Proc. Indian Acad. Sci. (Math. Sci.), Vol. 110, No. 3, August 2000, pp. 305–314.

C^2-rational cubic spline involving tension parameters

M SHRIVASTAVA and J JOSEPH

Department of Mathematics and Computer Science, R D University, Jabalpur 482 001, India

MS received 8 September 1998; revised 15 January 2000

Abstract. In the present paper, C^1-piecewise rational cubic spline function involving tension parameters is considered which produces a monotonic interpolant to a given monotonic data set. It is observed that under certain conditions the interpolant preserves the convexity property of the data set. The existence and uniqueness of a C^2-rational cubic spline interpolant are established. The error analysis of the spline interpolant is also given.

Keywords. Interpolation; rational spline; tension parameter; monotonicity; convexity; continuity.

1. Introduction

Interpolation techniques play a very important role in obtaining solutions of various problems that arise in many areas of scientific computation. Generally interpolant is preferred which preserves some of the characteristics of the function to be interpolated. In order to tackle such situations, a variety of shape preserving interpolation methods have been discussed in the literature for interpolating given sets of monotonic and convex data.

Shape preserving cubic spline interpolants have been obtained by Fritsch and Carlson in [4]. Gregory and Delbourgo, being motivated by the work of Fritsch and Carlson have introduced piecewise rational interpolatory splines (cf. [5]). Using rational functions, problem of obtaining a convex interpolant to convex data has been investigated by Delbourgo (cf. [3]). Shape preserving properties of the rational (cubic/quadratic) spline interpolant have been studied in [2]. In order to control the shape of the rational interpolant in a more efficient manner, Delbourgo and Gregory (cf. [2]) have introduced tension parameters in the definition of the C^1-rational cubic spline function. The tension parameters have been so chosen that it provides the desired geometric shape to the rational interpolant. For a designer these tension parameters act as intuitive tools for manipulating the shape of the curve. A C^2-piecewise rational (cubic/cubic) Bézier curve involving two tension parameters which is used to interpolate the given monotonic data is described in [6]. Shape preserving (cubic/linear) rational interpolants to monotone and convex functions have been studied in [7].

In the present paper, a rational spline interpolant is constructed which matches given data values and at the same time preserves certain geometric features namely the monotonicity and convexity properties of the function to be interpolated. In fact, we obtain a rational (cubic/linear) spline interpolant involving two tension parameters when values of the function and its first derivative are given at the knots.

Introducing two parameters r_i and t_i, we construct a C^1-rational (cubic/linear) spline interpolant and obtain its error bounds in § 2. In § 3, we study the convexity and monotonicity of this rational spline interpolant. Existence of a unique C^2-rational spline interpolant has been discussed in § 4. In § 5, we consider a numerical example and discuss the impact of variation of parameters r_i and t_i on the shape of the interpolant. Some remarks are given in § 6.

2. The rational spline interpolant

Let $P = \{x_i\}_{i=1}^{n}$ where $a = x_1 < x_2 < \cdots < x_n = b$, be a partition of the interval $[a, b]$, let f_i, $i = 1, \ldots, n$ be the function values at the data points. We set

$$h_i = x_{i+1} - x_i, \quad \Delta_i = (f_{i+1} - f_i)/h_i \tag{2.1}$$

and

$$\theta = (x - x_i)/h_i. \tag{2.2}$$

Further, we set

$$s(x) = P_i(\theta)/Q_i(\theta), \tag{2.3}$$

where

$$\begin{aligned} P_i(\theta) =& r_i f_i (1-\theta)^3 + \{(2r_i + t_i) f_i + r_i h_i d_i\}\theta(1-\theta)^2 \\ &+ \{(r_i + 2t_i) f_{i+1} - t_i h_i d_{i+1}\}\theta^2(1-\theta) + t_i f_{i+1}\theta^3 \end{aligned} \tag{2.4}$$

and

$$Q_i(\theta) = r_i + (t_i - r_i)\theta. \tag{2.5}$$

Here we choose parameters r_i and t_i in such a manner that

$$r_i, t_i > 0 \quad \text{and} \quad t_i > r_i \tag{2.6}$$

which ensures a strictly positive denominator in the rational spline. It is easily seen that

