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On the generalized Hankel–Clifford transformation of arbitrary order

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Abstract. Two generalized Hankel–Clifford integral transformations verifying a mixed Parseval relation are investigated on certain spaces of generalized functions for any real value of their orders $(\alpha - \beta)$.

Keywords. Generalized Hankel–Clifford transformation; Parseval relation; generalized functions (distributions).

1. Introduction

The conventional Hankel transformation defined by

$$h_\mu\{f(x)\}(y) = \int_0^\infty \sqrt{xy}\, J_\mu(xy) f(x) \mathrm{d}x \quad (0 < y < \infty) \tag{1}$$

was extended by [6] to certain generalized function of slow growth through a generalization of Parseval's equation. Later on [1] extended (1) to a class of generalized functions by the kernel method, which is a more natural extension of (1). In [3] the Hankel–Clifford transformations of order $\mu \geq 0$, defined by

$$(h_{1,\mu} f)(y) = y^\mu \int_0^\infty (xy)^{-\mu/2} J_\mu(2\sqrt{xy}) f(x) \mathrm{d}x \tag{2}$$

and

$$(h_{2,\mu} f)(\,) = \int_0^\infty x^\mu (xy)^{-\mu/2} J_\mu(2\sqrt{xy})(x) \mathrm{d}x \tag{3}$$

has been extended to a certain space of generalized functions. Recently the simple generalization of (2) and (3) called the generalized Hankel–Clifford transformation of order $\alpha - \beta \geq -1/2$ are defined by

$$F_1(y) = (h_{1,\alpha,\beta} f)(y) = y^{-\alpha-\beta} \int_0^\infty C_{\alpha,\beta}(xy) f(x) \mathrm{d}x, \tag{4}$$

$$F_2(y) = (h_{2,\alpha,\beta} f)(y) = \int_0^\infty x^{-\alpha-\beta} C_{\alpha,\beta}(xy) f(x) \mathrm{d}x, \tag{5}$$

where $C_{\alpha,\beta}(xy) = (xy)^{(\alpha+\beta)/2} J_{\alpha-\beta}(2\sqrt{xy})$ is a generalized Bessel–Clifford function of first kind of order $(\alpha - \beta)$, satisfying the differential equation $xy'' - (\alpha + \beta - 1)y' + (\alpha\beta x^{-1} + 1)y = 0$, which can be inverted by formulae

$$f(y) = (h^{-1}_{1,\alpha,\beta}F_1)(y) = y^{-\alpha-\beta}\int_0^\infty C_{\alpha,\beta}(xy)F_1(x)\mathrm{d}x, \tag{6}$$

$$f(y) = (h^{-1}_{2,\alpha,\beta}F_2)(y) = \int_0^\infty x^{-\alpha-\beta}C_{\alpha,\beta}(xy)F_2(x)\mathrm{d}x, \tag{7}$$

respectively. In symbols $h^{-1}_{1,\alpha,\beta} = h_{1,\alpha,\beta}$ and $h^{-1}_{2,\alpha,\beta} = h_{2,\alpha,\beta}$, if $\alpha - \beta \geq -1/2$. The distributional aspects of the transformations (6) and (7) have been discussed by the kernel method in [2].

In this paper the two transformations (4) and (5) are simultaneously investigated in certain spaces of distributions, by the procedure of the adjoint operator for which we exploit a Parseval equation involving (4) and (5) so that we define the first distributional transformation as the adjoint operator of the second one, and conversely for $(\alpha-\beta) \geq -1/2$ as well as for negative real values of order $(\alpha - \beta)$ similar to [4] and [5].

2. Preliminary results

We shall need the operational formulae given by Malgonde [2]

$$D^r[x^{-\alpha}C_{\alpha,\beta}(x)] = (-1)^r x^{-\alpha}C_{\alpha,\beta-r}(x), \tag{8}$$

$$D^r[x^{-(\beta-r)}C_{\alpha,\beta-r}(x)] = x^{-\beta}C_{\alpha,\beta}(x), \tag{9}$$

and the asymptotic behaviours

$$C_{\alpha,\beta}(x) = O(|x|^{\alpha}) \text{ as } x \to 0^+ \tag{10}$$

and

$$C_{\alpha,\beta}(x) = O(x^{(\alpha+\beta)/2-1/4}) \text{ as } x \to \infty. \tag{11}$$

Under certain assumptions, the Parseval equation for (4) given by

$$\int_0^\infty x^{\alpha+\beta}f(x)g(x)\mathrm{d}x = \int_0^\infty y^{\alpha+\beta}F_1(y)G_1(y)\mathrm{d}y \tag{12}$$

holds.

The corresponding Parseval equation for (5) takes the form

$$\int_0^\infty x^{-(\alpha+\beta)}f(x)g(x)\mathrm{d}x = \int_0^\infty y^{-(\alpha+\beta)}F_2(y)G_2(y)\mathrm{d}y. \tag{13}$$

Note that (4) and (5) are two variants of the Hankel transformation in the form considered by Tricomi. For $\alpha = 0$, $\beta = -\mu$ these reduce to Hankel–Clifford transformations given by (2) and (3). In the sequel I denotes the interval $0 < x < \infty$ and $L(I)$ represents the space of all functions $f(x)$ that are integrable Lebesgue on I. By invoking Fubini's theorem we can establish the following theorem.

Theorem 1. *Let $\alpha - \beta \geq -1/2$, suppose $x^\alpha f(x)$ and $y^\beta G_2(Y)$ belongs to $L(I)$. If $F_1(y) = (h_{1,\alpha,\beta}f)(y)$ and $g(x) = (h^{-1}_{2,\alpha,\beta}G_2(y))(x)$ then*

$$\int_0^\infty f(x)g(x)\mathrm{d}x = \int_0^\infty F_1(y)G_2(y)\mathrm{d}y. \tag{14}$$

Note that (14) does not involve any weight function, in contrast with expressions (12) and (13). Moreover, (14) relates both the Hankel–Clifford transforms (4) and (5). For this reason, (14) will be named as the mixed Parseval equation concerning generalized Hankel–Clifford transforms (4) and (5), and will play an important role throughout this paper.

