1 of [8]. However the techniques used there differ from our approach.

2. Description of the random model

We consider a quantum spin-$\frac{1}{2}$ system with spins located at the vertices of an infinite lattice $\mathbb{Z}^\nu$. The interaction between spins of course is taken to be random. The kinematical structure of the spin system is described by a quasi-local (UHF) algebra $\mathcal{A}$ with a generating net of $C^\star$-subalgebras $\{\mathcal{A}_\Lambda\}$, constructed over the finite subsets Λ of $\mathbb{Z}^\nu$. Besides this, we also have a natural action α of the lattice $\mathbb{Z}^\nu$ as *-automorphisms $a \mapsto \alpha_a$ of $\mathcal{A}$. For a detailed description see § 6.2.4 in [10] and § 6.2.1 in [4].

3. Random interactions

Before introducing random interactions, one defines the notion of measurability of Banach space valued functions on a measure space $(\Omega, \mathcal{S}, m)$, where Ω is a set, $\mathcal{S}$ a sigma algebra and m a sigma-finite measure on Ω.

DEFINITION 3.1

Let $(\Omega, \mathcal{S}, m)$ be a measure space. A function $f : \Omega \to B$ where B is a Banach space, is said to be weakly measurable if, for every $\phi \in B^\star$, the map $\omega \mapsto \phi(f(\omega))$ is $\mathcal{S}$-measurable. f is said to be strongly measurable if, there exists a sequence of countably valued functions strongly convergent to f almost everywhere on Ω [7].

In case m is a finite measure, then we may replace 'countably valued' in the above definition by 'simple'. It can be shown that the notions of strong and weak measurability are equivalent if B is separable.

From now on, let $(\Omega, \mathcal{S}, P)$ be a complete probability space.

DEFINITION 3.2

Let $\mathcal{F}$ be the collection of all finite subsets of $\mathbb{Z}^\nu$. A random interaction is a map $\Phi : \mathcal{F} \times \Omega \to \mathcal{A}$ such that, for each $\omega \in \Omega$, $\Phi(X, \omega)$ is a self-adjoint element in the $C^\star$-subalgebra $\mathcal{A}_X$ and $\omega \mapsto \Phi(X, \omega)$ is strongly measurable for every $X \in \mathcal{F}$.

Now given the random interaction Φ, for finite $\Lambda \subseteq \mathbb{Z}^\nu$, the Hamiltonian associated with the spins confined to the region Λ is given by a Hermitian (self adjoint) element

$$H(\Lambda, \omega) = \sum_{X \subseteq \Lambda} \Phi(X, \omega),$$

for each realization $\omega \in \Omega$. Clearly, $H(\Lambda, \omega)$ is strongly measurable since each $\Phi(X, \omega)$ is strongly measurable on Ω.

Next, we introduce a measure preserving group of automorphisms $\{T_a\}_{a \in \mathbb{Z}^\nu}$ on the probability space Ω, and consider the class of only those random interactions Φ which satisfy the following condition:

$$\Phi(X + a, T_{-a}\omega) = \alpha_a(\Phi(X, \omega)),$$

see [5]. Clearly, $H(\Lambda + a, T_{-a}\omega) = \alpha_a(H(\Lambda, \omega)) \forall \omega \in \Omega$ and $a \in \mathbb{Z}^\nu$.

DEFINITION 3.3

Let Φ be a random interaction. Then Φ is said to be a finite range interaction if, the set

$$\Delta_\omega = \{x \in \mathbb{Z}^\nu \mid \exists X \ni x;\ \text{such that } 0 \in X, \text{ and } \Phi(X, T_a\omega) \neq 0, \text{ for some } a \in \mathbb{Z}^\nu\}$$

is a finite subset of $\mathbb{Z}^\nu$ for almost every $\omega \in \Omega$.

Remark. Clearly, whenever $X - X \not\subseteq \Delta_\omega$, $\Phi(X, \omega) = 0$.

The definition given above yields the following result.

