Proc. Indian Acad. Sci. (Math. Sci.), Vol. 110, No. 3, August 2000, pp. 335–345

Suppression of instability in rotatory hydromagnetic convection

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MS received 13 January 1999; revised 21 December 1999

Abstract. Recently discovered hydrodynamic instability [1], in a simple Bénard configuration in the parameter regime $T_0\alpha_2 > 1$ under the action of a nonadverse temperature gradient, is shown to be suppressed by the simultaneous action of a uniform rotation and a uniform magnetic field both acting parallel to gravity for oscillatory perturbations whenever $(\mathcal{Q}\sigma_1/\pi^2 + \mathcal{J}/\pi^4) > 1$ and the effective Rayleigh number $\mathcal{R}(1 - T_0\alpha_2)$ is dominated by either $27\pi^4(1 + 1/\sigma_1)/4$ or $27\pi^4/2$ according as $\sigma_1 \geq 1$ or $\sigma_1 \leq 1$ respectively. Here T_0 is the temperature of the lower boundary while α_2 is the coefficient of specific heat at constant volume due to temperature variation and $\sigma_1, \mathcal{R}$, $\mathcal{Q}$ and $\mathcal{J}$ respectively denote the magnetic Prandtl number, the Rayleigh number, the Chandrasekhar number and the Taylor number.

Keywords. Bénard convection; hydrodynamic instability; hydromagnetic instability; oscillatory perturbations.

1. Introduction

The study of thermal convection in a layer of fluid heated from below has many physical applications, notable in astrophysics and geophysics, as well as in the industrial processes. The problem of onset of convection in such a layer of fluid confined between two infinite horizontal planes, known as Bénard problem, has been extensively investigated both experimentally and theoretically. The Bénard problem was studied mathematically for the first time by Lord Rayleigh [9] for the idealized case of two free boundaries. Rayleigh's theory shows that the gravity-dominated thermal instability in liquid layer heated underside, depends upon the Rayleigh number which is proportional to the uniform temperature difference maintained between the lowermost and uppermost temperatures of the layers of the concerning liquid. Banerjee *et al* [1] presented a modified analysis of thermal instability of a liquid layer heated underside by emphasizing and utilizing the point, on the basis of the experimental results of Schmidt and Milverton [10], that linear theoretical explanation of the phenomenon of gravity-dominated thermal instability in a liquid layer heated underside should depend not only upon the Rayleigh number which is proportional to the uniform temperature difference maintained across the layer but also upon another parameter that takes care of a relatively hotter or a cooler layer under almost identical conditions. It was found that Rayleigh's utilization of the Boussinesq approximation overlooks a term in the equation of heat conduction which is on account of the variations in specific heat at constant volume due to the variations in temperature and which is such that in usual circumstances it cannot be ignored if the Boussinesq approximation were to be consistently and relatively more accurately applied throughout the analysis. The essential argument on which this term found a place in their modified theory is that it is the temperature differences that were of moderate amounts but not necessarily the temperature itself and an incorporation of this term into the calculations adequately completes the qualitative and quantitative gaps in Rayleigh's theory as pointed out earlier. Further they showed that the reformulated equations of the Bénard convection breakdown in the parameter regime $T_0\alpha_2 > 1$ and predict the existence of some new phenomenon. In particular the existence of the hydrodynamic instability in a single diffusive bottom heavy system is mathematically derived as an outcome of the reformulated equations in the parameter regime $T_0\alpha_2 > 1$.

The stability investigations of the Bénard problem in the framework of various external force fields assume importance not only on account of being a meaningful mathematical extension of the problem but also because of its importance in the problems of meteorology, oceanography and various other fields of practical importance. The effects of the action of a uniform magnetic field/a uniform rotation acting parallel to gravity on the Bénard problem has been investigated by Chandrasekhar [2] and others and it is shown that in some respect their individual/combined effects are remarkably alike, namely they both inhibit the onset of instability and elongate the cells which appear at the marginal stability for certain ranges of values of the parameters involved. Another interesting point brought out by Chandrasekhar's analysis which is, in general qualitative agreement with the experimental results of Nakagawa ([6,7] and [8]) Fultz, Nakagawa and Frenzen [4] and others is that, in both the problems the marginal state could either be stationary or oscillatory in character for which sufficient conditions are obtained. The work of Eltayeb [3] is also concerned with the combined effect of rotation and magnetic field on simple Bénard problem. Gupta *et al* [5] also investigated the problem under the joint influence of a rotation and a magnetic field especially with a view to derive bounds for the complex growth rate for an arbitrary osciallatory perturbation which may be neutral or unstable.

The aim of the present paper is to show how the instability reported by Banerjee *et al* is suppressed for oscillatory perturbations by the simultaneous application of a uniform vertical rotation and a uniform vertical magnetic field.