Along this paper we follow the notation and terminology of Zemanian [6]. Thus $D(I)$, $E(I)$, $D'(I)$ and $E'(I)$ denote well known testing function spaces and their duals.

3. The testing function spaces $\mathbb{H}_{\beta}(I)$, $\mathbb{S}_{\alpha}(I)$ and their duals

Let β by any real number. $\mathbb{H}_{\beta}(I)$ is the space of all infinitely differentiable functions defined on I such that

$$\gamma^{\beta}_{m,k}(\phi) = \sup_{x \in I} |x^m D^k x^{\beta} \phi(x)| \tag{15}$$

exists for each pair of non-negative integers m and k. We equip with this space topology generated by the countable multinorm (15). Thus $\mathbb{H}_{\beta}(I)$ is a Frechet space.

If $\phi(x)$ admits the expansion

$$\phi(x) = x^{-\beta}[a_0 + a_1 x + a_2 x^2 + \cdots + a_p x^p + O(x^p)] \tag{16}$$

near the origin, a result analogous to Zemanian [6] can be deduced.

Lemma 2.1. *$\phi(x)$ is a member of $\mathbb{H}_{\beta}(I)$ if and only if $\phi(x)$ is an infinitely differentiable function, $\phi(x)$ has the form* (16) *in some vicinity of the origin and $D^k\phi(x)$ is of rapid descent when $x \to \infty$ for each $k = 0, 1, 2, \ldots$.*

$\mathbb{H}'_{\beta}(I)$ is the dual space of $\mathbb{H}_{\beta}(I)$. We consider only in $\mathbb{H}'_{\beta}(I)$ the weak topology ([6], p. 21). $\mathbb{H}'_{\beta}(I)$ is too complete. We point out the properties:

(i) The inclusions $D(I) \subset \mathbb{H}_{\beta}(I) \subset E(I)$ hold. $E'(I)$ is a subspace of $\mathbb{H}'_{\beta}(I)$.

(ii) The mapping $x^{-1}\phi(x) \to \phi(x)$ is an isomorphism from $\mathbb{H}_{\beta-1}(I)$ onto $\mathbb{H}_{\beta}(I)$. Therefore the operator $f(x) \to x^{-1}f(x)$ defined by

$$\langle x^{-1}f(x),\ \phi(x)\rangle = \langle f(x), x^{-1}\phi(x)\rangle,$$

$f \in \mathbb{H}'_{\beta}(I)$, $\phi(x) \in \mathbb{H}'_{\beta-1}(I)$ is an isomorphism from $\mathbb{H}'_{\beta}(I)$ on to $\mathbb{H}'_{\beta-1}(I)$.

(iii) $\mathbb{H}_{\beta-r}(I)$ is a subspace of $\mathbb{H}_{\beta}(I)$, for any position integer r.

On the other hand, $\mathbb{S}_{\alpha}(I)$ consists of all infinitely differentiable functions $\psi(x)$ defined on I so that

$$\rho^{\alpha}_{m,k}(\psi) = \sup_{x \in I} |x^m D^k x^{-\alpha}\psi(x)| \tag{17}$$

exists for each pair of non-negative integers m and k. With the topology generated by (17), $S_{\alpha}(I)$ is also a Frechet space. Assume that $\psi(x)$ takes the form

$$\psi(x) = x^{\alpha}[b_0 + b_1 x + b_2 x^2 + \cdots + b_p x^p + O(x^p)] \tag{18}$$

as $x \to 0^+$. We can then prove the following lemma.

Lemma 2.2. *$\psi(x)$ belongs to $\mathbb{S}_{\alpha}(I)$ if and only if $\psi(x)$ is infinitely differentiable, $\psi(x)$ is of the form* (18) *as $x \to 0^+$ and $D^k\psi(x)$ is of rapid descent as $x \to \infty$, for each $k = 1, 2, 3, \ldots$.*

(iv) The inclusions $D(I) \subset \mathbb{S}_\alpha(I) \subset E(I)$ hold. $E'(I)$ is a subspace of $S'_\alpha(I)$.

(v) The mapping $\psi(x) \to x\psi(x)$ is an isomorphism from $\mathbb{S}_\alpha(I)$ onto $\mathbb{S}_{\alpha+1}(I)$. Therefore, $f(x) \to xf(x)$ defined by $\langle xf(x), \Psi(x)\rangle = \langle f(x), x\psi(x)\rangle$, $f \in \mathbb{S}'_{\alpha+1}(I)$, $\psi(x) \in \mathbb{S}_\alpha(I)$, is an isomorphism from $\mathbb{S}'_{\alpha+1}(I)$ onto $\mathbb{S}'_\alpha(I)$.

(vi) $\mathbb{S}_{\alpha+r}(I)$ is a subspace of $\mathbb{S}_\alpha(I)$, for any positive integer r.

We next discuss the following differential operators:

$$R_\beta = x^{-\beta+1}Dx^\beta, \tag{19}$$

$$Q_\alpha = x^{-\alpha}Dx^\alpha, \tag{20}$$

$$R^*_\beta = -x^\beta Dx^{-\beta+1}, \tag{21}$$

$$Q^*_\alpha = -x^\alpha Dx^{-\alpha}. \tag{22}$$

When these operators act on these spaces $\mathbb{H}_\beta(I)$ and $\mathbb{S}_\alpha(I)$, one verifies the following lemma.