Lemma 3.4. *Let Φ be a random interaction. Then $\Delta_\omega = \Delta_{T_b\omega}$ for all $b \in Z^\nu$.*

Proof. Proof follows from the definition of Δ_ω. △

4. Random evolution

For a finite spin system confined to a region $\Lambda \subseteq \mathbb{Z}^\nu$, and for $\omega \in \Omega$, the equation of motion is given by

$$\frac{\mathrm{d}A_t^\Lambda(\omega)}{\mathrm{d}t} = i[H(\Lambda,\omega), A_t^\Lambda(\omega)], \quad A_t^\Lambda(\omega) \in \mathcal{A}_\Lambda.$$

This yields the time evolution given by $\tau_t^\Lambda(\omega)(A) = A_t^\Lambda(\omega) = \mathrm{e}^{iH(\Lambda,\omega)t} A \mathrm{e}^{-iH(\Lambda,\omega)t}$ for $\omega \in \Omega$ and for all $A \in \mathcal{A}_\Lambda$. Clearly, for each $\omega \in \Omega$, $\tau_t^\Lambda(\omega)$ is a one-parameter group of *-automorphisms of $\mathcal{A}_\Lambda$. Since the spin system consists of infinite number of spins, the construction of the time evolution of a fixed observable $A \in \mathcal{A}_{\Lambda_0}$ where $\Lambda_0 \subseteq \mathbb{Z}^\nu$, involves taking the limit of $\tau_t^\Lambda(\omega)(A)$ as $\Lambda \to \infty$ (the collection $\mathcal{F}$ of all finite subsets Λ of $\mathbb{Z}^\nu$ ordered by inclusion is a directed set).

Next we construct a family of strongly continuous one-parameter groups of *-automorphisms which determine the evolution of the spin system. To this end, we have the following theorem.

Theorem 4.1. *Let Φ be a finite range random interaction of the quantum spin system on a lattice $\mathbb{Z}^\nu$, satisfying*

$$\sup_{a \in \mathbb{Z}^\nu} \left(\sum_{X \ni 0} ||\Phi(X, T_a\omega)|| \right) < \infty$$

almost everywhere. Then, for almost every $\omega \in \Omega$, there exists a strongly continuous, one-parameter group of *-automorphisms $\tau_t(\omega)$ of $\mathcal{A}$ such that,

$$\lim_{\Lambda \to \infty} \tau_t^\Lambda(\omega)(A) = \tau_t(\omega)(A), \quad \forall\, A \in \mathcal{A}$$

and uniformly, for t in compacts, where $\tau_t^\Lambda(\omega)(A) = \mathrm{e}^{iH(\Lambda,\omega)t} A\, \mathrm{e}^{-iH(\Lambda,\omega)t}$. $\tau_t(\omega)$ is called the evolution group of the spin system whenever the limit exists.

Proof. Since $\Phi(X + a, T_{-a}\omega) = \alpha_a(\Phi(X,\omega))$, $\forall a \in \mathbb{Z}^\nu$, it is clear that whenever Δ_ω is finite,

$$P_\phi(\omega)(x) = \sum_{X \ni x} ||\Phi(X,\omega)|| = \sum_{Y \ni 0} ||\Phi(Y, T_x\omega)|| < \infty.$$

On appealing to Proposition 6.2.3 in [4], there exists a derivation $\delta(\omega)$ of $\mathcal{A}$ such that, the domain of $\delta(\omega)$

$$D(\delta(\omega)) = \bigcup_{\Lambda \subseteq \mathbb{Z}^\nu} \mathcal{A}_\Lambda; \quad \delta(\omega)(A) = i \sum_{X \cap \Lambda \neq \emptyset} [\Phi(X,\omega), A] \quad \text{for } A \in \mathcal{A}_\Lambda,$$

and $\delta(\omega)$ is norm-closeable with norm closure $\overline{\delta}(\omega)$. Next, we shall show that $D(\delta(\omega))$ is a dense set of analytic elements for $\overline{\delta}(\omega)$. Take $A \in \mathcal{A}_{\Lambda_0}$. Whenever Δ_ω is a finite set and

