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The algebra of G-relations

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Abstract. In this paper, we study a tower $\{A_n^G(d) : n \geq 1\}$ of finite-dimensional algebras; here, G represents an arbitrary finite group, d denotes a complex parameter, and the algebra $A_n^G(d)$ has a basis indexed by 'G-stable equivalence relations' on a set where G acts freely and has $2n$ orbits.

We show that the algebra $A_n^G(d)$ is semi-simple for all but a finite set of values of d, and determine the representation theory (or, equivalently, the decomposition into simple summands) of this algebra in the 'generic case'. Finally we determine the Bratteli diagram of the tower $\{A_n^G(d) : n \geq 1\}$ (in the generic case).

Keywords. Equivalence relations; Bratteli diagrams; G-relations.

1. Introduction

Let R_n denote the set of equivalence relations on the set $[n] = \{1, 2, \ldots, n\}$, and let ρ_n denote the cardinality of R_n. By convention, $n = 0, 1, 2, \ldots$ and $\rho_0 = 1$. Easy counting arguments show that the first few values of the sequence $\{\rho_n : n \geq 0\}$ are given by 1, 1, 2, 5, 15, 52, 203, ..., and that the sequence satisfies the recursion relation

$$\rho_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \rho_k, \quad \forall n \geq 0. \tag{1.1}$$

Given $P, Q \in R_n$, we shall say that $P \leq Q$ if any two P-related indices are necessarily Q-related – or equivalently, if every Q-equivalence class is a union of P-equivalence classes. Clearly, if $P_{\min}$ is the trivial relation all of whose equivalence classes are singletons, and if $P_{\max}$ is the equivalence relation with just one equivalence class, then $P_{\min} \leq P \leq P_{\max}$, $\forall\, P$. It is not hard to see that R_n is a lattice with respect to this order structure. (For instance, if $n = 4$, $P = \{\{1, 2\}, \{3, 4\}\}$ and $Q = \{\{1\}, \{2, 3, 4\}\}$, then $P \vee Q = P_{\max}$ and $P \wedge Q = \{\{1\}, \{2\}, \{3, 4\}\}$. We shall, as above, sometimes equate an equivalence relation with the set of its equivalence classes.)

Further, if $P \in R_n$, we shall write $||P||$ for the number of equivalence classes in P. Before proceeding further, we record a simple fact as a lemma, for convenience of future reference.

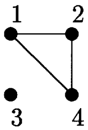
Lemma 1. *If* $P, Q \in R_n$, *then*,

(a) $||P \vee Q|| \leq ||P||$;
(b) *if* $||P \vee Q|| = ||P||$ *and* $P \neq Q$, *then* $||P|| < ||Q||$.

Proof. (a) Follows from the fact that every $P \vee Q$-equivalence class is a union of P-equivalence classes. (b) The hypothesis is seen to imply that no two indices which are

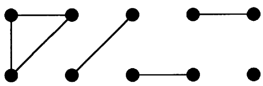
P-inequivalent can be $P \vee Q$-equivalent; this implies that $P \vee Q = P$. On the other hand, every Q-equivalence class is contained in a P-equivalence class, and the assumption that $P \neq Q$ says that at least one P-equivalence class must be the union of two or more Q-equivalence classes, and the proof is complete. □

We think of an element of R_{2n} as a diagram, thus: we think of the $2n$ elements of $[2n]$ listed in two rows of n elements each, with the jth point from the left on the top (resp., bottom) row indexed by j (resp., $n+j$); and connect every pair of indices which are equivalent under the relation. For instance, the relation in R_4, whose equivalence classes are $\{1,2,4\}$ and $\{3\}$ is represented by the picture

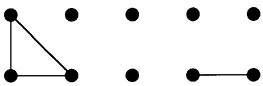


We will be interested in the vector space with R_{2n} as basis, which will be equipped with the structure of a $\mathbb{C}$-algebra, with the definition of the product of basis vectors involving a complex parameter d. Rather than giving a precise and rigorous definition, we shall describe the prescription for an example.

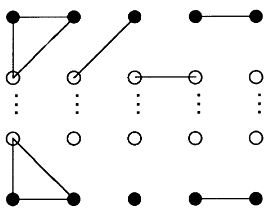
For instance, suppose $n = 5, P = \{\{1,2,6\},\{3,7\},\{4,5\},\{8,9\},\{10\}\}$ and $Q = \{\{1,6,7\},\{2\},\{3\},\{4\},\{5\},\{8\},\{9,10\}\}$; according to our diagrammatic notation, we have:

$P =$ 

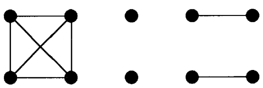
and

$Q =$ 

In order to define the product PQ, first concatenate the pictures (with P on top and Q at the bottom), and identify the intermediate levels of points as indicated:



then introduce a power of d equal to the number of 'components' in the grand picture which are entirely contained in the two middle levels, and then forget the two middle levels altogether, to finally obtain:

$PQ = d^2 \cdot$ 

It is relatively painless to verify that this definition yields a finite-dimensional associative $\mathbb{C}$-algebra (of dimension ρ_{2n}), which we denote by $A_n(d)$. This algebra has a multiplicative identity, i.e., the equivalence relation which has n equivalence classes, namely $\{k, n+k\}, 1 \le k \le n$.

As a trivial example, $A_1(d)$ has a basis consisting of 2 elements–say $1 = \{\{1,2\}\}$ and $P = \{\{1\},\{2\}\}$ – where 1 is the multiplicative identity, and $P^2 = dP$.

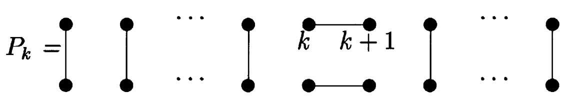
The process of 'adding a single vertical line at the right extreme' yields an injective map from R_{2n} into R_{2n+2}, which is easily seen to linearly extend to a multiplicative (identity-preserving) homomorphism of $A_n(d)$ into $A_{n+1}(d)$; further, since this map sends a basis injectively into a basis, it is necessarily a monomorphism. We thus have a tower

$$A_1(d) \subset A_2(d) \subset \cdots A_n(d) \subset \cdots$$

of finite-dimensional $\mathbb{C}$-algebras.

Remark 2. The tower $\{A_n(d) \colon n \ge 0\}$ has several interesting subtowers.

(a) The Temperley–Lieb algebra: Consider the subalgebra $T_n(d)$ of $A_n(d)$ consisting of the linear span of those equivalence relations P which satisfy two conditions: (i) each P-equivalence class contains precisely two elements; and most importantly, (ii) P admits a diagram–as in the above discussions–which is *planar*, i.e., the diagram has no crossings and is a planar diagram contained in the rectangle bounded by the $2n$ points. It is clear that the inclusion of $A_n(d)$ into $A_{n+1}(d)$ maps $T_n(d)$ into $T_{n+1}(d)$; and it is a fact that $T_n(d)$ is generated as a unital algebra by the elements $P_1, P_2, \ldots, P_{n-1}$, where, for $1 \le k < n$,



It is to be noted that $P_k^2 = dP_k$ and that $P_k P_{k\pm1} P_k = P_k$; so, if we define $e_k = d^{-1}P_k$, then the e_k's are idempotents which satisfy $e_k e_{k\pm1} e_k = d^{-2} e_k$.

(b) The group algebra $\mathbb{C}\Sigma_n$ of the symmetric group sits naturally as a subalgebra of $A_n(d)$ as follows: given $\sigma \in \Sigma_n$, let P_σ denote the equivalence relation, whose equivalence classes are $\{\{\sigma(k), k+n\} : 1 \le k \le n\}$. It is fairly easy to verify that this map is multiplicative, meaning that $P_\sigma \cdot P_\tau = P_{\sigma\tau}$. The following little observation, which we call (c) below for the sake of future reference, is proved easily.

(c) The following conditions on an element $P \in R_{2n}$ are equivalent: (i) P is an invertible element of $A_n(d)$; (ii) there exists a (necessarily unique) permutation $\sigma \in \Sigma_n$ such that $P = P_\sigma$, as in (b) above; (iii) P has precisely n 'through classes' in the sense of Definition 3 below.

DEFINITION 3

If $P \in R_{2n}$, a *through class of* P is an equivalence class A of P such that $A \cap \{1, 2, \ldots, n\} \neq \emptyset$ and $A \cap \{n+1, n+2, \ldots, 2n\} \neq \emptyset$. We write $t(P)$ for the number of through classes of P.

For instance, in the preceding example illustrating the definition of the product, we have $t(P) = 2$ and $t(Q) = t(d^{-2} \cdot PQ) = 1$. This is an instance of a general fact which has consequences for much of the following discussion.

Lemma 4. *Suppose $P, Q \in R_{2n}$, and $P \cdot Q = d^r \cdot S$, for some r and some $S \in R_{2n}$. Then,*

$$t(S) \leq \min\{t(P), t(Q)\}.$$

Proof. This follows easily from the definitions. □

COROLLARY 5

For $0 \leq k \leq n$, define I_k^n to be the set of those equivalence relations with exactly k through classes; and let I_k be the linear subspace spanned by $\cup_{r \leq k} I_r^n$; then

$$\{0\} = I_{-1} \subset I_0 \subseteq \cdots \subseteq I_k \subseteq I_{k+1} \subseteq \cdots \subseteq I_n = A_n(d) \tag{1.2}$$

is a filtration of $A_n(d)$ by two-sided ideals.

Proof. Obvious. □

Before concluding this introduction, we shall briefly dwell on the manner in which we would like to think of elements of R_{2n}, viz., as consisting of three pieces of information: (a) the 'top', (b) the data on how the top is connected to the bottom, and (c) the 'bottom'.

Thus, suppose $P \in R_{2n}$ and $t(P) = k$. Focus attention first on the top set of n points in the diagram representing P; we can naturally associate an element $P^+ \in R_n$, together with an unordered collection $\{C_j : 1 \leq j \leq k\}$ of 'distinguished' P^+-equivalence classes–these corresponding precisely to the intersections of through-classes of P and the top line. This is the motivation for the following definition.

DEFINITION 6

In the sequel, the symbol S_k^n will denote the set of symbols of the form $\underline{R} = (R; \{C_j(\underline{R}): 1 \leq j \leq k\})$, where $R \in R_n$ and $\{C_j(\underline{R}) : 1 \leq j \leq k\}$ is an unordered collection of k distinct 'distinguished' R-equivalence classes. (Note, in particular, that if $\underline{R} \in S_k^n$, then $||R|| \geq k$.)

We shall want to encode an element P of R_{2n}, for which $t(P) = k$, as a triple $(\underline{P^+}, \rho, \underline{P^-})$, where $\underline{P^\pm} \in S_k^n$, and ρ is a permutation, in such a way that (i) the P-equivalence classes which are entirely contained in $\{1, 2, \ldots, n\}$ are the same as the P^+-equivalence classes other than $C_j(\underline{P^+}), 1 \leq j \leq k$; (ii) the P-equivalence classes which are entirely contained in $\{n+1, n+2, \ldots, 2n\}$ are the same as the sets $(n + C) = \{n + m : m \in C\}$ where C is a P^--equivalence class other than the $C_j(\underline{P^-})$, $1 \leq j \leq k$; and (iii) the k through-classes of P are given by $C_j = C_{\rho(j)}(P^+) \cup (n + C_j(\underline{P^-}))$, $\forall\ 1 \leq j \leq k$. (In order to make precise sense of the permutation ρ, we should first choose some 'canonical ordering' of the collection of distinguished classes for each element of S_k^n; we will say no more on this here, since we will elaborate on it later.)

We record a lemma (which is a consequence of the definitions) for later reference; we omit the simple proof.

Lemma 7. *Suppose $P_1 \cdot P_2 = d^l Q$, with $P_j, Q \in R_{2n}$ and $l \in \mathbb{Z}_+$.*

(a) *Assume that $t(P_1) = t(Q)$. Then, $\underline{P_1^+} = \underline{Q^+}$.*
(b) *Dually, if $t(P_2) = t(Q)$, then $\underline{P_2^-} = \underline{Q^-}$.*

After this paper was written up, the authors discovered that the algebras $A_n(d)$ have been extensively studied by Paul Martin–see [M, M1]; he calls them *partition algebras* and even discusses their representation theory in the 'non-generic case'. We, on the other hand, discuss only the case of 'generic d' when the algebras are semisimple, but we consider the 'equivariant case'. Specifically, for any finite group G, we consider an algebra $A_n^G(d)$–which has a basis of 'G-stable equivalence relations'–and show (Theorem 21) that these algebras are 'generically semisimple' and obtain (Theorem 33) the Bratteli diagram for the tower $\{A_n^G(d) : n \geq 1\}$.

2. *G*-relations

We shall now consider an 'equivariant version' of the above analysis. (We should perhaps mention that one of the reasons for our study of these algebras is the hope that we might be able to tie them up with the theory of 'planar algebras' developed by Jones (see [J1]); this will be discussed elsewhere.)