$$\begin{aligned} s^{(1)}(x) =& \{[r_i + (t_i - r_i)\theta]\{-3r_i f_i (1-\theta)^2 + \{(2r_i + t_i) f_i \\ &+ r_i h_i d_i\}[(1-\theta)^2 - 2\theta(1-\theta)] + \{(r_i + 2t_i) f_{i+1} \\ &- t_i h_i d_{i+1}\}[2\theta(1-\theta) - \theta^2] + 3t_i f_{i+1}\theta^2\} \\ &+ (r_i - t_i)[r_i f_i (1-\theta)^3 + \{(2r_i + t_i) f_i + r_i h_i d_i\}\theta(1-\theta)^2 \\ &+ \{(r_i + 2t_i) f_{i+1} - t_i h_i d_{i+1}\}\theta^2(1-\theta) \\ &+ t_i f_{i+1}\theta^3]\}/h_i[r_i + (t_i - r_i)\theta]^2. \end{aligned} \tag{2.7}$$

We observe that

$$\begin{aligned} &s(x_i) = f_i, \qquad s(x_{i+1}) = f_{i+1}, \\ &s^{(1)}(x_i) = d_i, \quad s^{(1)}(x_{i+1}) = d_{i+1}, \end{aligned} \tag{2.8}$$

where d_i's denote the derivative values at the knots x_i. These derivative parameters are usually not given and can be determined by using the methods as discussed in [1].

Let $e = s - f$ denote the error function. We prove the following theorem.

Theorem 2.1. *Given a function f, let s be its piecewise rational (cubic/linear) spline interpolant satisfying the interpolatory conditions* (2.8). *Then for* $x \in [x_i, x_{i+1}]$, *the following hold.*

(a) *If* $f \in C^4[a,b]$, *then*

$$\| e \| \le \frac{h_i}{4\underline{r}} \max\{\bar{r}|f_i^{(1)} - d_i|, \bar{t}|f_{i+1}^{(1)} - d_{i+1}|\} + \frac{\bar{t}}{384\underline{r}}\{h_i^4 \| f^{(4)} \| + 4h_i^3 \| f^{(3)} \| .$$

(b) *If* $f \in C^1[a,b]$, *then*

$$\| e \| \le \frac{13}{9\underline{r}}(\bar{r} + \bar{t})\omega(f,h) + \frac{\bar{t}}{4\underline{r}} h_i \max\{|d_i|, |d_{i+1}|\},$$

where $\bar{r} = \max r_i$, $\bar{t} = \max t_i$, $\underline{r} = \min r_i$ *and* $\| \cdot \|$ *denotes the uniform norm on* $[x_i, x_{i+1}]$.

Proof. We set $x(\theta) = (x_i + \theta h_i)$ and $f(x) = F_i(\theta), 0 \le \theta \le 1$ in $[x_i, x_{i+1}]$. Then we observe that, in $[x_i, x_{i+1}]$,

$$e(x) = f(x) - s(x) = F_i(\theta) - P_i(\theta)/Q_i(\theta). \tag{2.9}$$

Let us take

$$\begin{aligned} R_i(\theta) &= r_i f_i (1-\theta)^3 + [(2r_i + t_i) f_i + r_i h_i f_i^{(1)}]\theta(1-\theta)^2 \\ &\quad + [(r_i + 2t_i) f_{i+1} - t_i h_i f_{i+1}^{(1)}]\theta^2(1-\theta) + t_i f_{i+1}\theta^3, \end{aligned} \tag{2.10}$$

and

$$s_i(\theta) = F_i(\theta) Q_i(\theta) = f(x_i + \theta h_i)[r_i + (t_i - r_i)\theta]. \tag{2.11}$$