$$\sup_{a\in\mathbb{Z}^\nu}\left(\sum_{X\ni 0}||\Phi(X,T_a\omega)||\right)<\infty,$$

we have

$$(\overline{\delta}(\omega))^n(A)=i^n\sum_{X_1\cap S_0\neq\emptyset}\cdots\sum_{X_n\cap S_{n-1}\neq\emptyset}[\Phi(X_n,\omega),[\ldots[\Phi(X_1,\omega),A]]],$$

where

$$S_0=\Lambda_0\quad\text{and}\quad S_j=\Lambda_0\cup\bigcup_{i=1}^{j}X_i,\quad\text{for}\quad j\geq 1.$$

Now, if

$$[\Phi(X_j,\omega),[\ldots[\Phi(X_1,\omega),A]]]\neq 0,$$

then

$$|X_i|\leq|\Delta_\omega|,\quad\forall\, i=1,2,\ldots,j\quad\text{and therefore,}\quad|S_j|\leq j|\Delta_\omega|+|\Lambda_0|.$$

Here $|\cdot|$ denotes the cardinality of a set. Therefore we get

$$\begin{aligned}||(\overline{\delta}(\omega))^n(A)||&\leq 2^n||A||\sum_{x_1\in S_0}\sum_{X_1\ni x_1}\cdots\sum_{x_n\in S_{n-1}}\sum_{X_n\ni x_n}||\Phi(X_n,\omega)||\cdots||\Phi(X_1,\omega)||\\&\leq 2^n||A||\sum_{x_1\in S_0}\sum_{X_1-x_1\ni 0}\cdots\sum_{x_n\in S_{n-1}}\sum_{X_n-x_n\ni 0}||\Phi(X_n-x_n,T_{x_n}\omega)||\\&\quad\cdots||\Phi(X_1-x_1,T_{x_1}\omega)||\\&\leq 2^n||A||\prod_{i=1}^{n}((i-1)|\Delta_\omega|+|\Lambda_0|)\left(\sup_{a\in\mathbb{Z}^\nu}\left(\sum_{X\ni 0}||\Phi(X,T_a\omega)||\right)\right)^n\\&\leq||A||\mathrm{e}^{|\Lambda_0|}2^n n!\left(\sup_{a\in\mathbb{Z}^\nu}\left(\sum_{X\ni 0}||\Phi(X,T_a\omega)||\right)\right)^n\mathrm{e}^{n|\Delta_\omega|}.\end{aligned}$$

This establishes that A is an analytic element for $\overline{\delta}(\omega)$, [2] with radius of analyticity independent of A. Therefore, it follows from Proposition 6.2.3 in [4] and the assumptions made in this theorem that for almost every $\omega\in\Omega$, $\overline{\delta}(\omega)$ is the generator of a strongly continuous one parameter group of *-automorphisms $\tau_t(\omega)$ of $\mathcal{A}$ such that,

$$\tau_t^\Lambda(\omega)(A)\to\tau_t(\omega)(A),\quad\forall\, A\in\mathcal{A},$$

where convergence above is uniform in t on compact sets. △

From now on let $\mu\times P$ be the complete product measure on $\mathbb{R}\times\Omega$, μ being the Lebesque measure on $\mathbb{R}$.

PROPOSITION 4.2

Let $\tau_t(\omega)$ be the strongly continuous, one-parameter group of *-automorphisms of $\mathcal{A}$, constructed above. Then, $(t,\omega)\mapsto\tau_t(\omega)(A)$ is strongly, jointly measurable in t and ω, for all $A\in\mathcal{A}$.