We begin with some notation. For a set X, we shall write $R(X)$ for the set of all equivalence relations on X–so that $R([n])$ is what was denoted by R_n in the last section. Suppose now that a group G acts on the set X; clearly then, we have a natural action of G on $R(X)$–given by $g \cdot R = \{(g \cdot x, g \cdot y) : (x, y) \in R\}$ whenever $R \in R(X)$ and $g \in G$. Call a relation G-*stable* if it is fixed by every element of G, and let $R^G(X)$ denote the set of all G-stable equivalence relations.

We shall only consider the case when G acts freely on X; when G and X are finite, the case we shall be concerned with, this amounts to assuming that $X = G \times \{1, 2, \ldots, n\}$ and that the action is defined by $g \cdot (x, i) = (g \cdot x, i)$. We shall denote this set by X_n in the sequel.

First consider the case $n = 1$. In the following lemma and elsewhere in this paper, the symbol $\coprod$ always denotes 'disjoint union'.

Lemma 8. (a) *If H is a subgroup of G, then the partition $G = \coprod g_i H$ of G into distinct left H-cosets yields a G-relation on X_1.* (b) *Every G-relation on X_1 arises as in* (a) *above.* (*Thus, there is a natural bijection between $R^G(X_1)$ and the set of subgroups of G.*)

Proof. (a) is clear; as for (b), let H denote the equivalence class of 1 (the identity of G) with respect to a G-stable relation. Suppose $h_1, h_2 \in H$; then, $1 \sim h_1 \Rightarrow h_1^{-1} \sim h_1^{-1}h_1 = 1$, and $h_1 h_2 \sim h_1 \sim 1$; so H is a subgroup. □

DEFINITION 9

Let $\mathcal{C} = \mathcal{C}(G)$ denote a collection of subgroups of G containing exactly one subgroup from each conjugacy class of subgroups; for each $H \in \mathcal{C}$, let $N(H)$ be the normaliser in G of H, and suppose

$$G = \coprod_{\kappa} N(H)\sigma_{\kappa}^{H}, \tag{2.3}$$

where we always assume that $\{\sigma_{\kappa}^{H} : 1 \leq \kappa \leq [G : N(H)]\}$ contains the identity element G. (Thus, we have chosen fixed coset-representatives for $N(H)\backslash G$, choosing the identity as the representative of the coset $N(H)$.)

Remark 10. (a) Suppose $P \in R^G(X_n)$. Then the equation

$$R^P = \{(i, j) : 1 \leq i, j \leq n, \exists g, h \in G \text{ such that } ((g, i), (h, j)) \in P\}$$

defines an element of $R([n])$; it follows from the definition that $[(g,i)]_P$ is a through-class of P if and only if $[i]_{R^P}$ is a through-class of R^P (provided n is even, so that this makes sense). (As above, we shall always use the notation $[i]_R$ to denote the R-equivalence class of the point i.) (b) For each $i \in [n]$, it follows from lemma 8 that there exists a unique subgroup, say $K_i(P)$, such that $((g,i),(h,i)) \in P \Leftrightarrow g \in hK_i(P)$. (c) If $((g,i),(h,j)) \in P$ are as in (a), then $gK_i(P)g^{-1} = hK_j(P)h^{-1}$, and in particular, the subgroups $K_i(P)$ and $K_j(P)$ are conjugate whenever $(i,j) \in R^P$. (Reason: fix $k_j \in K_j(P)$; then $[(1,i)]_P = [(g^{-1}h, j)]_P = [(g^{-1}hk_j, j)]_P$; similarly $[(1,j)]_P = [(h^{-1}g, i)]_P$, and hence $[(1,i)]_P = [(g^{-1}hk_jh^{-1}g, i)]_P$; i.e., $g^{-1}hK_j(P)h^{-1}g \subseteq K_i(P)$; the reverse inclusion follows identically.) (d) Thus, each $P \in R^G(X_n)$ determines a function

$$[n]/R^P \ni C \mapsto H_C^P \in \mathcal{C},$$

where H_C^P is the unique element of $\mathcal{C}$ which is conjugate to $K_i(P)\ \forall i \in C$. Further, for each $C \in [n]/R^P$, we shall consistently use the notation $i(C) = \min\{i : i \in C\}$.

PROPOSITION 11

Let $P \in R^G(X_n)$ and R^P be as above. Then,

(a) *there exists a unique function $\phi^P : [n] \to \coprod_{H\in\mathcal{C}} H\backslash G$, which satisfies the following conditions for all R^P-equivalence classes C:*

  (i) $\phi^P(i) \in H_C^P\backslash G\ \forall i \in C$;

  (ii) $\cup_{i\in C}(\phi^P(i) \times \{i\})$ *is a P-equivalence class*; (*here, we think of an element of $H\backslash G$ naturally as a subset of G*); *and*

  (iii) $\phi^P(i(C)) = H_C^P\sigma_C^P$, *where* $\sigma_C^P \in \{\sigma_\kappa^{H_C^P} : 1 \leq \kappa \leq [G : N(H_C^P)]\}$.

(b) *Conversely, suppose we are given (i) an $R \in R([n])$, (ii) a map $[n]/R \ni C \mapsto H_C \in \mathcal{C}$, and (iii) a map $\phi : [n] \to \coprod_{H\in\mathcal{C}} H\backslash G$, which satisfy*:

  (i)′ $\phi(i) \in H_C\backslash G$, *whenever i belongs to the R-equivalence class C*; *and*

  (iii)′ $\phi(i(C)) = H_C\sigma_C$, *where* $\sigma_C \in \{\sigma_\kappa^{H_C} : 1 \leq \kappa \leq [G : N(H_C)]\}\forall C$.

  Then, there exists a unique $P \in R^G(X_n)$ such that $R^P = R, H_C = H_C^P\ \forall C$, and $\phi^P = \phi$. Further, this relation P is defined by

$$((g,i),(h,j))) \in P \Leftrightarrow (i,j) \in R \text{ and } \phi(i)g^{-1} = \phi(j)h^{-1}. \tag{2.4}$$

(c) *If ψ^P is another function defined on $[n]$ and satisfying (a) (i), (ii), then for each R^P-equivalence class C, there exists a unique element $\omega_C^P \in N(H_C)/H_C$ such that $\psi^P(i) = \omega_C^P\phi^P(i)\ \forall i \in C$.*

Proof. (a) We first discuss uniqueness. Suppose we are given a function ϕ^P satisfying conditions (a) (i)–(iii). These conditions, and the definition of $K_i(P)$ shows that if $[i]_{R^P} = C$, then $\phi^P(i)$ is a left-coset of $K_i(P)$ as well as a right-coset of H_C^P. Suppose $\phi^P(i) = H_C^Pg = g_1K_i(P)$; then clearly $g^{-1}H_C^Pg = (\phi^P(i))^{-1}\phi^P(i) = K_i(P)$. In particular, this is true for $i = i(C)$, and since the σ_i^H are representatives of the distinct right-cosets of $N(H)$, it follows that there exists a unique σ_C^P satisfying condition (iii). Now if we define $D = [(\sigma_C^P, i(C))]_P$, we see from condition (ii) that for each $i \in C$, we must have $\phi^P(i) \times \{i\} = D \cap (G \times \{i\})$. This proves that the function ϕ^P is uniquely determined by the conditions (i)–(iii).

For existence, let us define σ_C^P and ϕ^P by the prescription forced by the discussion of last paragraph. We only need to verify that $\phi^P(i)$ is a right-coset of H_C. What is clear from the definition is that $\phi^P(i)$ is a left $K_i(P)$-coset; on the other hand, notice that $\phi^P(i(C))$ is invariant under the action of H_C^P, and that this is necessarily true also of D, and hence of $\phi^P(i)$, for each $i \in P$; thus, $\phi^P(i)$ is a left $K_i(P)$-coset, as also a union of right H_C^P-cosets; for reasons of cardinality, this forces $\phi^P(i)$ to be exactly one right H_C^P-coset, as desired. This proves existence.

(b) If the data of (b) (i)–(iii) satisfies (i)′, (iii)′, then equation (2.4) defines a G-stable equivalence relation. (Reason: the P-equivalence classes are just the 'sets of constancy' for the function $(g, i) \mapsto ([i]_R, H_{[i]_R} g^{-1})$.) The definition of P and of R^P implies that $R^P \subseteq R$; conversely, suppose $(i, j) \in R$; let $C = [i]_R = [j]_R$; since G acts transitively on $H_C \backslash G$, we can find $g \in G$ such that $\phi(i) g^{-1} = \phi(j)$; hence $((g, i), (1, j)) \in P$; this implies that $(i, j) \in R^P$. Thus, indeed $R = R^P$.

Let C be an R^P-equivalence class. By condition (iii)′, we have $\phi(i(C))\sigma_C^{-1} = H_C$, and hence, by definition of P,

$$\begin{aligned} ((\sigma_C, i(C)), (g, j)) \in P &\Leftrightarrow j \in C \text{ and } H_C = \phi(j) g^{-1} \\ &\Leftrightarrow j \in C \text{ and } \phi(j) = H_C g \\ &\Leftrightarrow j \in C \text{ and } g \in \phi(j), \end{aligned}$$

(since $\phi(j)$ is given to be a right-coset of H_C for $j \in C$). Hence ϕ also satisfies:

(ii)′ $D = \cup_{i \in C}(\phi(i) \times \{i\})$ is a P-equivalence class.

Then, for any $i \in C$, it follows from the definition of $K_i(P)$ that $\phi(i)$ is a left-coset of $K_i(P)$ as well as a right-coset of H_C; this means that the subgroups $K_i(P)$ and H_C are conjugate whenever $i \in C$. Thus, we see that $H_C^P = H_C \ \forall C$.

So, the function ϕ satisfies the conditions (a) (i)–(iii), and we deduce from the uniqueness assertion of (a) that $\phi = \phi^P$.

(c) If we set $D = \cup_{i \in C}(\psi^P(i) \times \{i\})$, we see as in the proof of (b) above that if C is any R^P-equivalence class, then $\psi^P(i(C))$ is a left-coset of $K_{i(C)}(P)$ as well as a right-coset of H_C^P; if $\psi^P(i(C)) = H_C^P g$, this means that $K_{i(C)} = g^{-1} H_C^P g$. We already know that $K_{i(C)} = (\sigma_C^P)^{-1} H_C^P \sigma_C^P$. This means that $g(\sigma_C^P)^{-1} \in N(H_C^P)$ and hence there exists a unique element $\omega_C^P \in N(H_C^P)$ such that $g = \omega_C^P \sigma_C^P$. The definitions show that

$$\psi^P(i(C)) = H_C^P g = H_C^P \omega_C^P \sigma_C^P = \omega_C^P H_C^P \sigma_C^P = \omega_C^P \phi^P(i(C)).$$

It is now easy to verify that the function defined by $\phi(i) = (\omega_{[i]_{R^P}}^P)^{-1} \psi^P(i)$ satisfies the three conditions (a) (i)–(iii), and an appeal to the uniqueness assertion of (a) completes the proof. □

Notice now that for any positive integer n, we may regard $R^G(X_n)$ as a subset of $R([n|G|])$; furthermore, if $P, Q \in R^G(X_{2n})$, and if $P \cdot Q = d^l S$, where the product is computed as in the algebra $A_{n|G|}(d)$, then it is easy to see that S corresponds to a G-stable equivalence relation on X_{2n}. Thus, the linear span of $R^G(X_{2n})$ is a subalgebra of $A_{2n|G|}(d)$.

DEFINITION 12

Let $A_n^G(d)$ denote the (finite-dimensional) algebra, with basis $R^G(X_{2n})$, obtained as above.

Remark 13. Let $P \in R^G(X_{2n})$; since G acts transitively on each $G \times \{j\}$, it is seen that if C is any through-class of R^P, then $G \times C$ is the disjoint union of $[G : H_C^P]$ many P-equivalence classes; and, as C varies over the through-classes of R^P, these exhaust all the through-classes of P; hence, if $t(P) = k$, then

$$k = \sum [G : H_C^P], \tag{2.5}$$

where the sum is over all through-classes C of R^P; in particular, if $t = t(R^P)$, then,

$$t \leq k \leq t|G|.$$

DEFINITION 14

(a) For $0 \leq k \leq n|G|$, define I_k^G to be the linear subspace of $A_n^G(d)$ spanned by $\{P \in R^G(X_{2n}) : t(P) \leq k\}$.
(b) If $P \in R^G(X_{2n})$, define $n^P : \mathcal{C} \to \mathbb{Z}_+ (= \{0, 1, \cdots\})$ by $n_P(H) = \#\{C : C$ is a through-class of R^P such that $H_C^P = H\}$ for all $H \in \mathcal{C}$.
(c) For $0 \leq k \leq n|G|$, let N_k denote the set of functions $\overline{n} : \mathcal{C} \to \mathbb{Z}_+$ which satisfy the conditions $\sum_H \overline{n}(H) \leq n$, and $k = \sum_H \overline{n}(H)[G : H]$. (Later, when we wish to vary n, we shall denote this object by the symbol $N_{n;k}$, since the definition also involves the inequality depending upon n.)
Let $N_{[n]} = \cup_{k=0}^{n|G|} N_k$.
(d) For arbitrary $\overline{n} \in N_{[n]}$, define $I(\overline{n}) = \{P \in R^G(X_{2n}) : n_P = \overline{n}\}$.