Lemma 2.3. (a) *The differential operator R_β is an isomorphism from $\mathbb{H}_\beta(I)$ into $\mathbb{H}_{\beta-1}(I)$, its inverse being*

$$R_\beta^{-1} = x^{-\beta}\int_\infty^x t^{\beta-1}\phi(t)\mathrm{d}t, \quad \phi \in \mathbb{H}_{\beta-1}(I). \tag{23}$$

(b) *The operator Q_α is a continuous linear mapping from $\mathbb{H}_{\beta-1}(I)$ into $\mathbb{H}_\beta(I)$.* (c) *The differential operator R^*_β is a continuous linear mapping from $\mathbb{S}_\alpha(I)$ into $\mathbb{S}_\alpha(I)$.* (d) *The differential operator Q^*_α is an automorphism from $\mathbb{S}_\alpha(I)$ onto $\mathbb{S}_\alpha(I)$, its inverse being*

$$(Q^*_\alpha)^{-1} = -x^\alpha\int_\infty^x t^{-\alpha}\phi(t)\mathrm{d}t. \tag{24}$$

Proof. Let $\phi \in \mathbb{H}_\beta(I)$. It can be seen that

$$\gamma^{\beta-1}_{m,k}[R_\beta\phi] = \gamma^\beta_{m,k+1}[\phi]. \tag{25}$$

On the other hand, if $\phi \in \mathbb{H}_{\beta-1}(I)$ we have

$$\gamma^\beta_{m,k}[R_\beta^{-1}\phi] = \gamma^{\beta-1}_{m,k-1}[\phi], \tag{26}$$

$k = 1, 2, 3, \ldots$ and for the case $k = 0$,

$$\gamma^\beta_{m,o}[R_\beta^{-1}\phi] \le \frac{\pi}{2}[\gamma^{\beta-1}_{m,o}(\phi) + \gamma^{\beta-1}_{m+2,o}[\phi]]. \tag{27}$$

To verify (b), note that

$$\gamma^\beta_{m,k}[Q_\alpha(\phi)] \le |\alpha - \beta + k + 1|\gamma^{\beta-1}_{m,k}[\phi] + \gamma^{\beta-1}_{m+1,k+1}[\phi]$$

holds for every $\phi \in \mathbb{H}_{\beta-1}(I)$.

A similar reasoning permits one to prove (c) and (d). But, if the same operators are considered acting on the spaces of generalized functions $\mathbb{H}'_\beta(I)$ and $\mathbb{S}'_\alpha(I)$, the following assertions can be derived as an immediate consequences of Lemma 2.3.

Lemma 2.4. (a′) *The generalized operator R^*_β, defined on $\mathbb{H}'_{\beta-1}(I)$ as the adjoint of R_β, that is, $\langle R^*_\beta f, \phi\rangle = \langle f, R_\beta\phi\rangle$, $f \in \mathbb{H}'_{\beta-1}(I)$, $\phi \in \mathbb{H}_\beta(I)$, is an isomorphism from $\mathbb{H}'_\beta(I)$ onto $H'_{\beta-1}(I)$.* (b′) *The generalized operator Q^*_α defined in $\mathbb{H}'_\beta(I)$, as usual by $\langle Q^*_\alpha f, \phi\rangle = \langle f, Q_\alpha\phi\rangle$, $f \in \mathbb{H}'_\beta(I)$, $\phi \in \mathbb{H}_{\beta-1}(I)$, is a continuous linear mapping of $\mathbb{H}'_\beta(I)$ into $H'_{\beta-1}(I)$.* (c′) *The generalized operator R_β, defined in $\mathbb{S}'_\alpha(I)$ as the adjoint of R^*_β in $\mathbb{S}_\alpha(I)$, namely, $\langle R_\beta f, \phi\rangle = \langle f, R^*_\beta\phi\rangle$, $f \in \mathbb{S}'_\alpha(I)$, $\phi \in \mathbb{S}_\alpha(I)$, is a continuous linear operator on $\mathbb{S}'_\alpha(I)$ into itself.* (d′) *The generalized operator Q_α defined in $\mathbb{S}'_\alpha(I)$ through $\langle Q_\alpha f, \phi\rangle = \langle f, Q^*_\alpha\phi\rangle$, $f \in \mathbb{S}'_\alpha(I)$, $\phi \in \mathbb{S}_\alpha(I)$, is an automorphism on $\mathbb{S}'_\alpha(I)$.*

Now assume that $\alpha - \beta \geq -1/2$, then $\mathbb{H}_\beta(I)$ can be identified with a subspace of $\mathbb{S}'_\alpha(I)$. Indeed, every member of $f \in \mathbb{H}_\beta(I)$ gives rise to a regular generalized function $f \in \mathbb{S}'_\alpha(I)$ by

$$\langle f, \phi\rangle = \int_0^\infty f(x)\phi(x)\mathrm{d}x, \quad \phi \in \mathbb{S}\alpha(I),$$

since the linear operator f satisfies

$$|\langle f, \theta\rangle| \leq \rho^\alpha_{0,0}(\phi)\int_0^\infty |x^{-\alpha} f(x)|\mathrm{d}x$$

for all $\phi \in \mathbb{S}_\alpha(I)$ and $f(x)$ is a function of rapid descent, that is, f is also continuous. Observe that the last integral exists is view of Lemma 2.1, because $f \in \mathbb{H}_\beta(I)$. Moreover, two members of $\mathbb{H}_\beta(I)$ which generates the same regular distribution in $\mathbb{S}'_\alpha(I)$ must be identical. These consideration justify the inclusion $\mathbb{H}_\beta(I) \subset \mathbb{S}'_\alpha(I)$. We can argue in a similar way to verify that $\mathbb{S}_\alpha(I) \subset \mathbb{H}'_\beta(I)$.

Remark 1. The generalized Kepinski operator, as given in [2] $K_{\alpha,\beta} = xD^2 - (\alpha + \beta - 1)D + \alpha\beta x^{-1} = R^*_\beta Q^*_\alpha$ is a continuous linear mapping from the spaces $\mathbb{S}_\alpha(I)$ and $\mathbb{H}'_\beta(I)$ into themselves. Analogously, $K_{\alpha,\beta} = xD^2 + (\alpha + \beta + 1)D + \alpha\beta x^{-1} = Q_\alpha R_\beta$ is a continuous linear mapping from the spaces $\mathbb{H}_\beta(I)$ and $\mathbb{S}'_\alpha(I)$ into themselves.