Proof. The proof is a consequence of theorem 4.1, if one notices that there exists a sequence $\{\Lambda_n\}$ of finite subsets increasing to $\mathbb{Z}^\nu$ i.e.,

$$\Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset \cdots, \quad \text{and} \quad \bigcup_{n=1}^{\infty} \Lambda_n = \mathbb{Z}^\nu. \qquad \triangle$$

It is seen in the case of quantum spin systems on a lattice $\mathbb{Z}^\nu$ with translation invariant interactions that, whenever the dynamics exists, the evolution group of *-automorphisms of the quasi-local algebra commutes with the action of the lattice $\mathbb{Z}^\nu$ on the algebra. Here we prove a variant of this property. Before we set about establishing this result the following fact is worth noting.

Lemma 4.3. *Let $\tau_t^\Lambda(\omega)$ be the one-parameter group of *-automorphisms associated with a finite $\Lambda \subseteq \mathbb{Z}^\nu$, where*

$$\tau_t^\Lambda(\omega)(A) = \mathrm{e}^{iH(\Lambda,\omega)t} A \mathrm{e}^{-iH(\Lambda,\omega)t}, \quad \forall\, A \in \mathcal{A}.$$

Then for all $a \in \mathbb{Z}^\nu$, we have

$$\alpha_a(\tau_t^\Lambda(\omega)(A)) = \tau_t^{\Lambda+a}(T_{-a}\omega)(\alpha_a(A)); \quad \forall A \in \mathcal{A}.$$

Proof. Using the fact that α_a is a *-automorphism, the lemma follows from functional calculus for $H(\Lambda, \omega)$, and the identity $H(\Lambda + a, T_{-a}\omega) = \alpha_a(H(\Lambda, \omega))$. $\triangle$

PROPOSITION 4.4

Let $\tau_t(\omega)$ be the evolution group of the spin system on an infinite lattice $\mathbb{Z}^\nu$. Then for all $a \in \mathbb{Z}^\nu$, we have

$$\tau_t(T_{-a}\omega)(\alpha_a(A)) = \alpha_a(\tau_t(\omega)(A)), \quad \forall\, A \in \mathcal{A}.$$

Proof. The proof follows from theorem 4.1, and the lemma established prior to this proposition. $\triangle$

In the proposition that follows, we establish an interesting algebraic property of the generators $\overline{\delta}(\omega)$ of the evolution groups $\tau_t(\omega)$.

PROPOSITION 4.5

Let $\tau_t(\omega)$ be the evolution group of the spin system and $D(\overline{\delta}(\omega))$ be the domain of its generator. Then for all $a \in \mathbb{Z}^\nu$, we have

$$\alpha_a(D(\overline{\delta}(\omega))) = D(\overline{\delta}(T_{-a}\omega)) \quad \text{and} \quad \alpha_a(\overline{\delta}(\omega))(A) = \overline{\delta}(T_{-a}\omega)(\alpha_a(A)),$$

for all $A \in D(\overline{\delta}(\omega))$.

Proof. Throughout the proof of this theorem $\delta^\Lambda(\omega)$ will denote the generator $i[H(\Lambda, \omega), .]$ of $\tau_t^\Lambda(\omega)$. Let $\{\Lambda_n\}$ be a sequence of finite subsets increasing to $\mathbb{Z}^\nu$. Since $\tau_t^{\Lambda_n}(\omega)(B) \to \tau_t(\omega)(B)$; $\forall B \in \mathcal{A}$, we conclude from preliminary 2.4 in [2] that, $\overline{\delta}(\omega)$ is the graph limit of $\delta^{\Lambda_n}(\omega)$. Hence, for $A \in D(\overline{\delta}(\omega))$, there exists a sequence $\{A_n\}$, where $A_n \in D(\delta^{\Lambda_n}(\omega))$ such that, $A_n \to A$ and $\delta^{\Lambda_n}(\omega)(A_n) \to \overline{\delta}(\omega)(A)$. This implies that $\alpha_a(A_n) \to \alpha_a(A)$ and