We observe that

$$\begin{aligned} |F_i(\theta) - P_i(\theta)/Q_i(\theta)| &\le [|F_i(\theta)Q_i(\theta) - R_i(\theta)| + |R_i(\theta) - P_i(\theta)|]/|Q_i(\theta)| \\ &= [|s_i(\theta) - R_i(\theta)| + |R_i(\theta) - P_i(\theta)|]/|Q_i(\theta)|. \end{aligned} \tag{2.12}$$

It can be verified that

$$\begin{aligned} &R_i(0) = s_i(0) = r_i f_i; \qquad R_i^{(1)}(0) = s_i^{(1)}(0) = (t_i - r_i) f_i + r_i h_i f_i^{(1)}; \\ &R_i(1) = s_i(1) = t_i f_{i+1}; \quad R_i^{(1)}(1) = s_i^{(1)}(1) = (t_i - r_i) f_{i+1} + t_i h_i f_{i+1}^{(1)}. \end{aligned}$$

Thus $R_i(\theta)$ is the cubic Hermite interpolant to $s_i(\theta)$ on $0 \le \theta \le 1$. Now bounding Cauchy remainder of $s_i(\theta)$, we get

$$\begin{aligned} |s_i(\theta) - R_i(\theta)| &\le \frac{1}{384} \max_{0\le\theta\le1} \left| \frac{\mathrm{d}^4}{\mathrm{d}\theta^4} s_i(\theta) \right| \\ &= \frac{1}{384} \max_{0\le\theta\le1} |F_i^{(4)}(\theta) Q_i(\theta) + 4F_i^{(3)}(\theta) Q_i^{(1)}(\theta)|, \end{aligned}$$

since $Q_i(\theta)$ is linear, so that $Q_i^{(2)}(\theta) = Q_i^{(3)}(\theta) = 0$. Now

$$\begin{aligned} |Q_i(\theta)| &= |r_i + (t_i - r_i)\theta| \le \max_i \{t_i\} = \bar{t}, \\ |Q_i^{(1)}(\theta)| &= |t_i - r_i| \le \max_i (t_i - r_i) \le \bar{t}, \end{aligned}$$

and

$$|F_i^{(j)}(\theta)| \leq h_i^j \parallel f^{(j)} \parallel .$$

Hence

$$|s_i(\theta) - R_i(\theta)| \leq \frac{\bar{t}}{384}\{h_i^4 \parallel f^{(4)} \parallel + 4h_i^3 \parallel f^{(3)} \parallel\}. \tag{2.13}$$

Also

$$\begin{aligned}|R_i(\theta) - P_i(\theta)| &\leq h_i\theta(1-\theta)\{|r_i(1-\theta)(f_i^{(1)} - d_i)| + |t_i\theta(d_{i+1} - f_{i+1}^{(1)}|\} \\ &\leq \frac{h_i}{4}\max\{\bar{r}|f_i^{(1)} - d_i|, \bar{t}|f_{i+1}^{(1)} - d_{i+1}|\}. \end{aligned} \tag{2.14}$$

Considering the denominator in (2.3) we find that

$$\min|Q_i(\theta)| = \min\{|r_i + (t_i - r_i)\theta|\} > \underline{r}. \tag{2.15}$$

Combining (2.13), (2.14) and (2.15) in (2.12) and (2.9) we get

$$\begin{aligned}\parallel e \parallel &\leq \frac{h_i}{4\underline{r}}\max\{\bar{r}|f_i^{(1)} - d_i|, \bar{t}|f_{i+1}^{(1)} - d_{i+1}|\} \\ &\quad + \frac{\bar{t}}{384\underline{r}}\{h_i^4 \parallel f^{(4)} \parallel + 4h_i^3 \parallel f^{(3)} \parallel\}.\end{aligned}$$

This completes the proof of (a).