4. The classical generalized Hankel–Clifford transformations on the space $\mathbb{H}_{\boldsymbol{\beta}}(\boldsymbol{I})$ and $\mathbb{S}_{\boldsymbol{\alpha}}(\boldsymbol{I})$

Under the restriction $(\alpha - \beta) \geq -1/2$ every member of $\mathbb{H}_\beta(I)$ fulfils the requirements of inversion theorem of (4) and consequently, the generalized Hankel–Clifford transformation $h_{1,\alpha,\beta}$ given by

$$(h_{1,\alpha,\beta}\phi)(y) = \Phi(y) = y^{-(\alpha+\beta)}\int_0^\infty C_{\alpha,\beta}(xy)\phi(x)\mathrm{d}x \tag{28}$$

exists for all $\phi \in \mathbb{H}_\beta(I)$. We next establish the main result.

Theorem 2. *Let $(\alpha - \beta) \geq -1/2$. The first generalized Hankel–Clifford transformation $h_{1,\alpha,\beta}$ is an automorphism on $\mathbb{H}_\beta(I)$.*

Proof. Let $\phi(x)$ be any member of $\mathbb{H}_\beta(I)$. Expression (28) defines obviously a linear operator on $\mathbb{H}_\beta(I)$. To prove its continuity, note that the smoothness of the integrand and the use of (8) and (9) yield

$$y^m D_y^k y^\beta \Phi(y) = y^m D_y^k y^\beta \left[y^{-(\alpha+\beta)} \int_0^\infty C_{\alpha,\beta}(xy)\phi(x)\mathrm{d}x \right]$$
$$= (-1)^k y^m \int_0^\infty \frac{\mathrm{d}^N}{\mathrm{d}x^N} (x^{-\beta+k+N} C_{\alpha+k+N,\beta}(xy)) x^\beta \phi(x)\mathrm{d}x$$

where N denotes an arbitrary positive integer. If we set $N = 2m$ and integrate by parts $2m$ times in the last integral, one can deduce

$$y^m D_y^k y^\beta \Phi(y) = \int_0^\infty (xy)^m C_{\alpha+k+2m,\beta}(xy) x^{-\beta+k+m} D_x^{2m} x^{(-\alpha-\beta)} \phi(x)\mathrm{d}x \tag{29}$$

since $|z^m C_{\alpha+k+2m,\beta}(z)| \le A_{m,k}$ on $0 < z < \infty$, $A_{m,k}$ being a positive constant. If n represents a positive integer no less than $-\beta + k + m$, it requires from (29) that

$$|y^m D_y^k y^\beta \Phi(y)| \le A_{m,k} \sum_{r=0}^{n+2} \binom{n+2}{r} \nu_{r,2m}^\beta(\phi) \int_0^\infty \frac{\mathrm{d}x}{(1+x)^2},$$

that is

$$\nu_{m,k}^\beta(\phi) \le A_{m,k} \sum_{r=0}^{n+2} \binom{n+2}{r} \nu_{r,2m}^\beta(\phi). \tag{30}$$

Expression (29) shows that $\Phi(y)$ is an infinitely differentiable function, whereas (30) implies that $h_{1,\alpha,\beta}$ is a continuous operator on $\mathbb{H}_\beta(I)$. Finally, by inversion theorem [2] we have $h_{1,\alpha,\beta}^2 \phi = \phi$ for all $\phi \in \mathbb{H}_\beta(I)$. This completes the proof of Theorem 2.

The second Hankel–Clifford transformation $h_{2,\alpha,\beta}$ defined by means of

$$(h_{2,\alpha,\beta}\psi)(y) = \Psi(y) = \int_0^\infty x^{-(\alpha+\beta)} C_{\alpha,\beta}(xy)\psi(x)\mathrm{d}x \tag{31}$$

exists for every $\psi \in \mathbb{S}_\alpha(I)$, by inversion theorem [2]. Through an argument similar to the one used in the proof of Theorem 2, we can assert the next theorem.

Theorem 3. *The second generalized Hankel–Clifford transformation $h_{2,\alpha,\beta}$ of order $(\alpha - \beta) \ge -1/2$ is an automorphism on $\mathbb{S}_\alpha(I)$.*

Now some interesting operational rules for the transformation $h_{1,\alpha,\beta}$ are obtained.

Lemma 4.1. *Let $\alpha - \beta \ge -1/2$. For all $\phi(x) \in \mathbb{H}_\beta(I)$, we have*

$$R_\beta h_{1,\alpha,\beta}(\phi) = h_{1,\alpha,\beta-1}(-x\phi), \tag{32}$$
$$h_{1,\alpha,\beta-1}(R_\beta\phi) = -y h_{1,\alpha,\beta}(\phi), \tag{33}$$
$$h_{1,\alpha,\beta}(Q_\alpha R_\beta \phi) = -y h_{1,\alpha,\beta}(\phi), \tag{34}$$
$$Q_\alpha R_\beta h_{1,\alpha,\beta}(\phi) = h_{1,\alpha,\beta}(-x\phi), \tag{35}$$

and for all $\phi(x) \in \mathbb{H}_{\beta-1}(I)$

$$h_{1,\alpha,\beta}(Q_\alpha \phi) = h_{1,\alpha,\beta-1}(\phi), \tag{36}$$
$$Q_\alpha h_{1,\alpha,\beta-1}(\phi) = h_{1,\alpha,\beta}(\phi). \tag{37}$$

Proof. Let $\phi \in \mathbb{H}_\beta(I)$. We may differentiate under the integral sign to obtain, in accordance with (8),

$$R_\beta h_{1,\alpha,\beta}\phi = y^{-\alpha-\beta+1}\int_0^\infty (-x)C_{\alpha,\beta-1}(xy)\phi(x)\mathrm{d}x = h_{1,\alpha,\beta-1}(-x\phi).$$

This proves (32). To see (33), the integration by parts and use of (9) yield

$$\begin{aligned} h_{1,\alpha,\beta-1}(R_\beta\phi) &= y^{-(\alpha+\beta-1)}\int_0^\infty x^{-\beta+1}C_{\alpha,\beta-1}(xy)D_x x^\beta\phi(x)\mathrm{d}x \\ &= y^{-(\alpha+\beta-1)}\left\{(xC_{\alpha,\beta-1}(xy)\phi)_{x\to 0}^{x\to\infty} + -\int_0^\infty x_0^{-\beta}C_{\alpha,\beta-1}(xy)D_x^\beta x\phi(x)\mathrm{d}x\right\}. \end{aligned}$$

The limit terms tend to zero since $\phi(x)$ is of rapid descent as $x \to \infty$ and, as $x \to 0^+$, $xC_{\alpha,\beta-1}(xy) = \infty(|x|^{\alpha+1})$ and $\phi(x) = 0(|x|^{-\beta})$ when $(\alpha - \beta) \geq -1/2$, this verifies (33). Similar manipulation allow us to obtain the remaining formulas.