2. Construction of the modified simplified equations governing the problem

Let the origin be taken on the lower boundary with the positive direction of the z-axis along the vertically upward direction. Let $z = d\,(> 0)$ denote the upper boundary and T_0 and $T_1 (< T_0)$ respectively denote the uniform temperatures of the lower and upper boundaries. Further, let the layer of fluid be in a state of uniform rotation with angular velocity $\vec{\Omega}$ and subject to a uniform magnetic field $\vec{H}$ such that $\vec{\Omega}$ and $\vec{H}$ are parallel to gravity. Then following Banerjee–Gupta *et al* [1] the modified simplified equations governing the rotatory hydromagnetic Bénard convection problem are given by

$$\partial u_j/\partial x_j = 0, \tag{1}$$

$$\begin{aligned}\partial u_i/\partial t + u_j \partial u_i/\partial x_j - \mu_e |\vec{H}|^2 (H_j \partial H_i/\partial x_j)/4\pi\rho_0 = (1 + \delta\rho/\rho_0) X_i \\ - \partial\{p/\rho_0 - |\vec{\Omega} \times \vec{r}|^2/2 + \mu_e |\vec{H}|^2/8\pi\rho_0\}/\partial x_i + 2\epsilon_{ijk} u_j \Omega_k + \nu_0 \nabla^2 u_i,\end{aligned} \tag{2}$$

$$(1 - T_0\alpha_2)\{\partial T/\partial t + \partial T/\partial x_j\} = K_0 \nabla^2 T, \tag{3}$$

$$\partial H_i/\partial t + u_j \partial H_i/\partial x_j = H_j \partial u_i/\partial x_j + \eta_0 \nabla^2 H_i, \tag{4}$$

$$\partial H_i/\partial x_j = 0 \tag{5}$$

and

$$\rho = \rho_0[1 - \alpha(T - T_0)], \tag{6}$$

where $x_j (j = 1, 2, 3)$ respectively denote the x, y and z coordinates; u_i, X_i, H_i, Ω_i $(i = 1, 2, 3)$ respectively denote the x, y and z components of velocity, external force, magnetic field and rotation; T denotes the temperature, ρ the density, p the pressure, ϵ_{ijk} the permutation tensor, μ_e the magnetic permeability, α is volume expansion, $\vec{r} = (x, y, z)$ is the position vector and ρ_0, ν_0, η_0 and K_0 stand for values of density, viscosity, magnetic diffusivity and thermal conductivity at the lower boundary $z = 0$.

Clearly, the initial stationary states whose stability we wish to examine is characterized by the following solutions for the velocity, temperature, magnetic field, density and pressure respectively:

$$u_j \equiv (0, 0, 0), T = T_0 - \beta z, H_i = (0, 0, H),$$
$$\rho = \rho_0[1 + \alpha(T_0 - T)] = \rho[1 + \alpha\beta z]$$

and

$$P = p - \rho_0|\vec{\Omega} \times \vec{r}|^2/2 + \mu_e|\vec{H}|^2/8\pi = P_0 - g\rho_0[z + \alpha\beta z^2/2], \tag{7}$$

where P_0 and ρ_0 are the values of P and ρ at the lower boundary $z = 0$ and $\beta = (T_0 - T_1)/d$ is the maintained uniform temperature gradient. Further, $(\Omega_i) = (0, 0, \Omega)$.

Let the initial stationary state described by equations (7) be slightly perturbed. Then the linearized perturbations equations on the basis of the normal mode resolution Chandrasekhar [2], wherein the desired solutions have x, y and t dependence of the form

$$\exp[i(k_x x + k_y y) + nt], \tag{8}$$

are as follows:

$$ik_x u + ik_y v + \mathrm{d}w/\mathrm{d}z = 0, \tag{9}$$

$$\rho_0 nu = -ik_x \delta P + \mu_0(\mathrm{d}^2/\mathrm{d}z^2 - k^2)u + (\mu_e H/4\pi)[\partial\hbar_x/\partial z - ik_x\hbar_z] + 2\rho_0\Omega v, \tag{10}$$

$$\rho_0 nv = -ik_y \delta P + \mu_0(\mathrm{d}^2/\mathrm{d}z^2 - k^2)v + (\mu_e H/4\pi)[\partial\hbar_y/\partial z - ik_y\hbar_z] - 2\rho_0\Omega u, \tag{11}$$

$$\rho_0 nw = -\mathrm{d}(\partial P)/\mathrm{d}z + \mu_0(\mathrm{d}^2/\mathrm{d}z^2 - k^2)w + g\alpha\rho_0\theta, \tag{12}$$

$$n(1 - T_0\alpha_2)\theta - (1 - T_0\alpha_2)\beta w = K_0(\mathrm{d}^2/\mathrm{d}z^2 - k^2)\theta, \tag{13}$$

$$n\hbar_x = H\mathrm{d}u/\mathrm{d}z + \eta_0(\mathrm{d}^2/\mathrm{d}z^2 - k^2)\hbar_x, \tag{14}$$

$$n\hbar_y = H\mathrm{d}v/\mathrm{d}z + \eta_0(\mathrm{d}^2/\mathrm{d}z^2 - k^2)\hbar_y, \tag{15}$$

$$n\hbar_z = H\mathrm{d}w/\mathrm{d}z + \eta_0(\mathrm{d}^2/\mathrm{d}z^2 - k^2)\hbar_z, \tag{16}$$