Further, from (2.3)–(2.5) we observe that

$$s(x) - f(x) = [P_i(\theta) - (r_i + (t_i - r_i)\theta)f(x)]/[r_i + (t_i - r_i)\theta],$$

so that

$$\begin{aligned}|s(x) - f(x)| &\leq [|r_i(1-\theta)^3 + (2r_i + t_i)\theta(1-\theta)^2||f_i - f(x)| \\ &\quad + |(2t_i + r_i)\theta^2(1-\theta) + t_i\theta^3||f_{i+1} - f(x)| \\ &\quad + h_i\theta(1-\theta)|r_i d_i(1-\theta) - t_i d_{i+1}\theta|]/\min\{|r_i + (t_i - r_i)\theta|\} \\ &\leq \frac{13}{9\underline{r}}(\bar{r} + \bar{t})\omega(f, h) + \frac{\bar{t}}{4\underline{r}}h_i \max\{|d_i|, |d_{i+1}|\}.\end{aligned}$$

Thus

$$\|e\| \leq \frac{13}{9\underline{r}}(\bar{r} + \bar{t})\omega(f, h) + \frac{\bar{t}}{4\underline{r}}h_i \max\{|d_i|, |d_{i+1}|\}, \tag{2.16}$$

which proves (b).

This completes the proof of Theorem 2.1.

3. Convexity and monotonicity of the interpolant

We shall now investigate the monotonicity and convexity preserving properties of the rational (cubic/linear) interpolant to a given monotonic or convex data.

3.1. *Monotonicity*

Let f be a monotonic increasing function in $[a, b]$ so that

$$f_1 \leq f_2 \leq \cdots \leq f_n, \text{ or equivalently } \Delta_i \geq 0. \tag{3.1}$$

We choose the derivative values d_i such that

$$d_i \geq 0, \quad i = 1, \ldots, n. \tag{3.2}$$

We observe that $s(x)$ is monotonic increasing if and only if for $x \in [a, b]$,

$$s^{(1)}(x) \geq 0. \tag{3.3}$$

A simple manipulation in (2.7) shows that for $x \in [x_i, x_{i+1}]$,

$$s^{(1)}(x) = [r_i^2 d_i (1-\theta)^3 + X_i \theta (1-\theta)^2 + Y_i \theta^2 (1-\theta) + t_i^2 d_{i+1} \theta^3]/[r_i + (t_i - r_i)\theta]^2, \tag{3.4}$$

where

$$X_i = 2r_i^2 \Delta_i + 4r_i t_i \Delta_i - r_i^2 d_i - 2r_i t_i d_{i+1}$$

and

$$Y_i = 2t_i^2 \Delta_i - 2r_i t_i d_i - t_i^2 d_{i+1} + 4r_i t_i \Delta_i.$$

We observe that the denominator of rational function $s^1(x)$ given in (2.7) is positive. Therefore considering the numerator in (3.4), we find that $s^{(1)}(x)$ is non-negative if $X_i \geq 0, Y_i \geq 0$ provided (3.2) is also satisfied. We observe that the sufficient condition that $X_i \geq 0$ and $Y_i \geq 0$ is

$$\frac{r_i}{t_i} \geq \frac{(d_{i+1} - \Delta_i)}{(\Delta_i - d_i)}. \tag{3.5}$$

Therefore $s^{(1)}(x)$ is non-negative if (3.5) holds.

We have thus proved the following theorem:

Theorem 3.1. *Given a monotonic increasing set of data satisfying* (3.1) *and the derivative values satisfying* (3.2), *there exists a monotone rational* (*cubic/linear*) *spline interpolant* $s \in C^1[a, b]$ *involving the tension parameters* r_i *and* t_i *which satisfies the interpolatory conditions* (2.8) *provided* (3.5) *holds.*

3.2. *Convexity*

Suppose the given data set is strictly convex then

$$\Delta_1 < \Delta_2 < \cdots < \Delta_{n-1}.$$

We choose derivative values $d_i \geq 0$ to be such that

$$0 \leq d_1 < \Delta_1 < d_2 < \cdots < \Delta_{i-1} < d_i < \Delta_i < \cdots < d_n. \tag{3.6}$$