Next we summarize operational calculus generated by the transform $h_{2,\alpha,\beta}$.

Lemma 4.2. *Let $\alpha - \beta \geq -1/2$. For all $\psi(x) \in \mathbb{S}_\alpha(I)$, we have*

$$Q_\alpha^* h_{2,\alpha,\beta}(\psi) = h_{2,\alpha,\beta-1}(\psi), \tag{38}$$

$$h_{2,\alpha,\beta-1}(Q_\alpha^*\psi) = h_{2,\alpha,\beta}(\psi), \tag{39}$$

$$h_{2,\alpha,\beta}(R_\beta^* Q_\alpha^*\psi) = -yh_{2,\alpha,\beta-1}(\psi), \tag{40}$$

$$R_\beta^* Q_\alpha^* h_{2,\alpha,\beta}(\psi) = h_{2,\alpha,\beta}(-x\psi). \tag{41}$$

For $\psi(x) \in \mathbb{S}_{\beta-1}(I)$,

$$h_{2,\alpha,\beta}(R_\beta^*\psi) = -yh_{2,\alpha,\beta-1}(\psi), \tag{42}$$

$$R_\alpha^* h_{2,\alpha,\beta-1}(\psi) = h_{2,\alpha,\beta-1}(-x\psi). \tag{43}$$

5. The distributional generalized Hankel–Clifford transformation $h'_{1,\alpha,\beta}$

Let $(\alpha - \beta) \geq -1/2$. We propose defining the distributional generalized Hankel–Clifford transformation $h'_{1,\alpha,\beta}$ on $\mathbb{S}'_\alpha(I)$ as the adjoint operator of $h_{2,\alpha,\beta}$ on $\mathbb{S}_\alpha(I)$, that is

$$\langle h'_{1,\alpha,\beta}f, \Phi\rangle = \langle f, h_{2,\alpha,\beta}\Phi\rangle \tag{44}$$

for all $f \in \mathbb{S}'_\alpha(I)$ and $\Phi \in \mathbb{S}_\alpha(I)$.

Note that by setting $\Phi = h_{2,\alpha,\beta}\phi$, $\phi \in \mathbb{S}_\alpha(I)$, (44) takes the form

$$\langle h'_{1,\alpha,\beta}f, h_{2,\alpha,\beta}\phi\rangle = \langle f, \phi\rangle, \quad f \in \mathbb{S}'_\alpha(I), \quad \phi \in \mathbb{S}_\alpha(I). \tag{45}$$

Hence (45) can be understood as a generalization of the mixed Parseval equation (14), as it happens in the extention of the Hankel transform to certain space of generalized functions ([6], p. 142). Note that definition (44) has a sense. Indeed, from Theorem 2 the following is inferred.

Theorem 3. *Let $(\alpha - \beta) \geq -1/2$. The distributional generalized Hankel–Clifford transformation $h'_{1,\alpha,\beta}$, as defined by* (44), *is an automorphism on $\mathbb{S}'_\alpha(I)$.*

Recall that $\mathbb{H}_\beta(I) \subset \mathbb{S}'_\alpha(I)$. If $f \in \mathbb{H}_\beta(I)$, then $h'_{1,\alpha,\beta}f$ exists and f gives rise to a regular member on $\mathbb{S}'_\alpha(I)$. We may write by (44)

$$\langle h'_{1,\alpha,\beta}f, \Phi\rangle = \langle f, h_{2,\alpha,\beta}\Phi\rangle = \int_0^\infty f(x)(h_{2,\alpha,\beta}\Phi)(x)\mathrm{d}x$$

for all $\Phi \in \mathbb{S}_\alpha(I)$. By applying (14) we can convert last integral into

$$\int_0^\infty (h_{1,\alpha,\beta}f)(y)\Phi(y)\mathrm{d}y = \langle h_{1,\alpha,\beta}f, \Phi\rangle.$$

Therefore, whatever $f \in \mathbb{H}_\beta(I)$

$$h'_{1,\alpha,\beta}f = h_{1,\alpha,\beta}f, \tag{46}$$

that is to say, the classical transformation $h_{1,\alpha,\beta}$ is a special case of the distributional transformation $h'_{1,\alpha,\beta}$ given by (44). From Lemma 4.2 we deduce immediately, in line with new definition (44), the following operational formulas.

Lemma 5.1. *Let* $(\alpha - \beta) \geq -1/2$. *For all* $f(x) \in \mathbb{S}'(I)$, *we have*

$$\begin{aligned} R_\beta h'_{1,\alpha,\beta}(f) &= h'_{1,\alpha,\beta-1}(-xf),\\ h'_{1,\alpha,\beta-1}(R_\beta f) &= -yh'_{1,\alpha,\beta}(f),\\ h'_{1,\alpha,\beta}(Q_\alpha R_\beta f) &= -yh'_{1,\alpha,\beta}(f),\\ Q_\alpha R_\beta h'_{1,\alpha,\beta}(f) &= h'_{1,\alpha,\beta}(-xf), \end{aligned}$$

and for all $f(x) \in \mathbb{S}_{\alpha+1}(I)$

$$\begin{aligned} h'_{1,\alpha,\beta}(Q_\alpha f) &= h'_{1,\alpha,\beta-1}(f),\\ Q_\alpha h'_{1,\alpha,\beta-1}(f) &= h'_{1,\alpha,\beta}(f). \end{aligned}$$

Since $\mathbb{H}_\beta(I) \subset \mathbb{S}'_\alpha(I)$ *and* $\mathbb{H}_{\beta-1}(I) \subset \mathbb{S}'_\alpha(I)$, *these results generalize those in Lemma* 4.1.