Thus, as in Corollary 5, it is true that $\{I_k^G : 0 \leq k \leq n|G|\}$ is a filtration of $A_n^G(d)$ by two-sided ideals.

Lemma 15. *For $0 \leq k \leq n|G|$ and arbitrary $\overline{n} \in N_k$, let $Q(\overline{n})$ denote the linear subspace spanned by $\pi(I(\overline{n}))$, where $\pi : I_k^G \to I_k^G / I_{k-1}^G$ is the quotient map; then, $Q(\overline{n})$ is an ideal in I_k^G / I_{k-1}^G, and further,*

$$I_k^G / I_{k-1}^G = \oplus_{\overline{n} \in N_k} Q(\overline{n}).$$

Proof. The lemma is a tautology when $k = 0$, so we may assume $k > 0$.

It should be clear that it is sufficient to prove that if $P_j \in I(\overline{n}_j), \overline{n}_j \in N_k, j = 1, 2$, if $P_1 \cdot P_2 = d^l Q$ in $A_n^G(d)$, and if $t(Q) = k$, then $\overline{n}_1 = \overline{n}_2 = n_Q$.

In view of Lemma 7, it suffices to observe that n_P is uniquely determined by $\underline{P^+}$ as well as by $\underline{P^-}$ – and this follows easily from the definitions. □

In order to arrive at a 'working description' of elements of these ideals, we shall first obtain an alternative way of encoding the 'tops' of elements $P \in R^G(X_{2n})$. On the one hand, we can forget that P is G-stable and represent the 'top' and 'bottom' of P, and just look at what we denoted by $\underline{P^\pm}$ at the end of § 1. Thus, for instance $\underline{P^+}$ is just the data P^+ of the equivalence relation obtained by restricting P to the top (i.e., $G \times [n]$), together with the data of which P^+-equivalence classes are contained in through-classes of P.

We wish to bring in the knowledge of G-invariance of P to encode this data differently. For this, the starting point is the observation – see Remark 13 – that through-classes of P are intimately tied with through-classes of R^P. We begin by trying to list the elements of the latter collection in a 'canonical order'.

If $n_P = \overline{n}$, and if $H \in \mathcal{C}$, then there exist $\overline{n}(H)$ many 'distinguished' R^{P^+}-equivalence classes C^+ for which $H_{C^+}^{P^+} = H$; let $\{C_{H,s}(\mathbf{P}^+) : 1 \leq s \leq \overline{n}(H)\}$ be the unique listing of these classes which satisfies

$$s < s' \Rightarrow i(C_{H,s}(\mathbf{P}^+)) < i(C_{H,s'}(\mathbf{P}^+)). \tag{2.6}$$

DEFINITION 16

For $\overline{n} \in N_{[n]}$, define $S(\overline{n})$ to be the collection of all symbols $\mathbf{P}^+ = (P^+; \{C_{H,s}(\mathbf{P}^+): 1 \leq s \leq \overline{n}(H), H \in \mathcal{C}\})$, where $P^+ \in R^G(X_n)$, and $\{C_{H,s}(\mathbf{P}^+) : 1 \leq s \leq \overline{n}(H), H \in \mathcal{C}\})$ is a collection of 'distinguished' R^{P^+}-equivalence classes such that (i) $H^{P^+}_{C_{H,s}(\mathbf{P}^+)} = H$ for all H, s, and (ii) the condition (2.6) is satisfied.

Thus, if $P \in R^G(X_{2n})$, and if $n_P = \overline{n}$, then the 'top' (resp., the 'bottom') of P determines an element $\mathbf{P}^+$ (resp., $\mathbf{P}^-$) of $S(\overline{n})$. Conversely, this $\mathbf{P}^+$ uniquely determines all the 'distinguished' classes of what we earlier called $\underline{P^+}$, since a P^+-equivalence class, say D^+, is contained in a through-class for P if and only if there exists a through-class, say C, of R^P such that $D^+ \subset (G \times C)$. Thus, what we have called $\mathbf{P}^+$ is nothing but another way of encoding what was earlier called $\underline{P^+}$ in case P is G-stable. Thus, in future, we shall freely use such expressions as 'let $\mathbf{P}^{\pm}$ denote the 'top' and 'bottom' of $P \in R^G(X_{2n})$'.

Lemma 17. *There exists a bijection*

$$I(\overline{n}) \ni P \stackrel{\zeta}{\mapsto} (\mathbf{P}^+, \rho(P), \mathbf{P}^-) \in S(\overline{n}) \times G(\overline{n}) \times S(\overline{n}),$$

where (i) $G(\overline{n}) = \prod_{H\in\mathcal{C}}((N(H)/H)^{\overline{n}(H)} \rtimes \Sigma_{\overline{n}(H)})$ *is the product (over the H's) of semi-direct-products (with respect to the natural permutation action of the second factor on the first), and (ii)* $\mathbf{P}^{\pm}$ *denote the 'top' and 'bottom' of P.*

Proof. Fix a $P \in I(\overline{n})$. For $Q \in \{P, P^+, P^-\}$, let ϕ^Q be the function associated to Q as in Proposition 11. By considering the through-classes of R^P, it is not hard to see that, for each fixed $H \in \mathcal{C}$, there is a unique permutation $\gamma_H \in \Sigma_{\overline{n}(H)}$ such that $\{(n+C_{H,s}(\mathbf{P}^-)) \cup C_{H,\gamma_H(s)}(\mathbf{P}^+) : 1 \leq s \leq \overline{n}(H)\}$ is precisely the collection of those R^P-through classes C for which $H^P_C = H$.

Notice next that the function defined on $[n]$ by $\psi^{P^-}(j) = \phi^P(n+j)$, satisfies the conditions of Proposition 11(c) (with P^- in place of the P there). Hence, by that proposition, for each R^{P^-}-equivalence class C^-, there exists a unique element $\omega^{P^-}_{C^-} \in N(H^{P^-}_{C^-})/H^{P^-}_{C^-}$ such that $\psi^{P^-}(j) = \omega^{P^-}_{C^-}\phi^{P^-}(j) \forall j \in C^-$. Set $\omega^H_s = \omega^{P^-}_{C_{H,\gamma_H^{-1}(s)}(\mathbf{P}^-)}$, for $1 \leq s \leq \overline{n}(H), H \in \mathcal{C}$.

Now define $\rho(P) = ((\rho(P)_H))_{H\in\mathcal{C}}$, where $\rho(P)_H \in (N(H)/H)^{\overline{n}(H)} \times \Sigma_{\overline{n}(H)}$ is defined by

$$\rho(P)_H = ((\omega^H_1, \ldots, \omega^H_{\overline{n}(H)}), \gamma_H).$$

Thus, we have defined the map ζ.

Conversely, suppose the triple $(\mathbf{P}^+, \rho(P), \mathbf{P}^-)$ is given, and suppose $\rho(P) = ((\rho(P)_H))_{H\in\mathcal{C}}$, where $\rho(P)_H = ((\omega^H_1, \ldots, \omega^H_{\overline{n}(H)}), \gamma_H)$. Then define:

(i) a relation $R \in R([2n])$ by demanding that its equivalence classes are: (a) the R^{P^+}-equivalence classes other than the $C_{H,s}(\mathbf{P}^+)$'s; (b) sets of the form $(n+C)$, where C is an R^{P^-}-equivalence class other than the $C_{H,s}(\mathbf{P}^-)$'s; and (c) $\{(n+C_{H,s}(\mathbf{P}^-)) \cup C_{H,\gamma_H(s)}(\mathbf{P}^+)) : 1 \leq s \leq \overline{n}(H), H \in \mathcal{C}\}$;

(ii) a map $[2n]/R \to \mathcal{C}$ by setting H_C to be equal to: (a) $H^{P^+}_C$, if C is an R^{P^+}-equivalence class other than the $C_{H,s}(\mathbf{P}^+)$'s; (b) $H^{P^-}_{C-n}$, if $(C-n)$ is an R^{P^-}-equivalence class other than the $C_{H,s}(\mathbf{P}^-)$'s; and (c) H if $C = (n+C_{H,s}(\mathbf{P}^-)) \cup C_{H,\gamma_H(s)}(\mathbf{P}^+)$ for some H, s; and

(iii) a map $\phi : [2n] \to \coprod_{H \in \mathcal{C}} H\backslash G$ by setting

$$\phi(k) = \begin{cases} \phi^{P^+}(k) & \text{if} \quad k \leq n \\ \phi^{P^-}(k) & \text{if} \quad k > n \text{ and } k-n \notin \cup_{H,s} C_{H,s}(\mathbf{P}^-) \\ \omega^H_{\gamma_H(s)} \phi^{P^-}(k-n) & \text{if} \quad k > n \text{ and } k-n \in C_{H,s}(\mathbf{P}^-) \end{cases} .$$

The data (i)–(iii) above satisfy the conditions (b) (i)′ and (iii)′ of Proposition 11 (with $2n$ instead of the n of the proposition) and therefore determine a unique $P \in R^G(X_{2n})$. It is easy to see that $P \in I(\overline{n})$. Set $\eta((\mathbf{P}^+, \rho(P), \mathbf{P}^-)) = P$.

The proof of the lemma is completed by verifying that the maps ζ and η are inverse to one another. □

In view of the above lemma, we shall feel free, in the sequel, to think of elements of $S(\overline{n}) \times G(\overline{n}) \times S(\overline{n})$ as elements of $I(\overline{n})$, and vice versa.

3. The structure of $A_n^G(d)$

We come now to the representation theory of $A_n^G(d)$.

PROPOSITION 18

Fix $0 \leq k \leq n|G|$ and $\overline{n} \in N_k$. Let $V(\overline{n})$ denote the $\mathbb{C}$-vector space with $S(\overline{n}) \times G(\overline{n})$ as basis.

(a) *The following prescription uniquely defines a representation $\pi_{(\overline{n})}$ of $A_n^G(d)$ on $V(\overline{n})$: temporarily fix an element $\mathbf{S}_0 \in S(\overline{n})$; let $P \in R^G(X_{2n})$, and $(\mathbf{S}, \sigma) \in S(\overline{n}) \times G(\overline{n})$, and suppose $P \cdot (\mathbf{S}, \sigma, \mathbf{S}_0) = d^l Q$ in the algebra $A_n^G(d)$; consider two cases now*:

  (i) *if $t(Q) = k$, then $Q \in I(\overline{n})$ and $Q = (\mathbf{S}_1, \sigma_1, \mathbf{S}_0)$ for a unique pair $(\mathbf{S}_1, \sigma_1) \in S(\overline{n}) \times G(\overline{n})$; in this case, define $\pi_{(\overline{n})}(P)(\mathbf{S}, \sigma) = d^l(\mathbf{S}_1, \sigma_1)$;*

  (ii) *if $Q \in I_{k-1}^G$, define $\pi_{(\overline{n})}(P)(\mathbf{S}, \sigma) = 0$.*

(b) *Let $P \in I(\overline{n})$ and $(\mathbf{S}, \sigma) \in S(\overline{n}) \times G(\overline{n})$. Suppose $P = (\mathbf{P}^+, \rho, \mathbf{P}^-)$. Then,*

$$\pi_{(\overline{n})}(P)(\mathbf{S}, \sigma) = D(\mathbf{P}^-, \mathbf{S})(\mathbf{P}^+, \rho\beta_{\mathbf{S}}^{\mathbf{P}^-}\sigma), \tag{3.7}$$

where the quantities $D(\mathbf{P}^-, \mathbf{S})$ and $\beta_{\mathbf{S}}^{\mathbf{P}^-}$ are most easily defined by considering two cases:

Case (i): *For each $H \in \mathcal{C}$ such that $\overline{n}(H) \neq 0$, there exist distinct $(R^{P^-} \vee R^S =)R^{P^- \vee S}$-equivalence classes, say $C_{H,s}, 1 \leq s \leq \overline{n}(H)$, and a (necessarily unique) permutation $\gamma_H \in \Sigma_{\overline{n}(H)}$ such that $C_{H,s}(\mathbf{S}) \cup C_{H,\gamma_H(s)}(\mathbf{P}^-) \subset C_{H,s}$ for each $1 \leq s \leq \overline{n}(H)$.*

In this case, define $D(\mathbf{P}^-, \mathbf{S}) = d^{||P^- \vee S|| - k}$, while $\beta_{\mathbf{S}}^{\mathbf{P}^-}$ is defined by the equation

$$(\mathbf{P}^-, 1, \mathbf{P}^-) \cdot (\mathbf{S}, 1, \mathbf{S}) = D(\mathbf{P}^-, \mathbf{S})(\mathbf{P}^-, \beta_{\mathbf{S}}^{\mathbf{P}^-}, \mathbf{S}). \tag{3.8}$$

Case (ii): *Suppose the conditions of Case* (i) *are not satisfied.*
In this case, define $D(\mathbf{P}^-, \mathbf{S}) = 0$ and $\beta_{\mathbf{S}}^{\mathbf{P}^-} = 1$.