$$ik_x\hbar_x + ik_y\hbar_y + \mathrm{d}w/\mathrm{d}z = 0, \tag{17}$$

$$\rho_0 n\zeta = \mu_0(\mathrm{d}^2/\mathrm{d}z^2 - k^2)\zeta + (\mu_e H/4\pi)\mathrm{d}\xi/\mathrm{d}z + 2\rho_0\Omega\mathrm{d}w/\mathrm{d}z, \tag{18}$$

$$n\xi = H\mathrm{d}\zeta/\mathrm{d}z + \eta_0(\mathrm{d}^2/\mathrm{d}z^2 - k^2)\xi, \tag{19}$$

where $\{u(z), v(z), w(z)\}, \theta(z), \delta\rho(z), \delta P(z)$ and $\{\hbar_x(z), \hbar_y(z), \hbar_z(z)\}$ are the perturbations in the velocity, temperature, initial density, pressure and magnetic field respectively, $k = (k_x^2 + k_y^2)^{1/2}$ is the wave number of perturbation, k_x and k_y being real, n is a constant which can be complex in general, ζ and ξ denote the vorticity and the current density respectively.

Multiplying equation (10) by ik_x and equation (11) by ik_y, adding the resulting equations and making use of equations (9) and (17), we have

$$\rho_0 n \mathrm{d}w/\mathrm{d}z = -k^2\delta P + \mu_0(\mathrm{d}^2/\mathrm{d}z^2 - k^2)\mathrm{d}w/\mathrm{d}z + (\mu_e H/4\pi)(\mathrm{d}^2/\mathrm{d}z^2 - k^2)\hbar_z - 2\rho_0\Omega\zeta. \tag{20}$$

Eliminating δP between eqs (20) and (12), it follows that

$$(\mathrm{d}^2/\mathrm{d}z^2 - k^2)(\mathrm{d}^2/\mathrm{d}z^2 - k^2 - n/\nu_0)w = g\alpha k^2\theta/\nu_0 - (\mu_e H/4\pi\rho_0\nu_0)(\mathrm{d}^2/\mathrm{d}z^2 - k^2)\mathrm{d}\hbar_z/\mathrm{d}z + (2\Omega/\nu_0)\mathrm{d}\zeta/\mathrm{d}z. \tag{21}$$

Further, eqs (13), (14)–(17), (18) and (19) can be written as

$$(\mathrm{d}^2/\mathrm{d}z^2 - k^2 - n/K_0)\theta = -(1 - T_0\alpha_2)\beta/K_0, \tag{22}$$

$$(\mathrm{d}^2/\mathrm{d}z^2 - k^2 - n/\eta_0)\hbar_z = -(H/\eta_0)\mathrm{d}w/\mathrm{d}z, \tag{23}$$

$$(\mathrm{d}^2/\mathrm{d}z^2 - k^2 - n/\nu_0)\zeta = -(\mu_e H/4\pi\rho_0\nu_0)\mathrm{d}\xi/\mathrm{d}z - (2\Omega/\nu_0)\mathrm{d}w/\mathrm{d}z, \tag{24}$$

$$(\mathrm{d}^2/\mathrm{d}z^2 - k^2 - n/\eta_0)\xi = -(H/\eta_0)\mathrm{d}\zeta/\mathrm{d}z. \tag{25}$$

Using the non-dimensional quantities defined by

$$z_* = z/d;\ a_* = k/d;\ \sigma_* = \nu_0/K_0;\ D_* = d\mathrm{d}/\mathrm{d}z;\ \rho_* = nd/K_0^2;\ \sigma_{1^*} = \nu_0/\eta_0;$$
$$\theta_* = \mathcal{R}_* a^2\theta/\beta d;\ W_* = d(w/K_0);\ \xi_* = \nu_0\eta_0\xi/d(2\Omega H K_0);\ \hbar_{z*} = \eta_0\hbar_z/HK_0;$$
$$\mathcal{R}_* = g\alpha\beta d^4/K_0\nu_0;\ \mathcal{Q}_* = \mu_e H^2 d^2/4\pi\rho_0\nu_0\eta_0;\ \mathcal{J}_* = 4\Omega^2 d^2/\nu_0. \tag{26}$$

where $D \equiv \mathrm{d}/\mathrm{d}z, \nu_0 = \mu_0/\rho_0$ and dropping the asterisks for convenience in writing, we have the following system of equations

$$(D^2 - a^2)(D^2 - a^2 - p/\sigma)W = \theta - \mathcal{Q}D(D^2 - a^2)\hbar_z + \mathcal{J}D\zeta, \tag{27}$$

$$(D^2 - a^2 - p)\theta = -\mathcal{R}(1 - T_0\alpha_2)a^2 W, \tag{28}$$

$$(D^2 - a^2 - p\sigma_1\sigma)\hbar_z = -DW, \tag{29}$$

$$(D^2 - a^2 - p/\sigma)\zeta = -\mathcal{Q}D\xi - DW, \tag{30}$$

$$(D^2 - a^2 - p\sigma_1/\sigma)\xi = -D\zeta. \tag{31}$$

Equations (27)–(31) together with the boundary conditions

$$W = 0 = \theta = DW = \zeta = D\xi = \hbar_z \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1,$$
$$\text{(rigid boundaries with regions outside perfectly conducting)} \tag{32}$$

constitute a double eigenvalue problem for p for prescribed values of $a^2, \mathcal{R}, \mathcal{Q}, \mathcal{J}$ and $T_0\alpha_2$ and a given state of the system is stable, neutral or unstable provided that the real part p_r of p is negative, zero or positive respectively. Further, if $p_r = 0$ implies that $p_i = 0$ for every wave number a, then the 'principle of exchange of stabilities' (PES) is valid, otherwise we have overstability at least when instability sets in as certain modes.