A simple calculation shows that for $x \in [x_i, x_{i+1}]$, from (2.7) we get

$$s^{(2)}(x) = [A_{2i}(1-\theta)^3 + B_{2i}\theta(1-\theta)^2 + C_{2i}\theta^2(1-\theta) + D_{2i}\theta^3]/h_i[r_i + (t_i - r_i)\theta]^3, \tag{3.7}$$

where

$$\begin{aligned}
A_{2i} &= 2r_i^3\Delta_i + 4r_i^2 t_i\Delta_i - 2r_i^3 d_i - 2r_i^2 t_i d_{i+1} - 2r_i^2 t_i d_i,\\
B_{2i} &= 6r_i^2 t_i(\Delta_i - d_i),\\
C_{2i} &= 6r_i t_i^2(d_{i+1} - \Delta_i) \text{ and}\\
D_{2i} &= 2t_i^3(d_{i+1} - \Delta_i) + 2r_i t_i^2(d_{i+1} - \Delta_i) + 2r_i t_i^2(-\Delta_i + d_i).
\end{aligned} \tag{3.8}$$

We observe that $s^{(2)}(x)$ is non-negative if each of A_{2i}, B_{2i}, C_{2i} and D_{2i} is non-negative. Since we are assuming that (3.6) holds, B_{2i} and C_{2i} are automatically positive. Thus the sufficient condition for the interpolant $s(x)$ to be convex is that $A_{2i} \geq 0$ and $D_{2i} \geq 0$.

Now $A_{2i} \geq 0$ if $2r_i^3(\Delta_i - d_i) - 2r_i^2 t_i(d_{i+1} - \Delta_i) + 2r_i^2 t_i(\Delta_i - d_i) \geq 0$,

$$\text{i.e., if } \frac{r_i}{t_i} \geq \frac{(d_{i+1} - \Delta_i)}{(\Delta_i - d_i)} \tag{A}$$

and $D_{2i} \geq 0$ if $2t_i^3(d_{i+1} - \Delta_i) + 2r_i t_i^2(-\Delta_i + d_i) \geq 0$,

$$\text{i.e., if } \frac{r_i}{t_i} \leq \frac{(d_{i+1} - \Delta_i)}{(\Delta_i - d_i)}. \tag{B}$$

Therefore the spline interpolant is convex if

$$\frac{r_i}{t_i} = \frac{(d_{i+1} - \Delta_i)}{(\Delta_i - d_i)}, \tag{3.9}$$

provided (2.6) and (3.6) hold.

Thus the spline interpolant is convex if (3.9) together with (2.6) and (3.6) holds.

We have thus proved the following theorem.

Theorem 3.2. *For a given set of strictly convex data, a convex rational* (*cubic/linear*) *spline interpolant $s \in C^1[a,b]$ involving the parameters r_i and t_i exists which satisfies the interpolatory conditions* (2.8), *with the derivative parameters d_i's satisfying* (3.6) *provided* (2.6) *and* (3.9) *hold.*

4. C^2-rational spline interpolant

For a given set of data points $\{(x_i, f_i)\}_{i=1}^n$, let s defined in § 2, represent a C^2-rational (cubic/linear) spline interpolant.

For $x \in [x_i, x_{i+1}]$, we have

$$\begin{aligned}
s^{(2)}(x) = {}& [A_{2i}(1-\theta)^3 + B_{2i}\theta(1-\theta)^2 + C_{2i}\theta^2(1-\theta)\\
&+ D_{2i}\theta^3]/h_i[r_i + (t_i - r_i)\theta]^3,
\end{aligned}$$

where A_{2i}, B_{2i}, C_{2i} and D_{2i} are given by (3.8). It is easy to see that

$$\begin{aligned}
s^{(2)}(x_{i-}) &= D_{2i-1}/h_{i-1}t_{i-1}^3\\
&= [2r_{i-1}t_{i-1}^2(d_i - 2\Delta_{i-1} + d_{i-1}) + 2t_{i-1}^3(d_i - \Delta_{i-1})]/h_{i-1}t_{i-1}^3\\
&= 2[r_{i-1}d_{i-1} - (2r_{i-1} + t_{i-1})\Delta_{i-1} + (r_{i-1} + t_{i-1})d_i]/h_{i-1}t_{i-1}
\end{aligned}$$

and

$$s^{(2)}(x_{i+}) = A_{2i}/h_i r_i^3$$
$$= 2[(r_i + 2t_i)\Delta_i - t_i d_{i+1} - (r_i + t_i)d_i]/h_i r_i.$$