Proof. (a) We only need to verify that $\pi_{(\overline{n})}(P_1 \cdot P_2) = \pi_{(\overline{n})}(P_1)\pi_{(\overline{n})}(P_2)$ for all $P_1, P_2 \in R^G(X_{2n})$. Suppose that $(\mathbf{S}, \sigma) \in S(\overline{n}) \times G(\overline{n})$, and $(P_1 \cdot P_2) \cdot (\mathbf{S}, \sigma, \mathbf{S}_0) = d^l Q$. Suppose $P_2 \cdot (\mathbf{S}, \sigma, \mathbf{S}_0) = d^{l_2} Q_2$.

First suppose $t(Q) = k$. It follows that also $t(Q_2) = k$. Deduce now from lemma 7(b) that $Q_2 = (\mathbf{S}_2, \sigma_2, \mathbf{S}_0)$ for some $(\mathbf{S}_2, \sigma_2)$, and that $Q_2 \in I(\overline{n})$. It is also seen – from the associativity of multiplication in $A_n^G(d)$ – that $P_1 \cdot Q_2 = d^{l-l_2}Q$; deduce, as before, that $Q \in I(\overline{n})$ and that $Q = (\mathbf{S}_1, \sigma_1, \mathbf{S}_0)$ for some $(\mathbf{S}_1, \sigma_1)$. Hence, we see that $\pi_{(\overline{n})}(P_1 \cdot P_2)(\mathbf{S}, \sigma) = d^l(\mathbf{S}_1, \sigma_1)$, while

$$\begin{aligned}\pi_{(\overline{n})}(P_1)\pi_{(\overline{n})}(P_2)(\mathbf{S}, \sigma) &= d^{l_2}\pi_{(\overline{n})}(P_1)(\mathbf{S}_2, \sigma_2)\\ &= d^l(\mathbf{S}_1, \sigma_1)\\ &= \pi_{(\overline{n})}(P_1 \cdot P_2)(\mathbf{S}, \sigma),\end{aligned}$$

as desired.

Next, suppose $Q \in I_{k-1}^G$, so that $\pi_{(\overline{n})}(P_1 \cdot P_2)(\mathbf{S}, \sigma) = 0$; then it must be the case that either (i) $Q_2 \in I_{k-1}^G$ or (ii) $Q_2 \in I(\overline{n})$, $Q_2 = (\mathbf{S}_2, \sigma_2, \mathbf{S}_0)$ for some $(\mathbf{S}_2, \sigma_2)$, and $P_1 \cdot (\mathbf{S}_2, \sigma_2, \mathbf{S}_0) \in I_{k-1}^G$. In either case, we have $\pi_{(\overline{n})}(P_1)\pi_{(\overline{n})}(P_2)(\mathbf{S}, \sigma) = 0$.

(b) If $P \in I(\overline{n})$ and $(\mathbf{S}, \sigma) \in S(\overline{n}) \times G(\overline{n})$, it is not hard to see that the following conditions are equivalent:

(α) The conditions of case (i) of (b) are satisfied;
(β) If $P \cdot (\mathbf{S}, \sigma, \mathbf{S}_0) = d^lQ$ in $A_n^G(d)$, then $t(Q) = k$;
(γ) $D(\mathbf{P}^-, \mathbf{S}) \neq 0$.

It is clearly enough to prove that eq. (3.7) is satisfied when the three equivalent conditions above are satisfied. If $P \in I(\overline{n})$ and $(\mathbf{S}, \sigma) \in S(\overline{n}) \times G(\overline{n})$, we thus need to verify (under the stated assumptions above) that

$$(\mathbf{P}^+, \rho, \mathbf{P}^-) \cdot (\mathbf{S}, \sigma, \mathbf{S}_0) = D(\mathbf{P}^-, \mathbf{S})(\mathbf{P}^+, \rho\beta_{\mathbf{S}}^{\mathbf{P}^-}\sigma, \mathbf{S}_0),$$

which we shall do, by considering several special cases.

Case 1: $\mathbf{P}^- = \mathbf{S}$ and $\sigma = 1$.

It is seen from the definition of the product in $A_n^G(d)$ that

$$(\mathbf{P}^+, \rho, \mathbf{S}) \cdot (\mathbf{S}, 1, \mathbf{S}_0) = D(\mathbf{S}, \mathbf{S})(\mathbf{P}^+, \rho, \mathbf{S}_0), \tag{3.9}$$

and eq. (3.7) is satisfied in this case, since (3.8) and the same reasoning, that goes in to justify (3.9), shows that $\beta_{\mathbf{S}}^{\mathbf{S}} = 1$.

We note for future reference that, in the same way, we obtain, for arbitrary $\mathbf{S}_1, \mathbf{S}_2 \in S(\overline{n})$ and $\sigma \in G(\overline{n})$:

$$\begin{aligned}(\mathbf{S}_1, \sigma, \mathbf{S}_2) &= \frac{1}{D(\mathbf{S}_1, \mathbf{S}_1)}(\mathbf{S}_1, 1, \mathbf{S}_1) \cdot (\mathbf{S}_1, \sigma, \mathbf{S}_2)\\ &= \frac{1}{D(\mathbf{S}_2, \mathbf{S}_2)}(\mathbf{S}_1, \sigma, \mathbf{S}_2) \cdot (\mathbf{S}_2, 1, \mathbf{S}_2).\end{aligned} \tag{3.10}$$

Case 2: $\mathbf{P}^- = \mathbf{S}$ and σ is arbitrary.

Thus, we have to verify that

$$(\mathbf{P}^+, \rho, \mathbf{P}^-) \cdot (\mathbf{P}^-, \sigma, \mathbf{S}_0) = D(\mathbf{P}^-, \mathbf{P}^-)(\mathbf{P}^+, \rho\sigma, \mathbf{S}_0), \tag{3.11}$$

and this is really the heart of the computation.

Let us write $P_1 = (\mathbf{P}^-, \sigma, \mathbf{S}_0)$ and $P \cdot P_1 = D(\mathbf{P}^-, \mathbf{P}^-)Q$. Since we are assuming that the conditions $(\alpha) - (\gamma)$ are satisfied, we know from (a) that $Q \in I(\overline{n})$. Suppose

$Q = (\mathbf{Q}^+, \phi, \mathbf{Q}^-)$. We know from Lemma 7 ((a) and (b)) that $\mathbf{Q}^+ = \mathbf{P}^+, \mathbf{Q}^- = \mathbf{S}_0$. Thus, we only need to show that $\phi = \rho\sigma$.

Suppose $\rho = ((\rho_H))_{H\in\mathcal{C}}$, where $\rho_H = ((\omega_1^H, \ldots, \omega_{\overline{n}(H)}^H), \gamma_H)$; and that similarly, $\sigma = ((\sigma_H))_{H\in\mathcal{C}}$, where $\sigma_H = ((\nu_1^H, \ldots, \nu_{\overline{n}(H)}^H), \kappa_H)$: thus, for each $H \in \mathcal{C}$, we have $\omega_s^H, \nu_s^H \in N(H)/H, 1 \leq s \leq \overline{n}(H)$, and $\gamma_H, \kappa_H \in \Sigma_{\overline{n}(H)}$.

The construction in the proof of Proposition 17, when unravelled, says that the group element ρ is related to the relation $P \in R^G(X_{2n})$ by the following requirement, and that ρ is determined by this requirement:

For all $H \in \mathcal{C}, 1 \leq s \leq \overline{n}(H)$, we have:

$$[(\sigma^{P^+}_{C_{H,s}(P^+)}, i(C_{H,s}(P^+)))]_P \supset (\omega_s^H \sigma^{P^-}_{C_{H,\gamma_H^{-1}(s)}(P^-)} \times \{i(C_{H,\gamma_H^{-1}(s)}(P^-))\}).$$

Similarly, we see that for all $H \in \mathcal{C}, 1 \leq s \leq \overline{n}(H)$:

$$[(\sigma^{P^-}_{C_{H,t}(P^-)}, i(C_{H,t}(P^-)))]_{P_1} \supset (\nu_t^H \sigma^{S_0}_{C_{H,\kappa_H^{-1}(t)}(S_0)} \times \{i(C_{H,\kappa_H^{-1}(t)}(S_0))\}).$$

Now, set $t = \gamma_H^{-1}(s)$ in the last inclusion, and use the G-invariance of the relation P_1 to deduce that for all H and s, we have:

$$\begin{aligned}&[(\omega_s^H \sigma^{P^-}_{C_{H,\gamma_H^{-1}(s)}(P^-)} \times i(C_{H,\gamma_H^{-1}(s)}(P^-)))]_{P_1}\\ &\quad \supseteq (\omega_s^H \nu^H_{\gamma_H^{-1}(s)} \sigma^{S_0}_{C_{H,\kappa_H^{-1}(\gamma_H^{-1}(s))}(S_0)} \times \{i(C_{H,\kappa_H^{-1}(\gamma_H^{-1}(s))}(S_0))\}).\end{aligned}$$

Hence, we see that for all H, s, we have:

$$\begin{aligned}&[(\sigma^{P^+}_{C_{H,s}(P^+)}, i(C_{H,s}(P^+)))]_Q\\ &\quad \supset (\omega_s^H \nu^H_{\gamma_H^{-1}(s)} \sigma^{Q^-}_{C_{H,\kappa_H^{-1}(\gamma_H^{-1}(s))}(Q^-)} \times \{i(C_{H,(\gamma_H\kappa_H)^{-1}(s)}(Q^-))\}).\end{aligned}$$

Since this property determines the group element ϕ, we see that $\phi = ((\phi_H))$, with $\phi_H = ((\chi_1^H, \ldots, \chi_{\overline{n}(H)}^H), \lambda_H)$, where $\chi_s^H = \omega_s^H \nu^H_{\gamma_H^{-1}(s)}$ and $\lambda_H = \gamma_H\kappa_H$; in other words, $\phi_H = \rho_H\sigma_H$, the product being computed in the semi-direct product. (This is the reason for introducing the semi-direct products.)

Case 3: $\mathbf{P}^-, \rho, \mathbf{S}, \sigma$ arbitrary.

Compute as follows:

$$\begin{aligned}(\mathbf{P}^+, \rho, \mathbf{P}^-) \cdot (\mathbf{S}, \sigma, \mathbf{S}_0) &= \frac{(\mathbf{P}^+, \rho, \mathbf{P}^-) \cdot (\mathbf{P}^-, 1, \mathbf{P}^-) \cdot (\mathbf{S}, 1, \mathbf{S}) \cdot (\mathbf{S}, \sigma, \mathbf{S}_0)}{D(\mathbf{P}^-, \mathbf{P}^-)D(\mathbf{S}, \mathbf{S})}\\ &= \frac{D(\mathbf{P}^-, \mathbf{S})(\mathbf{P}^+, \rho, \mathbf{P}^-) \cdot (\mathbf{P}^-, \beta_{\mathbf{S}}^{\mathbf{P}^-}, \mathbf{S}) \cdot (\mathbf{S}, \sigma, \mathbf{S}_0)}{D(\mathbf{P}^-, \mathbf{P}^-)D(\mathbf{S}, \mathbf{S})}\\ &= D(\mathbf{P}^-, \mathbf{S})(\mathbf{P}^+, \rho\beta_{\mathbf{S}}^{\mathbf{P}^-}\sigma, \mathbf{S}_0),\end{aligned}$$

where we have used both the equations (3.10) in the first step, the definition of β (see equation (3.8)) in the second step, and equation (3.11) twice in the last step. □

The next lemma is needed to ensure that that the algebra $A_n^G(d)$ is semisimple at all but a finite number of values of d.

Lemma 19. *Let $C = ((c^i_j))$ be a square matrix and suppose $c^i_j = d^{n^i_j}$, where d is a complex parameter, and the matrix $((n^i_j))$ satisfies the following conditions*:

(i) $n^i_j \in \{-\infty, 0, 1, 2, \ldots\}$,
(ii) $n^i_i \geq \max\{0, n^i_j\}\ \forall i, j$; *and*
(iii) *if $i \neq j$ and $n^i_j = n^i_i$, then $n^i_i < n^j_j$.*

Then det *C is a monic polynomial in d; in particular, the matrix C is non-singular when we substitute all but finitely many possible complex numbers for the parameter d.*

Proof. We shall show that the monomial in d obtained as the 'diagonal product' of C corresponding to any permutation σ which is distinct from the identity permutation, has degree strictly smaller than the degree of the 'main diagonal product' (which corresponds to the identity permutation).