$\alpha_a(\delta^{\Lambda_n}(\omega)(A_n)) \to \alpha_a(\overline{\delta}(\omega)(A))$. Now, since $\alpha_a(\delta^{\Lambda_n}(\omega)(A_n)) = \delta^{\Lambda_n+a}(T_{-a}\omega)(\alpha_a(A_n))$, we have $\delta^{\Lambda_n+a}(T_{-a}\omega)(\alpha_a(A_n)) \to \alpha_a(\overline{\delta}(\omega)(A))$. Since $\tau_t^{\Lambda_n+a}(T_{-a}\omega)(B)$ converges to $\tau_t(T_{-a}\omega)(B)$, for all $B \in \mathcal{A}$, preliminary 2.4 in [2] implies that $\overline{\delta}(T_{-a}\omega)$ is the graph limit of $\delta^{\Lambda_n+a}(T_{-a}\omega)$. Therefore, one concludes that $\alpha_a(A) \in D(\overline{\delta}(T_{-a}\omega))$ and $\alpha_a(\overline{\delta}(\omega))(A) = \overline{\delta}(T_{-a}\omega)(\alpha_a(A))$. Conversely, it can be shown that if $A \in D(\overline{\delta}(T_{-a}\omega))$ then $\alpha_{-a}(A) \in D(\overline{\delta}(\omega))$. This completes the proof of the proposition. △

In the next section, we study the Arveson spectrum of the evolution group $\tau_t(\omega)$, and report an interesting ergodic property of the Arveson spectrum.

5. Arveson spectrum

Here we introduce the notion of Arveson spectrum. If $\mathcal{A}$ is a $C^\star$-algebra and $t \mapsto \gamma_t$, $t \in \mathbb{R}$, a strongly continuous, one-parameter group of *-automorphisms of the $C^\star$-algebra, then the Bochner integral

$$\int_{-\infty}^{\infty} f(t)\gamma_t(A)\mathrm{d}t = \Gamma(f)(A); \quad A \in \mathcal{A}, \quad f \in L^1(\mathbb{R}),$$

defines a representation of $L^1(\mathbb{R})$ into the bounded operators on $\mathcal{A}$. Next, we have the following definition.

DEFINITION 5.1

The Arveson spectrum $\mathrm{Sp}(\gamma)$ of γ is a subset of the dual group $\hat{\mathbb{R}}$ of $\mathbb{R}$ defined as

$$\mathrm{Sp}(\gamma) = \{\sigma \in \mathbb{R} | \hat{f}(\sigma) = 0, \quad \forall f \in \ker\Gamma\},$$

where $\hat{f}$ is the Fourier transform of f.

It can be shown that $s \in \mathrm{Sp}(\gamma)$, if and only if, $|\hat{f}(s)| \le ||\Gamma(f)||$, for all $f \in L^1(\mathbb{R})$ vide [9].

The following definition is in order.

DEFINITION 5.2

Let $(\Omega, \mathcal{S}, P)$ be a probability space and J some index set. If T_j is a measure preserving automorphism of Ω, for each $j \in J$, then the action of T_j's is said to be ergodic if, for $A \in \mathcal{S}$, $P(A) = 0$ or 1 whenever $T_jA = A$, for all $j \in J$.

Our aim is to show that the Arveson spectrum of the evolution group $\tau_t(\omega)$ is almost surely independent of ω. To this end, we have the following theorem.

Theorem 5.3. *Let $\tau_t(\omega)$ be the strongly continuous, one-parameter group of *-automorphisms of $\mathcal{A}$, which determines the evolution of the spin system. If the action of the measure preserving group of automorphisms $\{T_a\}$ is ergodic, then the Arveson spectrum $Sp(\tau(\omega))$ of $\tau_t(\omega)$ is almost surely independent of ω.*

Proof. For $s \in \mathbb{R}$, let $E_s = \{\omega \in \Omega : ||\Gamma(\omega)(f)|| \ge |\hat{f}(s)| \forall f \in L^1(\mathbb{R})\}$, where

$$\Gamma(\omega)(f)(A) = \int_{-\infty}^{\infty} f(t)\tau_t(\omega)(A)\mathrm{d}t, \quad \forall A \in \mathcal{A}.$$