3. Mathematical analysis

We prove the following lemma and theorems;

Lemma. $(p, W, \theta, \hbar_z, \xi, \zeta)$, $p = p_r + ip_i$, $p_r \geq 0$, $p_i \neq 0$, $\mathcal{R} < 0$, $T_0\alpha_2 > 1$, $\mathcal{Q} > 0$, $\mathcal{J} > 0$ *is a solution of equations* (27)–(32), *then* $(\mathcal{Q}\sigma_1/\pi^2 + \mathcal{J}/\pi^4) > 1$.

Proof. Multiplying equation (27) by W^* (the complex conjugate of W) throughout, integrating the resulting equation over the vertical range of z and using eqs (28)–(31), we have

$$\int W^*(D^2 - a^2)(D^2 - a^2 - p/\sigma)W\mathrm{d}z = -1/[\mathcal{R}(1 - T_0\alpha_2)]a^2 \times \int \theta(D^2 - a^2 - p^*)\theta^*\mathrm{d}z - \mathcal{Q}\int (D^2 - a^2)\hbar_z(D^2 - a^2 - p^*\sigma_1/\sigma)\hbar_z^*\mathrm{d}z + \mathcal{J}\int \zeta(D^2 - a^2 - p^*/\sigma)\zeta^*\mathrm{d}z + \mathcal{Q}\mathcal{J}\int \xi^*(D^2 - a^2 - p\sigma_1/\sigma)\xi\mathrm{d}z. \quad (33)$$

The limits of integration in the above equation and subsequently will be omitted for sake of convenience in writing.

Integrating various terms of eq. (33) by parts for an appropriate number of times and making use of relevant boundary conditions given by eq. (32), we have

$$\int (|D^2W|^2 + 2a^2|DW|^2 + a^4|W|^2)\mathrm{d}z + p/\sigma\int (|DW|^2 + a^2|W|^2)\mathrm{d}z + \mathcal{J}\int (|D\zeta|^2 + a^2|\zeta|^2)\mathrm{d}z + \mathcal{J}p^*/\sigma\int |\zeta|^2\mathrm{d}z + \mathcal{Q}\left[\int (|D^2\hbar_z|^2 + 2a^2|D\hbar_z|^2 + a^4|\hbar_z|^2)\mathrm{d}z + p^*\sigma_1/\sigma\int (|D\hbar_z|^2 + a^2|\hbar_z|^2)\mathrm{d}z\right] + \mathcal{Q}\mathcal{J}\left[\int (|D\xi|^2 + a^2|\xi|^2 + (p\sigma_1/\sigma)|\xi|^2)\mathrm{d}z\right] = [1/[\mathcal{R}(1 - T_0\alpha_2)]a^2]\int [|D\theta|^2 + (a^2 + p^*)|\theta|^2]\mathrm{d}z. \quad (34)$$

Equating the real and imaginary parts of both sides of equation (34) and cancelling $p_i(\neq 0)$ throughout from the imaginary parts, we have

$$\int (|D^2W|^2 + 2a^2|DW|^2 + a^4|W|^2)\mathrm{d}z + p_r/\sigma\int (|DW|^2 + a^2|W|^2)\mathrm{d}z + \mathcal{J}\int (|D\zeta|^2 + a^2|\zeta|^2)\mathrm{d}z + \mathcal{Q}\int (|D^2\hbar_z|^2 + 2a^2|D\hbar_z|^2 + a^4|\hbar_z|^2)\mathrm{d}z + \mathcal{Q}\mathcal{J}\int (|D\xi|^2 + a^2|\xi|^2)\mathrm{d}z + p_r/\sigma\left[\int (|D\hbar_z|^2 + a^2|\hbar_z|^2)\mathrm{d}z + \mathcal{J}\int |\zeta|^2\mathrm{d}z + \mathcal{Q}\mathcal{J}\sigma_1\int |\xi|^2\mathrm{d}z - [\sigma/[\mathcal{R}(1 - T_0\alpha_2)]a^2]\int |\theta|^2\mathrm{d}z\right] = 1/[\mathcal{R}(1 - T_0\alpha_2)]a^2\int (|D\theta|^2 + a^2|\theta|^2)\mathrm{d}z \quad (35)$$

and

$$1/\sigma\int (|DW|^2 + a^2|W|^2)\mathrm{d}z + \mathcal{Q}\mathcal{J}\sigma_1/\sigma\int |\xi|^2\mathrm{d}z + 1/[\mathcal{R}(1 - T_0\alpha_2)]a^2\int |\theta|^2\mathrm{d}z = \mathcal{Q}\sigma_1/\sigma\int (|D\hbar_z|^2 + a^2|\hbar_z|^2)\mathrm{d}z + \mathcal{J}/\sigma\int |\zeta|^2\mathrm{d}z. \quad (36)$$