Since any such σ is expressible as a product of disjoint cycles, and since we have assumed that $n^i_i \geq 0$ (so that there is no problem of multiplying by 0), it is enough to (consider the case when σ is just a cycle, and) prove that if $i, j, k, \ldots, r, s$ is a collection of (two or more) distinct indices, then

$$(n^i_j + n^j_k + \cdots + n^r_s + n^s_i) < (n^i_i + n^j_j + \cdots + n^r_r + n^s_s). \tag{3.12}$$

However, we have termwise inequalities:

$$n^i_j \leq n^i_i,\ n^j_k \leq n^j_j, \ldots, n^r_s \leq n^r_r,\ n^s_i \leq n^s_s. \tag{3.13}$$

Since the hypothesis (ii) guarantees that the right side of (3.12) is a finite quantity (i.e., not equal to $-\infty$), the only way that the inequality (3.12) can fail to hold is that each of the inequalities in (3.13) is actually an equality; in that case, the assumption (iii) will imply that $n^i_i < n^j_j < \cdots < n^r_r < n^s_s < n^i_i$. This contradiction completes the proof of the lemma. □

PROPOSITION 20

Let $\overline{n} \in N_k$. The equation $(\Gamma(\tau))(\mathbf{R}, \rho) = (\mathbf{R}, \rho\tau^{-1})$ defines a representation Γ of $G(\overline{n})$ on $V(\overline{n})$. Let $\pi_{(\overline{n})}$ be the representation of $A^G_n(d)$ described in Proposition 18. *Then*,

(i) $\pi_{(\overline{n})}(A^G_n(d)) \subset \Gamma(G(\overline{n}))'$.
(ii) *Consider the matrix C with rows and columns indexed by $S(\overline{n}) \times G(\overline{n})$, defined – using the notation of Proposition* 18*(b) – by*

$$C((\mathbf{R}, \rho), (\mathbf{S}, \sigma)) = \delta_{\sigma, \rho\beta^{\mathbf{R}}_{\mathbf{S}}} D(\mathbf{R}, \mathbf{S}). \tag{3.14}$$

Then the matrix C satisfies the hypothesis of Lemma 19; *and if d is such that the matrix C is invertible*, *then*

$$\pi_{(\overline{n})}(A^G_n(d)) = \pi_{(\overline{n})}(\text{span } I(\overline{n})) = \Gamma(G(\overline{n}))'.$$

Proof. (i) Note that $\Gamma(\tau)(\mathbf{R}, \rho) = 1/D(\mathbf{S}_0, \mathbf{S}_0)\pi_{(\overline{n})}(\mathbf{R}, \rho, \mathbf{S}_0)(\mathbf{S}_0, \tau^{-1})$, for each $\tau \in G(\overline{n})$, and $(\mathbf{R}, \rho) \in S(\overline{n}) \times G(\overline{n})$; assertion (i) of the proposition is a consequence of the fact that 'left multiplication' commutes with 'right multiplication'. (ii) It is clear that $C((\mathbf{R}, \rho), (\mathbf{S}, \sigma)) = d^{N((\mathbf{R}, \rho), (\mathbf{S}, \sigma))}$, where N is the matrix defined by

$$N((\mathbf{R}, \rho), (\mathbf{S}, \sigma)) = \begin{cases} ||R \vee S|| - k & \text{if } D(\mathbf{R}, \mathbf{S}) \neq 0 \text{ and } \sigma = \rho\beta^{\mathbf{R}}_{\mathbf{S}} \\ -\infty & \text{otherwise} \end{cases}.$$

Notice first that $\beta^{\mathbf{R}}_{\mathbf{R}} = 1$, and that consequently,

$$N((\mathbf{R},\rho),(\mathbf{R},\rho)) = ||R|| - k \geq \max\{0, N((\mathbf{R},\rho),(\mathbf{S},\sigma))\}\ \forall(\mathbf{R},\rho),(\mathbf{S},\sigma);$$

thus N satisfies conditions (i) and (ii) of lemma 19.

Next, suppose $N((\mathbf{R},\rho),(\mathbf{R},\rho)) = N((\mathbf{R},\rho),(\mathbf{S},\sigma))$ for some $(\mathbf{R},\rho) \neq (\mathbf{S},\sigma)$. In particular, this means that the right side is not equal to $-\infty$, and hence, $D(\mathbf{R},\mathbf{S}) \neq 0$, $\sigma = \rho\beta^{\mathbf{R}}_{\mathbf{S}}$, and $||R \vee S|| = ||R||$. It follows that $R \vee S = R$, i.e., $S \leq R$.

Suppose, if possible, that $R = S$. The condition $D(\mathbf{R},\mathbf{S}) \neq 0$ is then seen to imply that $\mathbf{R} = \mathbf{S}$; then the condition $\sigma = \rho\beta^{\mathbf{R}}_{\mathbf{S}}$ is seen to imply (since $\beta^{\mathbf{R}}_{\mathbf{R}} = 1$) that $\sigma = \rho$; in other words, $(\mathbf{R},\rho) = (\mathbf{S},\sigma)$, contradicting the hypothesis; hence, indeed $R \neq S$.

Then it follows from Lemma 1(b) that $||R|| < ||S||$, and hence that

$$N((\mathbf{R},\rho),(\mathbf{R},\rho)) = ||R|| - t < ||S|| - t = N((\mathbf{S},\sigma),(\mathbf{S},\sigma)),$$

thereby completing the verification that C satisfies the conditions of lemma 19.

So, we assume, in the rest of this proof, that $d \in \mathbb{C}$ is such that the matrix C is invertible. We shall, in what follows, identify a linear operator, say T, on $V(\overline{n})$, with its matrix $((T^{(\mathbf{R},\rho)}_{(\mathbf{S},\sigma)}))$ with respect to the basis $S(\overline{n}) \times G(\overline{n})$. (Thus, $T(\mathbf{S},\sigma) = \sum_{(\mathbf{R},\rho)} T^{(\mathbf{R},\rho)}_{(\mathbf{S},\sigma)}(\mathbf{R},\rho)$.)

Now, the matrix of a typical element of $\Gamma(G(\overline{n}))'$ has the form

$$X((\mathbf{R},\rho),(\mathbf{S},\sigma)) = x^{(\rho\sigma^{-1})}(\mathbf{R},\mathbf{S}),$$

where $\{x^{(\tau)} : \tau \in G(\overline{n})\}$ is a collection of arbitrary matrices with rows and columns indexed by $S(\overline{n})$.

Hence, in order to prove (ii), it will suffice to prove that given an arbitrary collection $\{x^{(\tau)} : \tau \in G(\overline{n})\}$ of matrices with rows and columns indexed by $S(\overline{n})$, then there exist complex scalars $a(\mathbf{Q},\rho,\mathbf{R}), \mathbf{Q},\mathbf{R} \in S(\overline{n}), \rho \in G(\overline{n})$ such that

$$\left(\sum_{\mathbf{Q},\rho,\mathbf{R}} a(\mathbf{Q},\rho,\mathbf{R})\pi_{(\overline{n})}(\mathbf{Q},\rho,\mathbf{R})\right)((\mathbf{S}_1,\sigma_1),(\mathbf{S},\sigma)) = x^{(\sigma_1\sigma^{-1})}(\mathbf{S}_1,\mathbf{S}), \tag{3.15}$$

for all $(\mathbf{S}_1,\sigma_1),(\mathbf{S},\sigma) \in S(\overline{n}) \times G(\overline{n})$.

Fix $\mathbf{S}_1 \in S(\overline{n})$, and define $y^{(\mathbf{S}_1)}(\mathbf{S},\tau) = x^{(\tau)}(\mathbf{S}_1,\mathbf{S})$; due to the assumed invertibility of the matrix C, there exists a unique collection $\{z^{(\mathbf{S}_1)}(\mathbf{R},\rho) : (\mathbf{R},\rho) \in S(\overline{n}) \times G(\overline{n})\}$ of complex numbers such that

$$\sum_{\mathbf{R},\rho} z^{(\mathbf{S}_1)}(\mathbf{R},\rho)C((\mathbf{R},\rho),(\mathbf{S},\tau)) = y^{(\mathbf{S}_1)}(\mathbf{S},\tau), \tag{3.16}$$

for all $\mathbf{S}_1,\mathbf{S},\tau$.

Also note, from (3.7) and the definition of C, that

$$\pi_{(\overline{n})}((\mathbf{Q},\rho,\mathbf{R}))((\mathbf{S}_1,\sigma_1),(\mathbf{S},\sigma)) = \delta_{\mathbf{Q},\mathbf{S}_1}C((\mathbf{R},\rho),(\mathbf{S},\sigma_1\sigma^{-1})).$$

Now set $a(\mathbf{S}_1,\rho,\mathbf{R}) = z^{(\mathbf{S}_1)}(\mathbf{R},\rho)$, and compute as follows:

$$\begin{aligned}&\left(\sum_{\mathbf{Q},\rho,\mathbf{R}} a(\mathbf{Q},\rho,\mathbf{R})\pi_{(\overline{n})}(\mathbf{Q},\rho,\mathbf{R})\right)((\mathbf{S}_1,\sigma_1),(\mathbf{S},\sigma))\\ &\quad= \sum_{\mathbf{Q},\rho,\mathbf{R}} a(\mathbf{Q},\rho,\mathbf{R})\delta_{\mathbf{Q},\mathbf{S}_1}C((\mathbf{R},\rho),(\mathbf{S},\sigma_1\sigma^{-1}))\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\rho,\mathbf{R}} a(\mathbf{S}_1,\rho,\mathbf{R})C((\mathbf{R},\rho),(\mathbf{S},\sigma_1\sigma^{-1}))\\
&= \sum_{\rho,\mathbf{R}} z^{(\mathbf{S}_1)}(\mathbf{R},\rho)C((\mathbf{R},\rho),(\mathbf{S},\sigma_1\sigma^{-1}))\\
&= y^{(\mathbf{S}_1)}(\mathbf{S},\sigma_1\sigma^{-1})\\
&= x^{(\sigma_1\sigma^{-1})}(\mathbf{S}_1,\mathbf{S})
\end{aligned}
$$

and the proof is complete. □

The matrix that we called C in Proposition 20 really depends on $n, \overline{n}$ and d, and we shall write $C^n_{(\overline{n})}(d)$ (rather than merely C) when we wish to emphasize this dependence in the following; likewise, we shall, when desired, write $\Gamma^n_{(\overline{n})}$ for the representation of $G(\overline{n})$ that we called Γ in Proposition 20.

Theorem 21. *Suppose $d \in \mathbb{C}$ is such that $C^n_{(\overline{n})}(d)$ is invertible, for each $\overline{n} \in N_k, 0 \leq k \leq n|G|$. Then*

$$A^G_n(d) \cong \oplus_{\overline{n}\in N_{[n]}} \oplus_{\pi\in\widehat{G(\overline{n})}} (M_{d_\pi}(\mathbb{C}) \otimes M_{|S(\overline{n})|}(\mathbb{C})). \tag{3.17}$$

In particular, the algebras $A^G_n(d)$ are 'generically' semisimple.

Proof. Let us write $L^n_{(\overline{n})} = \Gamma^n_{(\overline{n})}(G(\overline{n}))'$; then, by Proposition 20 (ii), we have, for all $k, \overline{n}$,

$$\pi_{(\overline{n})}(A^G_n(d)) = \pi_{(\overline{n})}(\text{span } I(\overline{n})) = L^n_{(\overline{n})};$$

further, it is clear from the definition that the representation $\Gamma^n_{(\overline{n})}$ is equivalent to $R_{(\overline{n})} \otimes id_{\mathbb{C}^{|S(\overline{n})|}}$, where $R_{(\overline{n})}$ denotes the right regular representation of $G(\overline{n})$; it follows that $L^n_{(\overline{n})} \cong \mathbb{C}[G(\overline{n})] \otimes_{\mathbb{C}} M_{|S(\overline{n})|}(\mathbb{C})$, and hence that $\dim(L^n_{(\overline{n})}) = |G(\overline{n})| \cdot |S(\overline{n})|^2$; on the other hand, we also know that this is the dimension of $I(\overline{n})$ (since $I(\overline{n})$ has a basis indexed by $S(\overline{n}) \times G(\overline{n}) \times S(\overline{n})$), and consequently, we may conclude that $\pi_{(\overline{n})}$ maps (span $I(\overline{n})$) bijectively onto $L^n_{(\overline{n})}$.

Since each $L^n_{(\overline{n})}$ is clearly semisimple, the proposition will be proved once we establish the following isomorphism of $\mathbb{C}$-algebras:

$$\oplus_{\overline{n}\in N_{[n]}}\pi_{(\overline{n})} : A^G_n(d) \cong \oplus_{\overline{n}\in N_{[n]}} L^n_{(\overline{n})}.$$

Now $\dim A^G_n(d) = \dim(\oplus_{\overline{n}} L^n_{(\overline{n})})$, since $\coprod_{\overline{n}} I(\overline{n})$ is a basis of $A^G_n(d)$; so it suffices to prove surjectivity of $\oplus_{\overline{n}}\pi_{(\overline{n})}$.