Note that the classical operational formulae coincide with their respective distributional expressions.

$\mathbb{H}_{\alpha,\beta,a}$ is the space of all infinitely differentiable functions $\phi(x)$ defined on I for which

$$\gamma_k^{\alpha,\beta,a}(\phi) = \sup_{x\in I} |\mathrm{e}^{-ax}x^{-\alpha}\Delta^k_{\alpha,\beta}\phi(x)| < \infty, \quad k = 0, 1, 2, \ldots, \tag{47}$$

where $a > 0$, $(\alpha - \beta) \geq -1/2$ and $\Delta_{\alpha,\beta} = x^\beta D_x x^{\alpha-\beta+1}D_x x^{-\alpha} = xD_x^2 - (\alpha+\beta-1)D_x + \alpha\beta x^{-1}$ denotes the generalized Kepinski operator. $\mathbb{H}_{\alpha,\beta,a}$ is a Frechet space. In [2] we established that the kernel of (28) belongs to $\mathbb{H}_{\alpha,\beta,a}$ and defined the distributional generalized Hankel–Clifford transformation $F'_{\alpha,\beta}$ on the dual space $\mathbb{H}'_{\alpha,\beta,a}$ by the relation

$$\begin{aligned} F'_{\alpha,\beta}\{f\}(y) = F_1(y) &= \left\langle f(x), \left(\frac{y}{x}\right)^{-(\alpha+\beta)/2} J_{\alpha-\beta}(2\sqrt{xy})\right\rangle\\ &= \langle f(x), y^{-(\alpha+\beta)}C_{\alpha,\beta}(xy)\rangle \end{aligned} \tag{48}$$

where $f \in \mathbb{H}'_{\alpha,\beta,a}(I)$.

Lemma 5.2. *Let $(\alpha-\beta)\geq -1/2$. $\mathbb{S}_\alpha(I)$ is a subspace of $\mathbb{H}_{\alpha,\beta,a}$, the topology of $\mathbb{S}_\alpha(I)$ being stronger than that induced on it by $\mathbb{H}_{\alpha,\beta,a}$. Consequently restriction of $f\in\mathbb{H}'_{\alpha,\beta,a}$ to $\mathbb{S}_\alpha(I)$ is in $\mathbb{S}'_\alpha(I)$.*

Proof. Proof can be easily given [2].

Note that $(y/x)^{-(\alpha+\beta)/2}J_{\alpha-\beta}(2\sqrt{xy})$ belongs to $\mathbb{H}_{\alpha,\beta,a}$, [2] for all y fixed in $0<y<\infty$, but not to $\mathbb{S}_\alpha(I)$, because this function is not of rapid descent at infinity. Hence the above inclusion is proper.

Every member of $f\in\mathbb{H}'_{\alpha,\beta,a}$ admits, by Lemma 5.2, two distributional generalized Hankel–Clifford transformations given through (44) and (48). Our next objective is to show that these definitions agree.

Theorem 4. *Let $(\alpha-\beta)\geq -1/2$. If $f\in\mathbb{H}'_{\alpha,\beta,a}$ then the distributional generalized Hankel–Clifford transformation $F'_{\alpha,\beta}\{f\}(y)$ defined by* (48) *coincides with $h'_{1,\alpha,\beta}$ given by* (44), *in the sense of equality in $\mathbb{S}'_\alpha(I)$.*

Proof. Proof can be easily given [2].

Remark 2. If the distributional generalized Hankel–Clifford transformation $h'_{1,\alpha,\beta}$ were defined on $\mathbb{H}'_\beta(I)$, as usual in the available literature, by means of the adjoint of $h_{1,\alpha,\beta}$ on $\mathbb{H}_\beta(I)$, namely

$$\langle h'_{1,\alpha,\beta}f,\Phi\rangle=\langle f,h_{1,\alpha,\beta}\Phi\rangle,\quad f\in\mathbb{H}'_\beta(I),\quad \Phi\in\mathbb{H}_\beta(I) \tag{49}$$

the relation (46) between the classical and the distributional transforms must be replaced by

$$h'_{1,\alpha,\beta}f=y^{-(\alpha+\beta)}h_{1,\alpha,\beta}(x^{(\alpha+\beta)}f).$$

In other words, the classical transform $h_{1,\alpha,\beta}$ is not a special case of the distributional transform $h'_{1,\alpha,\beta}$. On the other hand, the operational formulas of $h_{1,\alpha,\beta}$ and $h'_{1,\alpha,\beta}$ would not coincide. Another drawback of this usual definition lies in the fact that Lemma 5.1 could not be established. Moreover (49) can not be understood as a generalization of Parseval equation (14) due to presence of the factor $x^{(\alpha+\beta)}$. These considerations show that our definition (44) is more adequate than (49).

Remark 3. Analogously, the distributional generalized Hankel–Clifford transformation $h'_{2,\alpha,\beta}$ can be defined on $\mathbb{H}'_\beta(I)$ as the adjoint of $h_{1,\alpha,\beta}$ on $\mathbb{H}_\beta(I)$, that is,

$$\langle h'_{2,\alpha,\beta}f,\Phi\rangle=\langle f,h_{1,\alpha,\beta}\Phi\rangle,\quad f\in\mathbb{H}'_\beta(I),\quad \Phi\in\mathbb{H}_\beta(I).$$

$h'_{2,\alpha,\beta}$ is an automorphism too on $\mathbb{H}'_\beta(I)$ by virtue of Theorem 2, provided that $(\alpha-\beta)\geq -1/2$. Since $\mathbb{S}_\alpha(I)\subset\mathbb{H}'_\beta(I)$ and $\mathbb{S}_\alpha(I)\subset\mathbb{H}'_{\beta-1}(I)$, the operational formulas for $h'_{2,\alpha,\beta}$ turn out to be those in Lemma 4.2.