Now multiplying both sides of equation (29) by $\hbar_z^*$, integrating the resulting equation over the vertical range of z a suitable number of times and making use of relevant boundary conditions (32), we have

$$\int (|D\hbar_z|^2 + a^2|\hbar_z|^2 + [p\sigma_1/\sigma]|\hbar_z|^2)\mathrm{d}z = -\int D\hbar_z^* W\mathrm{d}z. \tag{37}$$

Equating the real parts from both sides of the equation (37), we have

$$\begin{aligned}\int (|D\hbar_z|^2 + a^2|\hbar_z|^2 + [p_r\sigma_1/\sigma]|\hbar_z|^2)\mathrm{d}z &= \text{Real part of } \left[-\int D\hbar_z^* W\mathrm{d}z\right] \\ \leq \left|-\int D\hbar_z^* W\mathrm{d}z\right| \leq \left|\int D\hbar_z^* W\mathrm{d}z\right| &\leq \int |D\hbar_z^*||W|\mathrm{d}z = \int |D\hbar_z||W|\mathrm{d}z \\ \leq (1/2)\left[\int |W|^2\mathrm{d}z + \int |D\hbar_z|^2\mathrm{d}z\right]. \end{aligned} \tag{38}$$

Since, $p_r \geq 0$, therefore from inequality (38), we have

$$\int (|D\hbar_z|^2 + a^2|\hbar_z|^2)\mathrm{d}z < \int |W|^2\mathrm{d}z - a^2\int |\hbar_z|^2\mathrm{d}z < \int |W|^2\mathrm{d}z. \tag{39}$$

Further, since, $W(0) = 0 = W(1)$, we have the Rayleigh–Ritz inequality [10]

$$\int |W|^2\mathrm{d}z \leq 1/\pi^2 \int |DW|^2\mathrm{d}z. \tag{40}$$

Multiplying both sides of eq. (30) by ζ^*, integrating the resulting equation over the vertical range of z a suitable number of times and making use of relevant boundary conditions (32), we have

$$\int (|D\zeta|^2 + a^2|\zeta|^2 + [p/\sigma]|\zeta|^2)\mathrm{d}z = \mathcal{Q}\int \zeta^* D\xi\mathrm{d}z + \int \zeta^* DW\mathrm{d}z. \tag{41}$$

Equating the real parts from both sides of the equation (41), we have

$$\begin{aligned}\int (|D\zeta|^2 + a^2|\zeta|^2 + p_r/\sigma|\zeta|^2)\mathrm{d}z &= \text{Real part of } \left(\mathcal{Q}\int \zeta^* D\xi\mathrm{d}z + \int \zeta^* DW\mathrm{d}z\right) \\ &= \text{Real part of } \left(\mathcal{Q}\int \zeta^* D\xi\mathrm{d}z - \int D\zeta^* W\,\mathrm{d}z\right). \end{aligned} \tag{42}$$

Also

$$\int \zeta^* D\xi\mathrm{d}z = -\int \xi D\zeta^*\mathrm{d}z). \tag{43}$$

Now, substituting the value of $D\zeta^*$ from eq. (31) in eq. (43), integrating the resulting equation over the vertical range of z a suitable number of times and using the relevant boundary conditions (32), we have

$$\int \zeta^* D\xi\mathrm{d}z = -\int (|D\xi|^2 + a^2|\xi|^2 + [p^*\sigma_1/\sigma]|\xi|^2)\mathrm{d}z.$$

Therefore,

$$\text{Real part of } \int \zeta^* D\xi\mathrm{d}z = -\int (|D\xi|^2 + a^2|\xi|^2 + [p_r\sigma_1/\sigma]|\xi|^2)\mathrm{d}z. \tag{44}$$

Since, $p_r \geq 0$, it follows from eq. (44) that

$$\text{Real part of } \int \zeta^* D\xi \mathrm{d}z < 0. \tag{45}$$

Consequently, eq. (42) implies that

$$\int (|D\zeta|^2 + a^2|\zeta|^2 + p_r/\sigma|\zeta|^2)\mathrm{d}z < \text{Real part of}\left(-\int D\zeta^* W\, \mathrm{d}z\right)$$
$$\leq \left|\int D\zeta^* W \mathrm{d}z\right| \leq \int |D\zeta^*||W|\mathrm{d}z = \int |D\zeta||W|\mathrm{d}z < \left[\int |W|^2 \mathrm{d}z\right]^{1/2}$$
$$\times \left[\int |D\zeta|^2 \mathrm{d}z\right]^{1/2} \text{(using Schwartz-inequality).} \tag{46}$$