Therefore continuity of $s^{(2)}$ gives that

$$h_i r_i r_{i-1} d_{i-1}[h_{i-1}t_{i-1}(r_i + t_i) + h_i r_i(r_{i-1} + t_{i-1})]d_i$$
$$+ h_{i-1}t_i t_{i-1}d_{i+1} = h_i r_i(2r_{i-1} + t_{i-1})\Delta_{i-1} + h_{i-1}t_{i-1}(r_i + 2t_i)\Delta_i, \quad (4.1)$$

where the Δ_i's, $i = 1, \ldots, n$ are given by (2.1).

We observe that if d_1 and d_n are the given quantities then (4.1) represents a system of $(n-2)$ equations in $(n-2)$ unknowns, namely $d_2, \ldots, d_{n-1}$. Assume that $r_i \geq r > 0$ and $t_i \geq t > 0$, $i = 1, \ldots, n-1$, then it is easy to see that the coefficients of d_{i-1}, d_i and d_{i+1} are all positive and the excess of coefficient of d_i over the sum of those of d_{i-1} and d_{i+1} is $(h_{i-1} + h_i)r_i t_{i-1}$ which is clearly positive. Therefore the coefficient matrix of the system of equations (4.1) is diagonally dominant and is thus invertible. Therefore a unique solution for the system of equations (4.1) exists.

This establishes the following theorem.

Theorem 4.1. *Let f_i, $i = 1, \ldots, n$ be the given data-values. Then for the derivative parameters given at the end points namely d_i and d_n, there exists a unique C^2-piecewise rational (cubic/linear) spline interpolant satisfying the interpolatory conditions* (2.8) *provided that* (2.6) *holds.*

5. Numerical example

In this section, we construct a C^2-rational (cubic/linear) spline interpolant which involves two tension parameters r_i and t_i for a given set of function values. In view of (4.1), suppose that the function values f_i's at the knots and the derivative values at the end points namely d_1 and d_n are given. The derivative parameters d_i's, $i = 2, \ldots, n-1$ are unknown and these are determined by applying the C^2-continuity condition.

For $n = 14$; $\theta = j/q$, $q = 50$ and $j = 0, \ldots, q$; $h_i = h1 = 1$; $r_i = r$, $t_i = t$ such that $t > r$; $d_1 = d_n = 0$ and for the data values given as:

$$\{(x_i, f_i)\}_{i=1}^{14} = \{(122, 128); (122, 156); (150, 184); (178, 184);$$
$$(206, 156); (206, 128); (178, 100); (150, 100);$$
$$(122, 72); (122, 44); (150, 16); (178, 16);$$
$$(206, 44); (206, 72)\},$$

we obtain the C^2-rational cubic spline interpolant. Thus for different values of the tension parameters r and t, corresponding different graphs of the spline interpolant are obtained. From figure 1, we observe that as the values of the tension parameters r and t both are increased, we get considerable smooth curves of the C^2-rational (cubic/linear) spline interpolant. In fact, we observe the following:

(i) If only one parameter is taken into consideration as discussed in [6], i.e., if $r = 1$ and we have only one effective parameter t, then from figure 2 it can be seen that the curves obtained for different values of t are not smooth and as the value of t increases we get still worse curves.