We show that E_s is a measurable subset of Ω. Since $L^1(\mathbb{R})$ is separable, there exists a countable dense set $F = \{f_n | n = 1, 2, \ldots\}$ in $L^1(\mathbb{R})$. Hence, for each $f \in L^1(\mathbb{R})$, there exists a sequence f_{n_k} in F, converging to f in the L^1-norm. It follows from the properties of the Bochner integral that

$$\begin{aligned} | \, \|\Gamma(\omega)(f_{n_k})\| - \|\Gamma(\omega)(f)\| \, | &\leq \|\Gamma(\omega)(f_{n_k}) - \Gamma(\omega)(f)\| \\ &= \|\Gamma(\omega)(f_{n_k} - f)\| \\ &= \sup_{\|A\|=1} \left\| \int_{-\infty}^{\infty} (f_{n_k} - f)(t)\tau_t(\omega)(A)\mathrm{d}t \right\| \\ &\leq \sup_{\|A\|=1} \left(\|A\| \int_{-\infty}^{\infty} |(f_{n_k} - f)(t)|\mathrm{d}t \right) \\ &= \int_{-\infty}^{\infty} |(f_{n_k} - f)(t)|\mathrm{d}t \\ &= \|f_{n_k} - f\|_1. \end{aligned}$$

Therefore, $\|\Gamma(\omega)(f_{n_k})\|$ converges to $\|\Gamma(\omega)(f)\|$, for f_{n_k} converging to f, in the L^1-norm. In view of this, and the fact that F is dense in $L^1(\mathbb{R})$, we have

$$E_s = \bigcap_{n=1}^{\infty} E_s^n,$$

where $E_s^n = \{\omega \in \Omega | \, \|\Gamma(\omega)(f_n)\| \geq |\hat{f}_n(s)|\}$. In order to show that each of these E_s^n's is a measurable subset of Ω, it is sufficient to establish the measurability of the function $\omega \mapsto \|\Gamma(\omega)(f_n))\|$, for all $n = 1, 2, \ldots$. On appealing to Proposition 4.2, we conclude that for $f \in L^1(\mathbb{R})$ and $A \in \mathcal{A}$, $(t, \omega) \mapsto f(t)\tau_t(\omega)(A)$ is strongly, jointly measurable in t and ω. Moreover,

$$\begin{aligned} \int_{\mathbb{R}\times\Omega} \|f(t)\tau_t(\omega)(A)\|\mathrm{d}(\mu \times P)(t, \omega) &= \int_{\mathbb{R}\times\Omega} |f(t)| \, \|\tau_t(\omega)(A)\|\mathrm{d}(\mu \times P)(t, \omega) \\ &= \int_{\mathbb{R}} \int_{\Omega} \|A\| \, |f(t)|\mathrm{d}\mu(t)\mathrm{d}P(\omega) < \infty. \end{aligned}$$

Hence, it follows that $(t, \omega) \mapsto f(t)\tau_t(\omega)(A)$ is Bochner integrable on $\mathbb{R} \times \Omega$ [7]. Therefore, as a consequence of the analogue of Fubini's theorem for vector valued functions (see [7]), the map $\omega \mapsto \Gamma(\omega)(f)(A)$ is strongly measurable in ω. Hence, $\omega \mapsto \|\Gamma(\omega)(f)(A)\|$ is a measurable, real valued function on Ω. Thus it readily follows that for $f \in L^1(\mathbb{R})$, $\omega \mapsto \|\Gamma(\omega)(f)(A)\|$ is measurable for all $A \in \mathcal{A}$. Now $\mathcal{A}$ being a separable $C^\star$-algebra, we have for $c \geq 0$ and $f \in L^1(\mathbb{R})$,

$$\{\omega \in \Omega | \, \|\Gamma(\omega)(f)\| \leq c\} = \bigcap_{n=1}^{\infty} \{\omega \in \Omega | \, \|\Gamma(\omega)(f)(A_n)\| \leq c; \|A_n\| \leq 1\},$$