Since, $p_r \geq 0$, inequality (46) implies that $\int |D\zeta|^2 \mathrm{d}z < \int |W|^2 \mathrm{d}z$ which upon using Rayleigh–Ritz inequality $\int |D\zeta|^2 \mathrm{d}z \geq \pi^2 \int |\zeta|^2 \mathrm{d}z$, gives

$$\int |\zeta|^2 \mathrm{d}z < 1/\pi^2 \int |W|^2 \mathrm{d}z. \tag{47}$$

Using inequalities (39) (40) and (47) in eq. (36), we get

$$[\pi^2/\sigma - \{\mathcal{Q}\sigma_1/\sigma + \mathcal{J}/\sigma\pi^2\}] \int |W|^2 \mathrm{d}z + a^2/\sigma \int |W|^2 \mathrm{d}z$$
$$+ \frac{\mathcal{Q}\mathcal{J}\sigma_1}{\sigma} \int |\xi|^2 \mathrm{d}z + 1/[\mathcal{R}(1 - T_0\alpha_2)]a^2 \int |\theta|^2 \mathrm{d}z < 0. \tag{48}$$

It clearly follows from inequality (48) that $(\mathcal{Q}\sigma_1/\pi^2 + \mathcal{J}/\pi^4) > 1$. This completes the proof.

The above lemma proves that the oscillatory modes $(p_i \neq 0)$ of the problem under consideration will be stable $(p_r < 0)$ when $(\mathcal{Q}\sigma_1/\pi^2 + \mathcal{J}/\pi^4) \leq 1$, or equivalently, a necessary condition for the existence of oscillatory modes which may be neutral $(p_r = 0)$ or unstable $(p_r > 0)$ for the problem under consideration is that $(\mathcal{Q}\sigma_1/\pi^2 + \mathcal{J}/\pi^4) > 1$.

Theorem 1. $(p, W, \theta, \hbar_z, \xi, \zeta), p = p_r + ip_i, p_i \neq 0, (\mathcal{Q}\sigma_1/\pi^2 + \mathcal{J}/\pi^4) > 1, \mathcal{R} < 0, T_0\alpha_2 > 1$, $\mathcal{Q} > 0$, $\mathcal{J} > 0$ *is a solution of eqs* (27)–(32), *and* $\sigma_1 \geq 1$, $\mathcal{R}[1 - T_0\alpha_2] \leq 27\pi^4(1 + 1/\sigma_1)/4$, *then* $p_r < 0$.

Proof. Proceeding exactly as in the proof of lemma, we get eqs (35) and (36). If permissible, let $p_r \geq 0$. Multiplying eq. (36) by p_r and adding the resulting equation to eq. (35) we have

$$\int (|D^2 W|^2 + 2a^2|DW|^2 + a^4|W|^2)\mathrm{d}z$$
$$+ 2p_r/\sigma\left[\int (|DW|^2 + a^2|W|^2)\mathrm{d}z + \mathcal{Q}\mathcal{J}\sigma_1 \int |\xi|^2 \mathrm{d}z\right]$$
$$+ \mathcal{J} \int (|D\zeta|^2 + a^2|\zeta|^2)\mathrm{d}z + \mathcal{Q} \int (|D^2\hbar_z|^2 + 2a^2|D\hbar_z|^2 + a^4|\hbar_z|^2)\mathrm{d}z$$
$$+ \mathcal{Q}\mathcal{J} \int (|D\xi|^2 + a^2|\xi|^2)\mathrm{d}z = 1/[\mathcal{R}(1 - T_0\alpha_2)]a^2 \int (|D\theta|^2 + a^2|\theta|^2)\mathrm{d}z. \tag{49}$$

Multiplying eq. (28) by its complex conjugate and integrating over the range of z by parts an appropriate number of times, using the boundary conditions (32) and equating the real parts of the resulting equation, we obtain

$$\int (|D^2\theta|^2 + 2a^2|D\theta|^2 + a^4|\theta|^2)\mathrm{d}z + 2p_r \int (|D\theta|^2 + a^2|\theta|^2)\mathrm{d}z \times |p|^2 \int |\theta|^2\mathrm{d}z = [\mathcal{R}(1 - T_0\alpha_2)]^2 a^4 \int |W|^2\mathrm{d}z. \tag{50}$$

Since, $p_r \geq 0$, therefore eq. (50) implies that

$$\int (|D^2\theta|^2 + 2a^2|D\theta|^2 + a^4|\theta|^2)\mathrm{d}z = \int |(D^2 - a^2)\theta|^2\mathrm{d}z < [\mathcal{R}(1 - T_0\alpha_2)]^2 a^4 \int |W|^2\mathrm{d}z. \tag{51}$$