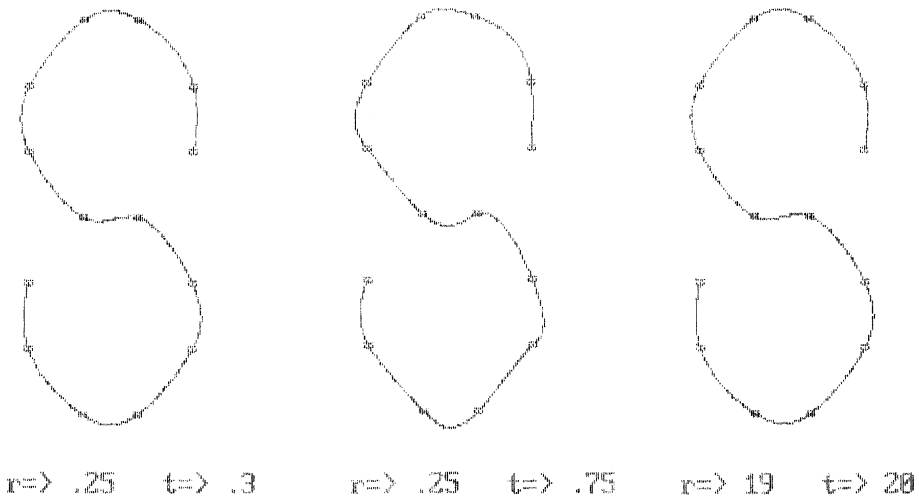

Figure 1.

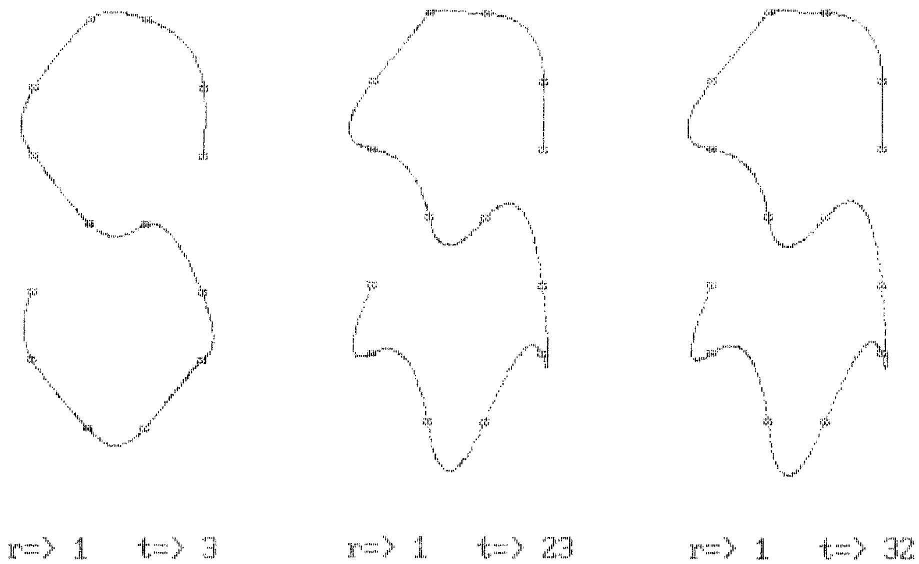

Figure 2.

(ii) From figures 3 and 4, we observe that when r and t are relatively different then the curves are not smooth.

(iii) When r and t are nearly equal and relatively larger in values then we have sufficiently smooth curves (see figures 1 and 4).

6. Remarks

6.1. The parameters r_i and t_i cannot be both zero since it leads to a trivial situation. For the choice $r_i = t_i$, the rational (cubic/linear) spline interpolant clearly reduces to usual cubic spline interpolating function values at the knots. This spline interpolant has been