where $\mathcal{U}_0 = \{A_n \in \mathcal{A} | n = 1, 2, \ldots\}$ is a dense subset of the closed unit ball in $\mathcal{A}$. This, coupled with the fact that $\omega \mapsto \|\Gamma(\omega)(f)(A_n)\|$ is a measurable function of ω for all $n = 1, 2, \ldots$, permits us to conclude that the set $\{\omega \in \Omega | \, \|\Gamma(\omega)(f)\| \leq c\}$, is a measurable subset of Ω. Since c is arbitrary, the function $\omega \mapsto \|\Gamma(\omega)(f)\|$ is a measurable function of ω. Thus, $\omega \mapsto \|\Gamma(\omega)(f)\|$ is measurable for all $f \in L^1(\mathbb{R})$. Therefore, $\omega \mapsto \|\Gamma(\omega)f_n\|$ is measurable $\forall n = 1, 2, \ldots$. Hence, each of these E_s^n's is a measurable subset of Ω. This proves conclusively that the set E_s is a measurable subset of Ω. Now, using the fact that the action of the measure preserving group of automorphisms $\{T_a\}$ is ergodic, we show that set E_s has measure either zero or one. It follows from the properties of the Bochner integral [10] and the fact that α_a is a *-automorphism of the $C^\star$-algebra $\mathcal{A}$ that, for $f \in L^1(\mathbb{R})$,

$$\begin{aligned}
||\Gamma(\omega)(f)|| &= \sup_{||A||=1}\left\| \int_{-\infty}^{\infty} f(t)\tau_t(\omega)(A)\mathrm{d}t \right\| \\
&= \sup_{||\alpha_a(A)||=1}\left\| \alpha_a\left(\int_{-\infty}^{\infty} f(t)\tau_t(\omega)(A)\mathrm{d}t \right) \right\| \\
&= \sup_{||\alpha_a(A)||=1}\left\| \int_{-\infty}^{\infty} f(t)\tau_t(T_{-a}\omega)(\alpha_a(A))\mathrm{d}t \right\| \\
&= ||\Gamma(T_{-a}\omega)(f)||,
\end{aligned}$$

for all $a \in \mathbb{Z}^\nu$. The penultimate step follows from Proposition 4.4. Therefore it is clear from the above equalities, that the set E_s is invariant under the action of the measure preserving group of automorphisms $\{T_a\}$. Since the action of the measure preserving group of automorphisms $\{T_a\}$ is assumed to be ergodic, it follows that the set E_s has measure either zero or one. Hence, s lies in the Arveson spectrum of $\tau_t(\omega)$ with probability either zero or one. Thus, the Arveson spectrum $\mathrm{Sp}(\tau(\omega))$ of $\tau_t(\omega)$ is almost surely independent of ω. △

DEFINITION 5.4

Let $(\mathcal{A}, \tau)$ be a $C^\star$-dynamical system, ρ a state over $\mathcal{A}$. Then for $\beta > 0$, ρ is said to be a (τ, β)-KMS state if, for any pair $A, B \in \mathcal{A}$, there exists a complex function $F_{A,B}$ which is analytic on the open strip $0 < \Im z < \beta$, uniformly bounded and continuous on the closed strip $0 \leq \Im z \leq \beta$ such that, $F_{A,B}(t) = \rho(A\tau_t(B))$ and $F_{A,B}(t + i\beta) = \rho(\tau_t(B)A)$.

Next, we shall show that for any family of $(\tau(\omega), \beta)$-KMS states the spectrum of the generator of the unitary group $U_t(\omega)$, which implements $\tau(\omega)$ in the GNS representation is almost surely independent of ω. To this end, we have the following proposition.

PROPOSITION 5.5

Let $\{\rho(\omega)\}$ be a family of $(\tau(\omega), \beta)$-KMS states. Also, let $H(\omega)$ be the generator of the strongly continuous, one-parameter group of unitaries $U_t(\omega)$ which implement $\tau_t(\omega)$ in the GNS representation. Then the spectrum $\sigma(H(\omega))$ of the generator $H(\omega)$ is almost surely independent of ω.