Further

$$\int (|D\theta|^2 + a^2|\theta|^2)\mathrm{d}z = \left|-\int \theta^*(D^2 - a^2)\theta\mathrm{d}z\right| \leq \left|\int \theta^*(D^2 - a^2)\theta\mathrm{d}z\right| \leq \int |\theta||(D^2 - a^2)\theta|\mathrm{d}z \leq \left[\int |\theta|^2\mathrm{d}z\right]^{1/2} \times \left[\int |(D^2 - a^2)\theta|^2\mathrm{d}z\right]^{1/2} \text{(using Schwartz-inequality)} \tag{52}$$

and

$$\int |D\theta|^2\mathrm{d}z = \left|-\int \theta^* D^2\theta\mathrm{d}z\right| \leq \left|\int \theta^* D^2\theta\mathrm{d}z\right| \leq \int |\theta^*||D^2\theta|\mathrm{d}z = \int |\theta||D^2\theta|\mathrm{d}z \leq \left[\int |\theta|^2\mathrm{d}z\right]^{1/2} \times \left[\int |D^2\theta|^2\mathrm{d}z\right]^{1/2} \text{(using Schwartz-inequality).} \tag{53}$$

Since, $\theta(0) = 0 = \theta(1)$, using Rayleigh–Ritz inequality, we have

$$\pi^2 \int |\theta|^2\mathrm{d}z \leq \int |D\theta|^2\mathrm{d}z. \tag{54}$$

Using inequality (53), inequality (52) implies that

$$\pi^4 \int |\theta|^2\mathrm{d}z \leq \int |D^2\theta|^2\mathrm{d}z. \tag{55}$$

Now combining inequalities (53) and (54), we have

$$\int (|D^2\theta|^2 + 2a^2|D\theta|^2 + a^4|\theta|^2)\mathrm{d}z \geq \int (\pi^4|\theta|^2 + 2a^2\pi^2|\theta|^2 + a^4|\theta|^2)\mathrm{d}z = (\pi^2 + a^2)^2 \int |\theta|^2\mathrm{d}z, \tag{56}$$

which combined with inequality (51) yields the inequality

$$(\pi^2 + a^2)^2 \int (|\theta|^2)\mathrm{d}z < [\mathcal{R}(1 - T_0\alpha_2)]^2 a^4 \int |W|^2\mathrm{d}z. \tag{57}$$

Hence, inequality (52) with the help of inequalities (51) and (57) gives

$$\int (|D\theta|^2 + a^2|\theta|^2)\mathrm{d}z < \mathcal{R}[(1 - T_0\alpha_2)]^2 a^4/(\pi^2 + a^2) \int |W|^2\mathrm{d}z. \tag{58}$$

Further, eq. (36) implies that

$$\mathcal{Q}\sigma_1 \int (|D\hbar_z|^2 + a^2|\hbar_z|^2)\mathrm{d}z + \mathcal{J} \int |\zeta|^2\mathrm{d}z > \int (|DW|^2 + a^2|W|^2)\mathrm{d}z$$
$$> (\pi^2 + a^2) \int |W|^2\mathrm{d}z$$

or

$$\mathcal{Q} \int (|D\hbar_z|^2 + a^2|\hbar_z|^2)\mathrm{d}z > (\pi^2 + a^2)/\sigma_1 \int |W|^2)\mathrm{d}z - \mathcal{J}/\sigma_1 \int |\zeta|^2\mathrm{d}z. \tag{59}$$

Also, we have the following inequalities which are derived in a manner analogous to the derivations of inequalities (54)–(56) (since $W(0) = 0 = W(1)$, $\hbar_z(0) = 0 = \hbar_z(1)$ and $\zeta(0) = 0 = \zeta(1)$):

$$\int (|D^2W|^2 + 2a^2|DW|^2 + a^4|W|^2)\mathrm{d}z \geq (\pi^2 + a^2)^2 \int |W|^2\mathrm{d}z,$$
$$\int (|D^2\hbar_z|^2 + 2a^2|D\hbar_z|^2 + a^4|\hbar_z|^2)\mathrm{d}z \geq (\pi^2 + a^2) \int (|D\hbar_z|^2 + a^2|\hbar_z|^2)\mathrm{d}z$$

and

$$\int (|D\zeta|^2 + a^2|\zeta|^2)\mathrm{d}z \geq (\pi^2 + a^2) \int |\zeta|^2\mathrm{d}z. \tag{60}$$

Equation (49) upon using inequalities (58)–(60) yields the following inequality

$$[(\pi^2 + a^2)^3/a^2 + (\pi^2 + a^2)^3/a^2\sigma_1 - [\mathcal{R}(1 - T_0\alpha_2)]] \int |W|^2\mathrm{d}z$$
$$+ \mathcal{J}(\pi^2 + a^2)^2(1 - 1/\sigma_1)/a^2 \int |\zeta|^2\mathrm{d}z + 2p_r(\pi^2 + a^2)/a^2\sigma$$
$$\times \left[\int (|DW|^2 + a^2|W|^2)\mathrm{d}z + \mathcal{QJ}\sigma_1 \int |\xi|^2\mathrm{d}z\right]$$
$$+ \mathcal{QJ}(\pi^2 + a^2)/a^2 \int (|D\xi|^2 + a^2|\xi|^2)\mathrm{d}z < 0. \tag{61}$$