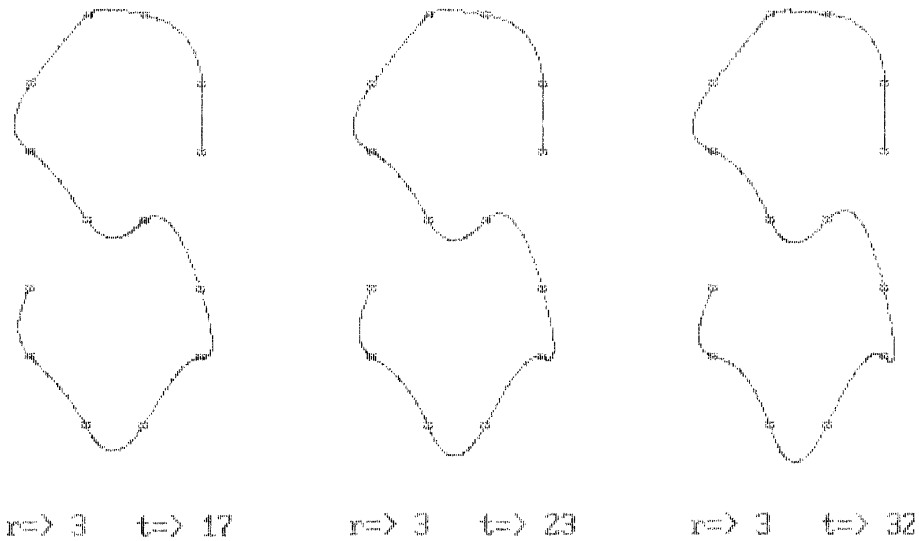

Figure 3.

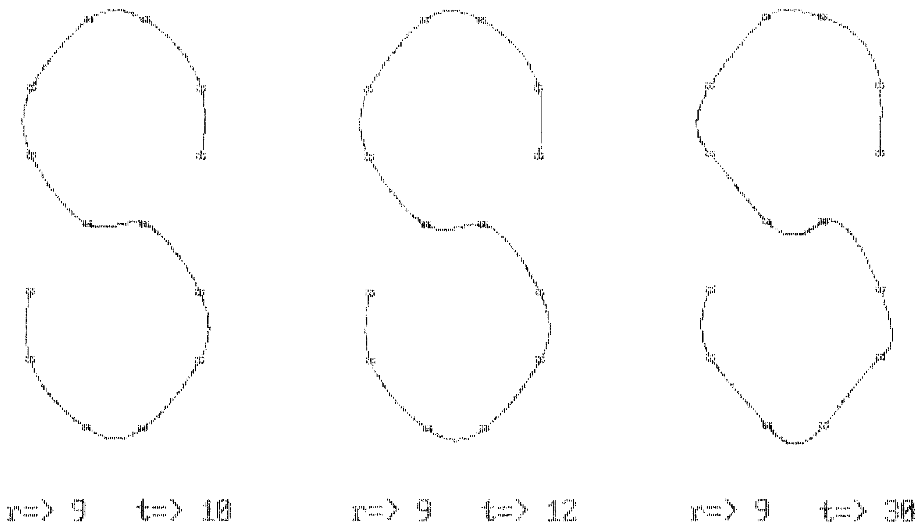

Figure 4.

studied in [4]. When either of the two parameters is zero, then the rational (cubic/linear) spline interpolant reduces to a rational (cubic/linear) interpolant which has been discussed in [7].

6.2. If the derivative parameters satisfy the inequality (3.6), then the C^1-rational (cubic/linear) spline interpolant is monotonic as well as convex, provided (3.9) holds.

6.3. We observe that as $r_i \to t_i$ and $h_i \to 0$, then (2.16) reduces to

$$\|e\| \leq \frac{\bar{t}}{4t} h_i \max\{|d_i|, d_{i+1}|\}.$$

References

[1] Delbourgo R and Gregory J A, The determination of derivative parameters for a monotonic rational quadratic interpolant. *IMA J. Numer. Anal.* **5** (1985) 397–406

[2] Delbourgo R and Gregory J A, Shape preserving piecewise rational interpolation. *SIAM J. Sci. Stat. Comput.* **6** (1985) 967–975

[3] Delbourgo R, Shape preserving interpolation to convex data by rational functions with quadratic numerator and linear denominator. *IMA J. Numer. Anal.* **9** (1989) 123–136

[4] Fritsch F N and Carlson R E, Monotone piecewise cubic interpolation. *SIAM J. Numer. Anal.* **17** (1980) 238–246

[5] Gregory J A and Delbourgo R, Piecewise rational quadratic interpolation to monotone data, *IMA J. Numer. Anal.* **2** (1982) 123–130

[6] Ismail M K, Monotonicity preserving interpolation using C^2-rational cubic Bezier curves, *Mathematical Methods in CAGD II* (eds) T Lyche and L L Schumaker (1992) pp. 343–350

[7] Shrivastava M, Piecewise rational interpolation to monotonic and convex data. *J. Comp. Appl. Math.* (communicated)