So suppose $\oplus_{\overline{n}} x_{(\overline{n})} \in \oplus_{\overline{n}} L^n_{(\overline{n})}$; we shall exhibit $\{a_{(\overline{m})} \in (\text{span } I(\overline{m})) : \overline{m} \in N_{[n]}\}$ such that $(\oplus_{\overline{n}}\pi_{(\overline{n})})(\sum_{\overline{m}} a_{(\overline{m})}) = \oplus_{\overline{n}} x_{(\overline{n})}$. Note that $\pi_{(\overline{n})}(I(\overline{m})) = 0$ whenever either (i) $l < k$, or (ii) $l = k$ and $\overline{m} \neq \overline{n}$ – where $\overline{m} \in N_l, \overline{n} \in N_k$; hence the $a_{(\overline{m})}$'s must satisfy

$$\sum_{l\geq k, \overline{m}\in N_l} \pi_{(\overline{n})}(a_{(\overline{m})}) = x_{(\overline{n})} \ \forall \ \overline{n} \in N_k, 0 \leq k \leq n.$$

Since we know that $\pi_{(\overline{n})}$ maps $I(\overline{n})$ onto $L^n_{(\overline{n})}$, we may inductively define the $a_{(\overline{m})}$'s by just requiring that if $\overline{n} \in N_k$, and if $a_{(\overline{m})}$ has been defined for all $\overline{m} \in N_l, l < k$, then

$$\pi_{(\overline{n})}(a_{(\overline{n})}) = x_{(\overline{n})} - \sum_{l>k, \overline{m}\in N_l} \pi_{(\overline{n})}(a_{(\overline{m})}).$$

□

4. The tower $\{A_n^G(d) \colon n = 1, 2, \ldots\}$

Henceforth, we make the blanket assumption that d satisfies the hypothesis of Theorem 21.

It is a consequence of that theorem that – in the notation of that theorem – the irreducible representations of $A_n^G(d)$ are parametrized by the set $\{(\overline{n}, \pi) : \overline{n} \in N_{[n]}, \pi \in \widehat{G(\overline{n})}\}$. For the sake of future computations, we wish to explicitly write out a model for the irreducible representation corresponding to $(\overline{n}, \pi)$. In the sequel, we write $\mathbb{C}S(\overline{n})$ for the $\mathbb{C}$-vector space with basis $S(\overline{n})$, with $S(\overline{n})$ as before.

Remark 22. (i) We wish to note here that although we used a 'reference element' $\mathbf{S}_0$ in defining the representation $\pi_{(n)}$ of Proposition 18, the definition is actually independent of the element $\mathbf{S}_0$ – at least under our blanket assumption that d satisfies the hypothesis of Theorem 21. This is because: (a) it is seen from eq. (3.7) that the definition of $\pi(P)$ is independent of $\mathbf{S}_0$ at least when $P \in I(\overline{n})$; and (b) for a semi-simple algebra, a representation is uniquely determined by its restriction to any ideal which acts 'non-degenerately'.

(ii) Further, as we shall wish to consider $A_n^G(d)$ for varying n, we shall use a subscript n for symbols used so far, to indicate the dependence on n; thus, we shall talk of $V_n(\overline{n})$, $S_n(\overline{n})$, etc.; also, we shall use the notation $N_{n;k}$ for what we have so far denoted by N_k (see Definition 14(c)).

PROPOSITION 23

Fix $\overline{n} \in N_{n;k}, 0 \leq k \leq n|G|, \pi \in \widehat{G(\overline{n})}$. *Let* V_π *denote the vector space on which* π *represents* $G(\overline{n})$, *and define* $V(\overline{n}, \pi) = \mathbb{C}S(\overline{n}) \otimes V_\pi$.

(a) *Then the following prescription uniquely defines the structure of an* $A_n^G(d)$*-module on* $V(\overline{n}, \pi)$: *let* $P \in R^G(X_{2n}), \mathbf{S} \in S(\overline{n})$; *by the definition of the representation* $\pi_{(\overline{n})}$ – *see Proposition* 18 – *there exists a unique scalar* $C(P, \mathbf{S})$ *and an element* $(\mathbf{S}_1, \sigma_1) \in S(\overline{n}) \times G(\overline{n})$ *such that*

$$\pi_{(\overline{n})}(P)(\mathbf{S}, 1) = C(P, \mathbf{S})(\mathbf{S}_1, \sigma_1),$$

where the 1 *on the left denotes the identity element of* $G(\overline{n})$; *then let*

$$P \cdot (\mathbf{S} \otimes v) = C(P, \mathbf{S})(\mathbf{S}_1 \otimes \pi(\sigma_1)v).$$

(b) $V(\overline{n}, \pi)$ *is irreducible as a module over the ideal* I_k^G (*and hence also as an* $A_n^G(d)$*-module*), *and further, if* $\overline{m} \in N_{n;l}$, *then* $I(\overline{m})$ *acts as* 0, *whenever either* (i) $l < k$, *or* (ii) $l = k$ *and* $\overline{m} \neq \overline{n}$.

(c) *The modules* $\{V(\overline{n}, \pi) : 0 \leq k \leq n|G|, \overline{n} \in N_{n;k}, \pi \in \widehat{G(\overline{n})}\}$ *are pairwise inequivalent.*

Proof. Suppose $X, Y \in R^G(X_{2n})$ and $\mathbf{S} \in S(\overline{n})$, and suppose that

$$\pi_{(\overline{n})}(Y)(\mathbf{S}, 1) = C(Y, \mathbf{S})(\mathbf{S}_1, \sigma_1); \text{ and}$$
$$\pi_{(\overline{n})}(X)(\mathbf{S}_1, 1) = C(X, \mathbf{S}_1)(\mathbf{S}_2, \sigma_2);$$

it follows that

$$\begin{aligned}\pi_{(\overline{n})}(XY)(\mathbf{S}, 1) &= \pi_{(\overline{n})}(X)\pi_{(\overline{n})}(Y)(\mathbf{S}, 1)\\ &= C(Y, \mathbf{S})\pi_{(\overline{n})}(X)(\mathbf{S}_1, \sigma_1)\end{aligned}$$

$$
\begin{aligned}
&= C(Y,\mathbf{S})\pi_{(\overline{n})}(X)\Gamma^{n}_{(\overline{n})}(\sigma_1^{-1})(\mathbf{S}_1, 1)\\
&= C(Y,\mathbf{S})\Gamma^{n}_{(\overline{n})}(\sigma_1^{-1})\pi_{(\overline{n})}(X)(\mathbf{S}_1, 1)\\
&= C(Y,\mathbf{S})\Gamma^{n}_{(\overline{n})}(\sigma_1^{-1})(C(X,\mathbf{S}_1)(\mathbf{S}_2, \sigma_2))\\
&= C(Y,\mathbf{S})C(X,\mathbf{S}_1)(\mathbf{S}_2, \sigma_2\sigma_1);
\end{aligned}
$$

it follows from this that $C(XY,\mathbf{S}) = C(Y,\mathbf{S})C(X,\mathbf{S}_1)$.

Now deduce from the definitions that

$$
\begin{aligned}
Y\cdot(\mathbf{S}\otimes v) &= C(Y,\mathbf{S})(\mathbf{S}_1\otimes \pi(\sigma_1)v);\\
X\cdot(\mathbf{S}_1\otimes w) &= C(X,\mathbf{S}_1)(\mathbf{S}_2\otimes \pi(\sigma_2)w);
\end{aligned}
$$

and hence that

$$
\begin{aligned}
XY\cdot(\mathbf{S}\otimes v) &= C(Y,\mathbf{S})C(X,\mathbf{S}_1)(\mathbf{S}_2\otimes \pi(\sigma_2\sigma_1)v)\\
&= C(Y,\mathbf{S})X\cdot(\mathbf{S}_1\otimes \pi(\sigma_1)v)\\
&= X\cdot(C(Y,\mathbf{S})(\mathbf{S}_1\otimes \pi(\sigma_1)v))\\
&= X\cdot(Y\cdot(\mathbf{S}\otimes v));
\end{aligned}
$$

this proves that the 'representation' is multiplicative; the verification of linearity is trivial.

(b) and (c) It is clear that I_{k-1} acts as 0 on $V(\overline{n},\pi)$. Further, if $\overline{m}\in N_{n;k}$, it is a consequence of Lemma 15 that $\pi_{(\overline{n})}(I(\overline{m})) = 0$ for $\overline{m}\neq\overline{n}$, and hence $I(\overline{m})$ also acts as 0 on $V(\overline{n},\pi)$, if $\overline{m}\neq\overline{n}$. On the other hand, $I(\overline{n})$ does not act as 0 on $V(\overline{n},\pi)$, since $\pi_{(\overline{n})}$ is injective on $I(\overline{n})$. It follows from the preceding statements that if $\overline{m}\in N_{n;l}$, then $V(\overline{n},\pi)$ and $V(\overline{m},\chi)$ are inequivalent $A_n^G(d)$-modules, unless $\overline{n}=\overline{m}$.

In order to complete the proof of the proposition, we shall verify – and this is clearly sufficient – that if $T: V(\overline{n},\pi)\to V(\overline{n},\chi)$ is an $I(\overline{n})$-linear map, where $\pi,\chi\in\widehat{G(\overline{n})}$, then

$$
T = \begin{cases}\lambda\, id_{V(\overline{n},\pi)} & \text{if } \pi=\chi,\\ 0 & \text{if } \pi \text{ is not equivalent to } \chi,\end{cases}
$$

for some $\lambda\in\mathbb{C}$.

Suppose $\{e_j : 1\le j\le d_\pi\}$ (resp., $\{f_i : 1\le i\le d_\chi\}$) is an orthonormal basis for V_π (resp., V_χ), and suppose

$$
T(\mathbf{S}\otimes e_j) = \sum_{\mathbf{S}_1,i} T^{(\mathbf{S}_1,i)}_{(\mathbf{S},j)}(\mathbf{S}_1\otimes f_i).
$$

Let $(\mathbf{Q},\rho,\mathbf{R})\in I(\overline{n})$. Computing $T((\mathbf{Q},\rho,\mathbf{R})\cdot(\mathbf{S}\otimes e_j))$ and $(\mathbf{Q},\rho,\mathbf{R})\cdot T(\mathbf{S}\otimes e_j))$, and equating coefficients of $\mathbf{Q_1}\otimes f_l$, we see that

$$
D(\mathbf{R},\mathbf{S})\sum_k \pi_j^k(\rho\beta^{\mathbf{R}}_{\mathbf{S}})T^{(\mathbf{Q_1},l)}_{(\mathbf{Q},k)} = \delta^{\mathbf{Q_1}}_{\mathbf{Q}}\sum_{\mathbf{S}_1,i} T^{(\mathbf{S}_1,i)}_{(\mathbf{S},j)}D(\mathbf{R},\mathbf{S}_1)\chi_i^l(\rho\beta^{\mathbf{R}}_{\mathbf{S}_1}), \tag{4.18}
$$

for all possible choices of $\mathbf{Q_1},\mathbf{Q},\mathbf{R},\mathbf{S},l,j,\rho$.

If $\mathbf{Q_1}\neq\mathbf{Q}$, set $R=S,\rho=1$ in eq. (4.18) to deduce that

$$
T^{(\mathbf{Q_1},l)}_{(\mathbf{Q},j)} = 0\ \forall l,\ j, \tag{4.19}
$$

whenever $\mathbf{Q_1}\neq\mathbf{Q}$.