Now, since the minimum value of $(\pi^2 + a^2)^3/a^2$ with respect to a^2 is $27\pi^4/4$, therefore it follows from inequality (61) that

$$[27\pi^4(1 + 1/\sigma_1)/4 - [\mathcal{R}(1 - T_0\alpha_2)]] \int |W|^2\mathrm{d}z$$
$$+ \mathcal{J}(\pi^2 + a^2)^2(1 - 1/\sigma_1)/a^2 \int |\zeta|^2\mathrm{d}z$$
$$+ 2p_r(\pi^2 + a^2)/a^2\sigma\left[\int (|DW|^2 + a^2|W|^2)\mathrm{d}z + \mathcal{QJ}\sigma_1 \int |\xi|^2\mathrm{d}z\right]$$
$$+ \mathcal{QJ}(\pi^2 + a^2)/a^2 \int (|D\xi|^2 + a^2|\xi|^2)\mathrm{d}z < 0 \tag{62}$$

which clearly is incompatible with the hypothesis of Theorem 1. Hence, if $\sigma_1 \geq 1$ and $[\mathcal{R}(1 - T_0\alpha_2)] \leq 27\pi^4(1 + 1/\sigma_1)/4$ then $p_r < 0$. This completes the proof.

Theorem 2. *$(p, W, \theta, \hbar_z, \xi, \zeta)$, $p = p_r + ip_i$, $p_i \neq 0$, $(\mathcal{Q}\sigma_1/\pi^2 + \mathcal{J}/\pi^4) > 1$, $\mathcal{R} < 0$, $T_0\alpha_2 > 1$, $\mathcal{Q} > 0$, $\mathcal{J} > 0$ is a solution of eqs* (27)–(32), *and $\sigma_1 \leq 1$, $\mathcal{R}[1 - T_0\alpha_2] \leq 27\pi^4/2$, then $p_r < 0$.*

Proof. Proceeding exactly as in the proof of the Theorem 1, eq. (36) can be written as

$$\begin{aligned} \mathcal{J}\int |\zeta|^2 dz &> \int (|DW|^2 + a^2|W|^2)dz - \mathcal{Q}\sigma_1 \int (|D\hbar_z|^2 + a^2|\hbar_z|^2)dz \\ &> (\pi^2 + a^2)\int |W|^2 dz - \mathcal{Q}\sigma_1 \int (|D\hbar_z|^2 + a^2|\hbar_z|^2)dz. \qquad (63) \end{aligned}$$

Now, using inequality (63) in place of inequality (59) and inequalities (58) and (60) in eq. (49), we have

$$\begin{aligned} &[(\pi^2 + a^2)^3/a^2 + (\pi^2 + a^2)^3/a^2 - [\mathcal{R}(1 - T_0\alpha_2)]]\int |W|^2 dz \\ &\quad + \mathcal{Q}(\pi^2 + a^2)^2(1 - \sigma_1/a^2)\int (|D\hbar_z|^2 + a^2|\hbar_z|^2)dz \\ &\quad + 2p_r(\pi^2 + a^2)/a^2\sigma\left[\int (|DW|^2 + a^2|W|^2)dz + \mathcal{Q}\mathcal{J}\sigma_1\int |\xi|^2 dz\right] \\ &\quad + \mathcal{Q}\mathcal{J}(\pi^2 + a^2)/a^2\int (|D\xi|^2 + a^2|\xi|^2)dz < 0, \qquad (64) \end{aligned}$$

which upon substituting the minimum value of $(\pi^2 + a^2)^3/a^2$ with respect to a^2, i.e. $27\pi^4/4$, reduces to

$$\begin{aligned} &[2(27\pi^4/4) - [\mathcal{R}(1 - T_0\alpha_2)]]\int |W|^2 dz \\ &\quad + \mathcal{Q}(\pi^2 + a^2)^2(1 - \sigma_1)/a^2\int (|D\hbar_z|^2 + a^2|\hbar_z|^2)dz \\ &\quad + 2p_r(\pi^2 + a^2)/a^2\sigma\left[\int (|DW|^2 + a^2|W|^2)dz + \mathcal{Q}\mathcal{J}\sigma_1\int |\xi|^2 dz\right] \\ &\quad + \mathcal{Q}\mathcal{J}(\pi^2 + a^2)/a^2\int (|D\xi|^2 + a^2|\xi|^2)dz < 0. \qquad (65) \end{aligned}$$

Inequality (65) is clearly incompatible with the hypothesis of Theorem 2. Hence, if $\sigma_1 \leq 1$ and $\mathcal{R}[1 - T_0\alpha_2] \leq 27\pi^4/2$, then $p_r < 0$. This completes the proof.

4. Conclusion

Theorems 1 and 2 show that the oscillatory modes of the rotatory magnetohydrodynamic modified Bénard convection problem can be suppressed by the simultaneous application of a uniform vertical rotation and a uniform vertical magnetic field in the parameter regime $\mathcal{R}[1 - T_0\alpha_2] \leq 27\pi^4(1 + 1/\sigma_1)/4$ or $27\pi^4/2$ according as $\sigma_1 \geq 1$ or $\sigma_1 \leq 1$.

Acknowledgement

Thanks are extended to the learned referee for his perspicacious comments on an earlier version of the paper.

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