6. The distributional generalized Hankel–Clifford transformation of arbitrary order

Let $(\alpha-\beta)$ be any fixed real number. We first define a certain transformation on $\mathbb{H}_\beta(I)$, which coincides with (4) whenever $(\alpha-\beta)\geq -1/2$. Let k be any positive integer such that $(\alpha-\beta+k)\geq -1/2$. For any $\Phi\in \mathbb{H}_\beta(I)$ set

$$\phi(x)=h_{1,\alpha,\beta,k}[\Phi(y)]=(-1)^k x^{-k} h_{1,\alpha,\beta-k} R_{\beta-(k-1)}\cdots R_{\beta-1}R_\beta\Phi(y) \tag{50}$$

$$\Phi(y)=h^{-1}_{1,\alpha,\beta,k}[\phi(x)]=(-1)^k R^{-1}_\beta R^{-1}_{\beta-1}\cdots R^{-1}_{\beta-(k-1)} R^{-1}_{\beta-1} h^{-1}_{1,\alpha,\beta-k} x^k\phi(x). \tag{51}$$

Lemma 6.1. *The transformation $h_{1,\alpha,\beta,k}$ as defined by* (50) *is an automorphism on $\mathbb{H}_\beta(I)$ whatever be the real number $(\alpha-\beta)$. Its inverse is $h^{-1}_{1,\alpha,\beta,k}$ as defined by* (51)*. Finally, on $\mathbb{H}_\beta(I)$, $h_{1,\alpha,\beta,k}$ coincides with $h_{1,\alpha,\beta}$ as defined by* (6) *whenever $(\alpha-\beta)\geq -1/2$.*

Proof. The first assertion follows from the fact that $\Phi\to R_{\beta-(k-1)}\cdots R_{\beta-1}R_\beta\Phi$ is an isomorphism from $\mathbb{H}_\beta(I)$ onto $\mathbb{H}_{\beta-k}(I)$, $\Phi\to h_{1,\alpha,\beta,k}\Phi$ is an automorphism on $\mathbb{H}_{\beta-k}(I)$, and $\phi\to x^{-k}\phi$ is an isomorphism from $\mathbb{H}_{\beta-k}(I)$ onto $\mathbb{H}_\beta(I)$.

By assumption, $\alpha+\beta+k\geq -1/2$. It is a classical fact that $h_{1,\alpha,\beta-k}$ is the inverse of itself when it acts on smooth functions in $L(I)$. Since $\mathbb{H}_{\beta-k}(I)\subset L(I)$, the second assertion follows from this fact and Lemma 2.1, (ii), Lemma 2.2.

For the third assertion, assume that $\Phi(y)\in\mathbb{H}_\beta(I)$ and $(\alpha-\beta)\geq -1/2$, and consider the case $k=1$;

$$\begin{aligned} h_{1,\alpha,\beta,1}\Phi &= -x^{-1}h_{1,\alpha,\beta-1}\Phi \\ &= -x^{-1}x^{-(\alpha+\beta-1)}\int_0^\infty C_{\alpha,\beta-1}(xy)(y^{-\beta+1}D_y y^\beta\Phi(y))\mathrm{d}y \end{aligned}$$

An integration by parts and the formula

$$D_y[y^{-\beta+1}C_{\alpha,\beta-1}(xy)]=y^{-\beta}C_{\alpha,\beta}(xy)$$

yield

$$= -x^{-1}x^{-(\alpha+\beta-1)}[y^{-\beta+1}C_{\alpha,\beta-1}(xy)(y^{-\beta}\Phi(y)]_0^\infty-\int_0^\infty y^{-\beta}C_{\alpha,\beta}(xy)y^\beta\Phi(y)\mathrm{d}y].$$

The limit terms are zero because $\Phi(y)$ is of rapid descent and $yC_{\alpha,\beta-1}(xy)$ remains bounded as $y\to\infty$, whereas, for $y\to 0^+$, $C_{\alpha,\beta}(x)=0(x^\alpha)$ where $(\alpha-\beta)\geq -1/2$. Thus,

$$h_{1,\alpha,\beta,1}(\Phi)=x^{-1}x^{-(\alpha+\beta-1)}\int_0^\infty C_{\alpha,\beta}(xy)\Phi(y)\mathrm{d}y=h_{1,\alpha,\beta}(\Phi).$$

The general statement for larger integral values of k follows by induction from this results. This ends the proof.

A consequences of Lemma 6.1 is that $h_{1,\alpha,\beta,k}=h_{1,\alpha,\beta,p}$ so long as the positive integers k and p are both larger than $-(\alpha-\beta)-1/2$. Indeed, assuming that $k>p$, we have $h_{1,\alpha,\beta-p,k+p}=h_{1,\alpha,\beta-p}$ according to the last statement of Lemma 6.1, and therefore, for $\Phi\in\mathbb{H}_\beta(I)$,

$$h_{1,\alpha,\beta,k}\Phi=(-1)^p x^{-p}h_{1,\alpha,\beta-p,k+p}R_{\beta-(p-1)}\cdots R_\beta\Phi=h_{1,\alpha,\beta,p}\Phi.$$

Since $h_{1,\alpha,\beta}\Phi=h^{-1}_{1,\alpha,\beta}\Phi$ whenever $\Phi\in\mathbb{H}_\beta(I)$ and $(\alpha-\beta)\geq -1/2$. Lemma 6.1 also implies that $h^{-1}_{1,\alpha,\beta,k}$ coincides with $h^{-1}_{1,\alpha,\beta}$ when $(\alpha-\beta)\geq -1/2$. Moreover, by virtue of the preceding paragraph, $h^{-1}_{1,\alpha,\beta,k}$ is independent of the choice of k so long as $\alpha-\beta+k\geq -1/2$.

If view of these results, it is reasonable to define the first generalized Hankel–Clifford transformation $h_{1,\alpha,\beta,k}$ for $(\alpha-\beta)<-1/2$ on $\Phi\in\mathbb{H}_\beta(I)$ by $h_{1,\alpha,\beta}\Phi=h_{1,\alpha,\beta,k}\Phi$ where k

is any positive integer no less than $-(\alpha-\beta)-1/2$. The generalized inverse Hankel–Clifford transformation $h_{1,\alpha,\beta}^{-1}$ is defined by $h_{1,\alpha,\beta}^{-1}\Phi = h_{1,\alpha,\beta,k}^{-1}\Phi$, $\Phi \in \mathbb{H}_{\beta}(I)$. As in the classical case, $h_{1,\alpha,\beta} = h_{1,\alpha,\beta}^{-1}$ when $(\alpha-\beta) \geq -1/2$, but this is not the case when $(\alpha-\beta) < -1/2$.