Writing $T_{\mathbf{Q}}$ for the matrix defined by $T_{\mathbf{Q}} = (((T_{\mathbf{Q}})^l_j))$, where $(T_{\mathbf{Q}})^l_j = T^{(\mathbf{Q},l)}_{(\mathbf{Q},j)}$, we next deduce – on setting $\mathbf{R} = \mathbf{S}$, $\mathbf{Q} = \mathbf{Q}_1$ in eq. (4.18) – that

$$T_{\mathbf{Q}}\pi(\rho) = \chi(\rho)T_{\mathbf{S}} \tag{4.20}$$

for all choices of $\rho, \mathbf{Q}, \mathbf{S}$. Set $\rho = 1$ in eq. (4.20) to find that $T_{\mathbf{Q}} = T_{\mathbf{S}} = T_0$ (say), for all $\mathbf{Q}, \mathbf{S}$; deduce next from (4.20) that T_0 intertwines the representations π and χ, thereby completing the proof. □

Since we wish to now look at the inclusion $A^G_n(d) \subset A^G_{n+1}(d)$, it will be necessary to write $S_n(\overline{n})$ for what we called $S(\overline{n})$ till now. Thus,

$$S_n(\overline{n}) = \{\mathbf{S} = (S; (\{C_{H,s}(\mathbf{S}) : H \in \mathcal{C}, 1 \leq s \leq \overline{n}(H)\}) \mid S \in R^G(X_n)\}.$$

Given $P \in R^G(X_{2n})$, define $\widetilde{P} \in R^G(X_{2n+2})$ by 'adding on a set of $|G|$-many vertical lines to the right of P'; more pedantically, if $P \in I_n(\overline{n})$ is given by $P = (\mathbf{P}^+, \rho, \mathbf{P}^-)$, with $\mathbf{P}^\pm \in S_n(\overline{n}), \overline{n} \in N_{[n]}$, then $\widetilde{P} = (\widetilde{\mathbf{P}^+}, \widetilde{\rho}, \widetilde{\mathbf{P}^-}) \in S_{n+1}(\overline{m})$, where (a) $\widetilde{P^\pm} = P^\pm \cup \{((g, n+1), (g, n+1)) : g \in G\}$, (b) $\overline{m}(H) = \overline{n}(H) + \delta_{H,\{1\}}$, and

$$C_{H,s}(\widetilde{\mathbf{P}^\pm}) = \begin{cases} \{n+1\} & \text{if } H = \{1\} \text{ and } s = \overline{n}(\{1\}) + 1 \\ C_{H,s}(\mathbf{P}^\pm) & \text{otherwise;} \end{cases}$$

and (c) with the natural identification of $G(\overline{n})$ as a subgroup of $G(\overline{m})$, we have simply $\widetilde{\rho} = \rho$. (Note that (i) G^t sits as the subgroup of G^{t+1} consisting of those elements with last co-ordinate equal to 1, (ii) Σ_t sits as the subgroup of Σ_{t+1} consisting of those permutations which fix $t+1$, (iii) the semi-direct product $G^t \rtimes_s \Sigma_t$ naturally embeds in $G^{t+1} \rtimes \Sigma_{t+1}$ in a manner that is consistent with (i) and (ii) above, and (iv) with $\overline{n}$ and $\overline{m}$ as above, there is a group K such that $G(\overline{n}) = K \rtimes (G^t \rtimes_s \Sigma_t)$ and $G(\overline{m}) = K \times (G^{t+1} \rtimes_s \Sigma_{t+1})$. Later, we shall need the analogous and slightly more general fact that if $\overline{n}, \overline{m} \in N_{[n]}$ and if $\overline{n}(H) \leq \overline{m}(H)\ \forall H \in \mathcal{C}$, then $G(\overline{n})$ may be regarded as a subgroup of $G(\overline{m})$.)

Given $H_0 \in \mathcal{C}$, we shall write 1_{H_0} for the function on $\mathcal{C}$ which is equal to one at H_0 and 0 elsewhere. In the sequel, we shall specify elements $\mathbf{S} \in S_{n+1}(\overline{m})$ thus: (a) by specifying the data of (i) an element $R^S \in R([n+1])$, (ii) a mapping $[n+1]/R \ni C \mapsto H^S_C \in \mathcal{C}$, and (iii) a map ϕ^S defined on $[n+1]$ and taking values in right-cosets of the H^S_C's satisfying the conditions of Proposition 11; and demanding that $S \in R^G(X_{n+1})$ is the unique element corresponding to the data (i)–(iii) as in Proposition 11; and (b) by specifying an explicitly labelled collection $\{C_{H,s}(\mathbf{S}) : 1 \leq s \leq \overline{m}(H), H \in \mathcal{C}\}$ of R^S-equivalence classes such that $C_{H,s}(\mathbf{S})$ is assigned to H under the assignment of (a)(ii), and such that the labelling satisfies the condition (2.6).

It will be convenient to have a 'standard' or 'reference' element of each $S_n(\overline{n})$; we specify such an element in the following definition.

DEFINITION 24

Once and for all, fix some total order on the class $\mathcal{C}$. Fix $n \in \{1, 2, \ldots\}$, and $\overline{n} \in N_{[n]}$. Then $\overline{n}$ uniquely specifies distinct elements $H_1, H_2, \ldots, H_l$ of $\mathcal{C}$ such that:

(i) $H_1 < H_2 < \cdots < H_l$ (with respect to the chosen total order on $\mathcal{C}$); and
(ii) $\overline{n}(H) \neq 0 \Leftrightarrow H \in \{H_j : 1 \leq j \leq l\}$.

Suppose $\overline{n}(H_j) = \nu_j$; set $\mu_j = \sum_{k=1}^{j} \nu_k$. Then, define $\mathbf{S}_0(n, \overline{n}) \in S_n(\overline{n})$ (which we shall simply abbreviate to $\mathbf{S}_0$ if $n, \overline{n}$ are clear from the context) as follows:

(a) (i) R^{S_0} is the 'identity' equivalence relation on $[n]$, all of whose equivalence classes are singletons;

(ii) $H^{S_0}_{\{k\}} = \begin{cases} H_j & \text{if } \mu_{j-1} < k \leq \mu_j \\ \{1\} & \text{if } \mu_l < k; \end{cases}$

(iii) $\phi^{S_0}(k) = H^{S_0}_{\{k\}} \forall k$; and

(b) $C_{H_j,s}(\mathbf{S}_0) = \{\mu_{j-1} + s\}$, for $1 \leq s \leq \nu_j, 1 \leq j \leq l$.

With a view to decomposing $V_{n+1}(\overline{n}, \chi)$ as an $A^G_n(d)$-module, we shall now proceed to construct several $A^G_n(d)$-linear maps from $V_n(\overline{m}, \pi)$ to $V_{n+1}(\overline{n}, \chi)$, for appropriate $\overline{m}$ and π. The basic idea behind the construction of these intertwiners is the old one that 'right-multiplications commute with left-multiplications'.

Recall, from Definition 9 that for every $H \in \mathcal{C}$, we have chosen a fixed set $\{\sigma^H_\kappa : 1 \leq \kappa \leq [G : N(H)]\}$ of coset-representatives for $N(H)\backslash G$.

Lemma 25. *Fix* $\overline{n} \in N_{[n+1]}, H_0 \in \mathcal{C}$ *such that* $\overline{n}(H_0) > 0$, *and* $\sigma \in \{\sigma^{H_0}_\kappa : 1 \leq \kappa \leq [G : N(H_0)]\}$.

(1) *If* $\mathbf{Q} \in S_n(\overline{n} - 1_{H_0})$, *define* $\alpha_{H_0,\sigma}(\mathbf{Q}) = \mathbf{S}$, *thus*:

(a) (i) $R^S = R^Q \cup \{(n+1, n+1)\}$,

(ii) $H^S_C = \begin{cases} H^Q_C & \text{if } C \subset [n] \\ H_0 & \text{if } C = \{n+1\} \end{cases}$

(iii) $\phi^S(i) = \begin{cases} \phi^Q(i) & \text{if } i \leq n \\ H_0\sigma & \text{if } i = n+1 \end{cases}$;

(b) $C_{H,s}(\mathbf{S}) = \begin{cases} C_{H,s}(\mathbf{Q}) & \text{if } H \neq H_0 \text{ or } H = H_0, 1 \leq s < \overline{n}(H_0) \\ \{n+1\} & \text{if } H = H_0, s = \overline{n}(H_0) \end{cases}$.

Then $\alpha_{H_0,\sigma}$ *is a* 1–1 *map of* $S_n(\overline{n} - 1_{H_0})$ *into* $S_{n+1}(\overline{n})$.

(2) *Conversely, if* $\mathbf{S} \in S_{n+1}(\overline{n})$, *and if the singleton* $\{n+1\}$ *is an* R^S*-equivalence class which is one of the 'distinguished classes' – meaning that* $\{n+1\} = C_{H_0,s_0}(\mathbf{S})$ (*for a necessarily unique* $H_0 \in \mathcal{C}$ *and a unique integer* s_0 *necessarily equal to* $\overline{n}(H_0)$) *– then there exists a unique* $H_0 \in \mathcal{C}$ (*namely the one just discussed*), *a unique* σ, *and a unique* $\mathbf{Q} \in S_n(\overline{n} - 1_{H_0})$ *such that* $\alpha_{H_0,\sigma}(\mathbf{Q}) = \mathbf{S}$.

(3) *Let* $\pi \in \widehat{G(\overline{n} - 1_{H_0})}, \chi \in \widehat{G(\overline{n})}$, *and suppose* $L : V_\pi \to V_\chi$ *is a non-zero* $G(\overline{n} - 1_{H_0})$*-linear operator.* (*Note that* $G(\overline{n} - 1_{H_0}) \subset G(\overline{n})$, *so the above sentence makes sense.*) *Then the equation*

$$(A_{H_0,\sigma}(L))(\mathbf{T} \otimes \xi) = \alpha_{H_0,\sigma}(\mathbf{T}) \otimes L\xi \tag{4.21}$$

defines a non-zero $A^G_n(d)$*-linear operator* $A_{H_0,\sigma}(L) : V_n(\overline{n} - 1_{H_0}, \pi) \to V_{n+1}(\overline{n}, \chi)$.

Proof. (1) and (2): It should be clear from the definitions that indeed $\alpha_{H_0,\sigma}$: $S_n(\overline{n} - 1_{H_0}) \to S_{n+1}(\overline{n})$. To complete the proof, we only need to verify injectivity. On the other hand, the statement (2) is also fairly obvious, and explicitly contains the specification of the range of the map $\alpha_{H_0,\sigma}$, as well as the assertion that any point in this range admits a unique pre-image, i.e., that $\alpha_{H_0,\sigma}$ is 1–1.

(3) We shall find the following notation useful: if $\overline{m} \in N_{[m]}$, we shall write $t(\overline{m}) = \sum_{H \in \mathcal{C}} \overline{m}(H)[G : H]$; thus, if $P \in I_m(\overline{m})$, then $t(P) = t(\overline{m})$. Also, let $J_m(\overline{m})$ denote

the linear subspace spanned by $(I_m(\overline{m}) \cup \bigcup_{\{\overline{m}':t(\overline{m}')<t(\overline{m})\}} I_m(\overline{m}'))$; it should be clear that $J_m(\overline{m})$ is an ideal in $A_m^G(d)$ which acts non-degenerately on the module $V_m(\overline{m}, \zeta)$.

Since $A_n^G(d)$ is semi-simple, it will suffice to show that $A_{H_0,\sigma}(L)$ is $J_n(\overline{n} - 1_{H_0})$-linear.

First, suppose $P \in I(\overline{m})$ for $\overline{m} \in N_{[n]}$ with $t(\overline{m}) < t(\overline{n} - 1_{H_0})$; then, we shall show that

$$(A_{H_0,\sigma}(L))(P \cdot (\mathbf{T} \otimes \xi)) = 0 = \tilde{P} \cdot (A_{H_0,\sigma}(L))(\mathbf{T} \otimes \xi)) \ \forall \mathbf{T} \in S_n(\overline{n} - 1_{H_0}), \xi \in V_\pi.$$

The fact that the left-side of the above equation is zero is a consequence of the fact that $t(P \cdot (\mathbf{T}, 1, \mathbf{S}_0)) \leq t(P) < t(\overline{n} - 1_{H_0})$, and such elements of the algebra act as zero on the module in question.

As for the right side, it suffices to verify that $t(\tilde{P} \cdot (\alpha_{H_0,\sigma}(\mathbf{T}), 1, \mathbf{S}_0)) < t(\overline{n})$; but notice that the first $n|G|$ strands of this product contribute at most $t(P)$ through-classes, while the last $|G|$ strands contribute exactly $[G : H_0]$ through-classes, and the sum of these two terms is, by hypothesis, less than $t(\overline{n})$.

We need now to verify that

$$(A_{H_0,\sigma}(L))(P \cdot (\mathbf{T} \otimes \xi)) = \tilde{P} \cdot (A_{H_0,\sigma}(L))(\mathbf{T} \otimes \xi)) \ \forall \mathbf{T} \in S_n(\overline{n} - 1_{H_0}), \ \xi \in V_\pi,$$

whenever $P \in I_n(\overline{n} - 1_{H_0})$. Thus, suppose $P = (\mathbf{Q}, \rho, \mathbf{R})$, where $\mathbf{Q}, \mathbf{R} \in S_n(\overline{n} - 1_{H_0})$ and $\rho \in G(\overline{n} - 1_{H_0})$. Then, if we write $g = \rho\beta_{\mathbf{T}}^{\mathbf{R}}$, we see that since $(\mathbf{Q}, \rho, \mathbf{R}) \cdot (\mathbf{T}, 1, \mathbf{S}_0) = D(\mathbf{R}, \mathbf{T})(\mathbf{Q}, g, \mathbf{S}_0)$, we have, by definition,

$$\begin{aligned}(A_{H_0,\sigma}(L))(P \cdot (\mathbf{T} \otimes \xi)) &= (A_{H_0,\sigma}(L))(D(\mathbf{R}, \mathbf{T})(\mathbf{Q} \otimes \pi(g)\xi)) \\ &= D(\mathbf{R}, \mathbf{T})\alpha_{H_0,\sigma}(\mathbf{Q}) \otimes L\pi(g)\xi. \end{aligned} \tag{4.22}$$

On the other hand, since

$$\tilde{P} \cdot (A_{H_0,\sigma}(L))(\mathbf{T} \otimes \xi)) = \tilde{P} \cdot (\alpha_{H_0,\sigma}(\mathbf{T}) \otimes L\xi)),$$

in order to evaluate the right side of this equation, we will need to first calculate $(\mathbf{Q}, \rho, \mathbf{R})\tilde{\ } \cdot (\alpha_{H_0,\sigma}(\mathbf{T}), 1, \mathbf{S}_0)$.