Proof. Let π_ω denote the representation associated with the $(\tau(\omega), \beta)$-KMS state $\rho(\omega)$, with cyclic vector Θ_ω. The unitary group $U_t(\omega)$ with generator $H(\omega)$ implements $\tau_t(\omega)$ in this representation π_ω. Now, for $f \in L^1(\mathbb{R})$, we have

$$\begin{aligned}
\Psi_\omega(f)\phi = \int_{-\infty}^{\infty} f(t)U_t(\omega)\phi\mathrm{d}t = 0, \quad \forall\ \phi \in \mathcal{H}_\omega \\
\Leftrightarrow \int_{-\infty}^{\infty} f(t)U_t(\omega)(\pi_\omega(A)\Theta_\omega)\mathrm{d}t = 0, \quad \forall\ A \in \mathcal{A}
\end{aligned}$$

$$
\begin{aligned}
&\Leftrightarrow \int_{-\infty}^{\infty} f(t)\pi_\omega(\tau_t(\omega)(A))\Theta_\omega \mathrm{d}t = 0, \quad \forall\, A \in \mathcal{A}\\
&\Leftrightarrow \left(\int_{-\infty}^{\infty} f(t)\pi_\omega(\tau_t(\omega)(A))\mathrm{d}t\right)\Theta_\omega = 0, \quad \forall\, A \in \mathcal{A}\\
&\Leftrightarrow \pi_\omega\left(\int_{-\infty}^{\infty} f(t)\tau_t(\omega)(A)\mathrm{d}t\right)\Theta_\omega = 0, \quad \forall\, A \in \mathcal{A}\\
&\Leftrightarrow \pi_\omega\left(\int_{-\infty}^{\infty} f(t)\tau_t(\omega)(A)\mathrm{d}t\right) = 0, \quad \forall\, A \in \mathcal{A}\\
&\Leftrightarrow \int_{-\infty}^{\infty} f(t)\tau_t(\omega)(A)\mathrm{d}t = 0, \quad \forall\, A \in \mathcal{A}.
\end{aligned}
$$

The first step follows from the fact that Θ_ω is a cyclic vector for $\pi_\omega(\mathcal{A})$. The second follows from the definition of $U_t(\omega)$. Since $\rho(\omega)$ is a KMS state, the separating character of the cyclic vector Θ_ω for $\pi_\omega(\mathcal{A})''$, accounts for the penultimate step. We arrive at the final step by virtue of the fact that the representation π_ω is faithful. Now, using the spectral theorem it is not very difficult to show that $\sigma(H(\omega)) = -\{s \in \mathbb{R} | \hat{f}(s) = 0, \forall\, f \in \ker \Psi(\omega)\}$. Therefore, we have $\sigma(H(\omega)) = -\{s \in \mathbb{R} | \hat{f}(s) = 0,\ \forall\, f \in \ker\, \Gamma(\omega)\}$, where $\Gamma(\omega)$ is as in the theorem proved above. Hence, the proof follows from the theorem proved above, where we have shown that the Arveson spectrum of $\tau(\omega)$ is almost surely independent of ω. △

6. Conclusion

In this paper we have studied the dynamics of a quantum spin glass through the spectral properties of a family of evolution groups $\{\tau_t(\omega)\}$ of a quantum spin system with random interactions. The almost sure independence of the Arveson spectrum $\mathrm{Sp}(\tau(\omega))$ of the evolution group $\tau_t(\omega)$ in a way suggests that the Arveson spectrum becomes independent of ω in the thermodynamic limit. Besides, given a family of $(\tau(\omega), \beta)$-KMS states $\{\rho(\omega)\}$, we demonstrated the almost sure independence of the spectrum of the generator of the group of unitaries which implement the evolution group $\tau(\omega)$ in the GNS representation.

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