Lemma 6.2. *Let $(\alpha-\beta)$ be any real number. For all $\phi(x) \in \mathbb{H}_{\beta}(I)$, we have*

$$\begin{aligned} R_{\beta}h_{1,\alpha,\beta,k}(\phi) &= h_{1,\alpha,\beta-1,k}(-x\phi),\\ h_{1,\alpha,\beta-1,k}(R_{\beta}\phi) &= -y\,h_{1,\alpha,\beta,k}(\phi),\\ h_{1,\alpha,\beta,k}(Q_{\alpha}R_{\beta}\phi) &= -y\,h_{1,\alpha,\beta,k}(\phi),\\ Q_{\alpha}R_{\beta}h_{1,\alpha,\beta,k}(\phi) &= h_{1,\alpha,\beta,k}(-x\phi). \end{aligned}$$

and for all $\phi(x) \in \mathbb{H}_{\beta-1}(I)$

$$\begin{aligned} h_{1,\alpha,\beta,k}(Q_{\alpha}\phi) &= h_{1,\alpha,\beta-1,k}(\phi),\\ Q_{\alpha}h_{1,\alpha,\beta-1,k}(\phi) &= h_{1,\alpha,\beta,k}(\phi). \end{aligned}$$

Now we define the second generalized Hankel–Clifford transformation of arbitrary order for $h_{2,\alpha,\beta}$.

Let $\alpha-\beta$ be any fixed real number and k any positive integer such that $\alpha-\beta+k \geq -1/2$. We define the transformation $h_{2,\alpha,\beta,k}$ on any $\psi \in \mathbb{S}_{\alpha}(I)$ by

$$h_{2,\alpha,\beta,k}[\Psi(y)] = \Psi(x) = h_{2,\alpha,\beta-k},(Q_{\alpha}^{*})^{k}\Psi(y) \tag{52}$$

and

$$\psi(y) = h_{2,\alpha,\beta,k}^{-1}[\psi(x)] = \{(Q_{\alpha}^{*})^{-1}\}^{k}h_{2,\alpha,\beta-k}\Psi(x). \tag{53}$$

Analogous conclusions as Lemma 6.1 is also valid.

Lemma 6.3. *The transformation $h_{2,\alpha,\beta,k}$ as defined by* (52) *is an automorphism on $\mathbb{S}_{\alpha}(I)$ whatever be the real number $\alpha-\beta$. Its inverse is $h_{2,\alpha,\beta,k}^{-1}$ as defined by* (53). *Finally, on $\mathbb{S}_{\alpha}(I)$, $h_{2,\alpha\beta,k}$ coincides with $h_{2,\alpha,\beta}$ as defined by* (5) *whenever $\alpha-\beta \geq -1/2$.*

We now turn to the distributional generalized Hankel–Clifford transformation $h'_{1,\alpha,\beta}$ of any arbitrary order $(\alpha-\beta)$ when it is acting on $\mathbb{S}'_{\alpha}(I)$. As before, k is any positive integer $\geq -(\alpha-\beta)-1/2$. Then the distributional generalized Hankel–Clifford transformation $h'_{1,\alpha,\beta}$ is defined on $\mathbb{S}'_{\alpha}(I)$ as the adjoint of $h_{2,\alpha,\beta,k}$ on $\mathbb{S}_{\alpha}(I)$ as

$$\langle h'_{1,\alpha,\beta}f,\Phi\rangle = \langle f, h_{2,\alpha,\beta,k}\Phi\rangle, \quad \text{for } f \in \mathbb{S}'_{\alpha}(I), \quad \Phi \in \mathbb{S}_{\alpha}(I). \tag{54}$$

Theorem 5. *The distributional Hankel–Clifford transformation of arbitrary order $(\alpha-\beta)$, defined in* (54), *is an automorphism onto $\mathbb{S}'_{\alpha}(I)$.*

This leads to the following transformation formulas:

Lemma 6.4. *Let $(\alpha-\beta)$ be any real number. For $f \in \mathbb{S}'_{\alpha}(I)_{\alpha}$,*

$$\begin{aligned} R_{\beta}h'_{1,\alpha,\beta}(\phi) &= h'_{1,\alpha,\beta-1}(-x\phi),\\ h'_{1,\alpha,\beta-1}(R_{\beta}\phi) &= -yh'_{1,\alpha,\beta}(\phi), \end{aligned}$$

$$h'_{1,\alpha,\beta}(Q_\alpha R_\beta \phi) = -y h'_{1,\alpha,\beta}(\phi),$$
$$Q_\alpha R_\beta h'_{1,\alpha,\beta}(\phi) = h'_{1,\alpha,\beta}(-x\phi).$$

and for all $\phi(x) \in \mathbb{S}_{\alpha+1}(I)$

$$h'_{1,\alpha,\beta}(Q_\alpha \phi) = h'_{1,\alpha,\beta-1}(\phi),$$
$$Q_\alpha h'_{1,\alpha,\beta-1}(\phi) = h'_{1,\alpha,\beta}(\phi).$$

Note. Analogously the distributional generalized Hankel–Clifford transformation $h'_{2,\alpha,\beta}$ can be defined on $\mathbb{H}'_\beta(I)$ as the adjoint of $h_{1,\alpha,\beta,k}$ on $\mathbb{H}_\beta(I)$ as $\langle h'_{2,\alpha,\beta} f, \Psi\rangle = \langle f, h_{1,\alpha,\beta,k}\Psi\rangle$ for $f \in \mathbb{H}'_\beta(I)$ and $\Psi \in \mathbb{H}_\beta(I)$.

Remark 4. It is proposed to apply the theory thus developed to solve some partial differential equations of generalized Kapinski type operator with distributional boundary conditions in subsequent papers.

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