To this end, it will be convenient to introduce the following element of $I_{n+1}(\overline{n})$, which we shall denote by $\tilde{\alpha}_{H_0,\sigma}$:

$$\tilde{\alpha}_{H_0,\sigma} = (\alpha_{H_0,\sigma}(\mathbf{S}_0(n, \overline{n} - 1_{H_0})), 1, \mathbf{S}_0(n + 1, \overline{n})).$$

The point is that

$$(\alpha_{H_0,\sigma}(\mathbf{S}), g, \mathbf{S}_0(n + 1, \overline{n})) = (\mathbf{S}, g, \mathbf{S}_0(n, \overline{n} - 1_{H_0}))\tilde{\ } \cdot \tilde{\alpha}_{H_0,\sigma}, \tag{4.23}$$

for all $\mathbf{S} \in S_n(\overline{n} - 1_{H_0})$ and any $g \in G(\overline{n} - 1_{H_0})$, where the g on the left side of the equation denotes g when thought of as an element of $G(\overline{n})$ (via the natural inclusion $G(\overline{n} - 1_{H_0}) \subset G(\overline{n})$). (Equation (4.23) is verified by looking at the picture represented by the product on the right side, noting that it does belong to $I_{n+1}(\overline{n})$, and checking that its three ingredients are indeed as given by the left side of (4.23).)

Hence,

$$\begin{aligned}(\mathbf{Q}, \rho, \mathbf{R})\tilde{\ } \cdot (\alpha_{H_0,\sigma}(\mathbf{T}), 1, \mathbf{S}_0) &= (\mathbf{Q}, \rho, \mathbf{R})\tilde{\ } \cdot (\mathbf{T}, 1, \mathbf{S}_0)\tilde{\ } \cdot \tilde{\alpha}_{H_0,\sigma} \\ &= D(\mathbf{R}, \mathbf{T})(\mathbf{Q}, g, \mathbf{S}_0)\tilde{\ } \cdot \tilde{\alpha}_{H_0,\sigma} \\ &= D(\mathbf{R}, \mathbf{T})(\alpha_{H_0,\sigma}(\mathbf{Q}), g, \mathbf{S}_0).\end{aligned}$$

Hence we may deduce that

$$(\mathbf{Q}, \rho, \mathbf{R})\tilde{}\cdot (A_{H_0,\sigma}(L))(\mathbf{T} \otimes \xi)) = (\mathbf{Q}, \rho, \mathbf{R})\tilde{}\cdot (\alpha_{H_0,\sigma}(\mathbf{T}) \otimes L\xi))$$
$$= D(\mathbf{R}, \mathbf{T})(\alpha_{H_0,\sigma}(\mathbf{Q}) \otimes \chi(g)L\xi). \qquad (4.24)$$

Since $g \in G(\overline{n} - 1_{H_0})$, we have $\chi(g)L = L\pi(g)$, and the lemma follows from equations (4.22) and (4.24). □

Lemma 26. *Fix* $\overline{n} \in N_{[n]}, H_0 \in \mathcal{C}$ *and* $\sigma \in \{\sigma_\kappa^{H_0} : 1 \le \kappa \le [G : N(H_0)]\}$.

(1) *If* $\mathbf{Q} \in S_n(\overline{n})$, *define* $\beta_{H_0,\sigma}(\mathbf{Q}) = \mathbf{S}$, *thus*:

(a) (i) $R^S = R^Q \cup \{(n+1, n+1)\}$;

(ii) $H^S_C = \begin{cases} H^Q_C & \text{if } C \subset [n] \\ H_0 & \text{if } C = \{n+1\} \end{cases}$;

(iii) $\phi^S(i) = \begin{cases} \phi^Q(i) & \text{if } i \le n \\ H_0\sigma & \text{if } i = n+1 \end{cases}$;

(b) $C_{H,s}(\mathbf{S}) = C_{H,s}(\mathbf{Q})$ for $H \in \mathcal{C}, 1 \le s \le \overline{n}(H)$.

Then $\beta_{H_0,\sigma}$ *is a* 1–1 *map of* $S_n(\overline{n})$ *into* $S_{n+1}(\overline{n})$.

(2) *Conversely, if* $\mathbf{S} \in S_{n+1}(\overline{n})$, *and if the singleton* $\{n+1\}$ *is an* R^S*-equivalence class which is not a 'distinguished class' – meaning that* $\{n+1\} \notin \{C_{H,s}(\mathbf{S}) : H \in \mathcal{C}, 1 \le s \le \overline{n}(H)\}$ *– then there exists a unique* $H_0 \in \mathcal{C}$, *a unique* σ, *and a unique* $\mathbf{Q} \in S_n(\overline{n})$ *such that* $\beta_{H_0,\sigma}(\mathbf{Q}) = \mathbf{S}$.

(3) *Let* $\pi \in \widehat{G(\overline{n})}$. *Then the equation*

$$B_{H_0,\sigma}(\mathbf{T} \otimes \xi) = \beta_{H_0,\sigma}(\mathbf{T}) \otimes \xi \qquad (4.25)$$

defines a non-zero $A^G_n(d)$*-linear operator* $B_{H_0,\sigma} : V_n(\overline{n}, \pi) \to V_{n+1}(\overline{n}, \pi)$.

Proof. The proof is almost identical to that of the last lemma, and so we shall say nothing more about the proof except that we would here want to look at the special element $\tilde{\beta}_{H_0,\sigma} \in I_{n+1}(\overline{n})$ defined by

$$\tilde{\beta}_{H_0,\sigma} = (\beta_{H_0,\sigma}(\mathbf{S}_0), 1, \mathbf{S}_0),$$

and the crucial identify it satisfies is

$$(\beta_{H_0,\sigma}(\mathbf{S}), g, \mathbf{S}_0) = (\mathbf{S}, g, \mathbf{S}_0)\tilde{}\cdot \tilde{\beta}_{H_0,\sigma}.$$ □

Remark 27. (1) Fix n and $\overline{n} \in N_{[n+1]}$. Consider two cases now:

Case 1: $\overline{n} \in N_{[n]}$. It is a consequence of lemma 25(2) and lemma 26(2) that if $H_i \in \mathcal{C}$, $\sigma_i \in \{\sigma_\kappa^{H_i} : 1 \le \kappa \le [G : N(H_i)]\}, 1 \le i \le 4$, if $\overline{n}(H_1), \overline{n}(H_2) > 0$, and if $(H_1, \sigma_1) \neq (H_2, \sigma_2)$ and $(H_3, \sigma_3) \neq (H_4, \sigma_4)$, then the four sets $\alpha_{H_1,\sigma_1}(S_n(\overline{n} - 1_{H_1}))$, $\alpha_{H_2,\sigma_2}(S_n(\overline{n} - 1_{H_2}))$, $\beta_{H_3,\sigma_3}(S_n(\overline{n})), \beta_{H_4,\sigma_4}(S_n(\overline{n}))$ are pairwise disjoint. In this case, define $W^0_{n+1}(\overline{n})$ to be the subspace of $\mathbb{C}S_{n+1}(\overline{n})$ spanned by $(\cup_{H,\sigma}\alpha_{H,\sigma}(S_n(\overline{n} - 1_H))) \cup (\cup_{H,\sigma}\beta_{H,\sigma}(S_n(\overline{n})))$.

Case 2: $\overline{n} \notin N_{[n]}$. Here also, it is true (and follows from lemma 25(2) and lemma 26(2)) that if $H_i \in \mathcal{C}, \sigma_i \in \{\sigma_\kappa^{H_i} : 1 \le \kappa \le [G : N(H_i)]\}, i = 1, 2$, if $\overline{n}(H_1), \overline{n}(H_2) > 0$, and if

$(H_1, \sigma_1) \neq (H_2, \sigma_2)$, then the sets $\alpha_{H_1,\sigma_1}(S_n(\overline{n} - 1_{H_1}))$ and $\alpha_{H_2,\sigma_2}(S_n(\overline{n} - 1_{H_2}))$ are disjoint. In this case, define $W^0_{n+1}(\overline{n})$ to be the subspace of $\mathbb{C}S_{n+1}(\overline{n})$ spanned by $(\cup_{H,\sigma}\alpha_{H,\sigma}(S_n(\overline{n} - 1_H)))$.

It also follows from lemma 25(2) and lemma 26(2) that if $\mathbf{S} \in S_{n+1}(\overline{n})$, then $\mathbf{S} \in W^0_{n+1}(\overline{n})$ if and only if $\{n+1\}$ is an R^S-equivalence class.

(2) Fix $\overline{n} \in N_{[n+1]}$ as above, and $\chi \in \widehat{G(\overline{n})}$. Also fix $H \in \mathcal{C}$ such that $\overline{n}(H) > 0$, and fix $\sigma \in \{\sigma^H_\kappa : 1 \leq \kappa \leq [G : N(H)]\}$. Consider V_χ as a $G(\overline{n} - 1_H)$-module and decompose into irreducible submodules; specifically, assume that there exist $G(\overline{n} - 1_H)$-linear maps $L_{\pi,j} : V_\pi \to V_\chi, 1 \leq j \leq m_\pi, \pi \in \widehat{G(\overline{n} - 1_H)}$ such that V_χ is the direct sum of the ranges of all these maps. Then, we wish to observe that:

$$\oplus_{j,\pi} \text{ range } A_{H,\sigma}(L_{\pi,j}) = \mathbb{C}(\alpha_{H,\sigma}(S_n(\overline{n} - 1_H))) \otimes V_\chi;$$

and the two sides of this equation represent an $A^G_n(d)$-submodule of $V_{n+1}(\overline{n}, \chi)$.

Reason: The sum on the left is a direct sum because, for any fixed π, j, the corresponding 'summand' is a subspace of $\mathbb{C}S_{n+1}(\overline{n}) \otimes L_{\pi,j}(V_\pi)$. This direct sum is, by definition, included in the space on the right. To prove the reverse inclusion, it is clearly sufficient to verify that any vector of the form $\alpha_{H,\sigma}(\mathbf{Q}) \otimes L_{\pi,j}\xi, \mathbf{Q} \in S_n(\overline{n} - 1_H)), \xi \in V_\pi$ belongs to the left side, but this is just $A_{H,\sigma}(L_{\pi,j})(\mathbf{Q} \otimes \xi)$.

(3) Clearly, for fixed $\overline{n} \in N_{[n]}, \chi \in \widehat{G(\overline{n})}, H \in \mathcal{C}, \sigma \in \{\sigma^H_\kappa : 1 \leq \kappa \leq [G : N(H)]\}$,

$$\text{range } B_{H,\sigma} = \mathbb{C}(\beta_{H,\sigma}(S_n(\overline{n}))) \otimes V_\chi;$$

and the two sides of this equation represent an $A^G_n(d)$-submodule of $V_{n+1}(\overline{n}, \chi)$.

(4) Define $W^0_{n+1}(\overline{n}, \chi) = W^0_{n+1}(\overline{n}) \otimes V_\chi$. It is now a consequence of (1)–(3) above that

$$W^0_{n+1}(\overline{n}, \chi) = (\oplus_{H,\sigma,j,\pi} \text{ range } A_{H,\sigma}(L_{\pi,j})) \oplus (\oplus_{H,\sigma} \text{ range } B_{H,\sigma}),$$

and that $W^0_{n+1}(\overline{n}, \chi)$ is an $A^G_n(d)$-submodule of $V_{n+1}(\overline{n}, \chi)$.

Before discussing further intertwiners, it will be convenient to describe some coset-representatives for some subgroups, and also to discuss a certain natural group action; we do so in the following lemma, whose proof we omit since it is an easy verification.

Lemma 28. *Fix* $\overline{m} \in N_{[m]}, H_0 \in \mathcal{C}$.

(1) *For* $1 \leq s \leq \overline{m}(H_0) + 1$, *and* $f \in H_0\backslash N(H_0)$, *define an element* $\sigma(s, f) \in G(\overline{m} + 1_{H_0})$ *as follows*:

$$\sigma(s, f) = ((\sigma(s, f))_H),$$

where

$$(\sigma(s, f))_H = \begin{cases} 1 & \text{if } H \neq H_0 \\ ((1, \ldots, f, 1, \ldots, 1); \lambda_s) & \text{if } H = H_0 \end{cases}$$

where the f *occurs in the* s*-th slot, and* λ_s *is the cycle* $(s, s+1, \ldots, \overline{m}(H_0) + 1)$.

Then, $G(\overline{m} + 1_{H_0}) = \coprod_{(s,f)} \sigma(s, f)G(\overline{m})$.

(2) *Consider the set* $\{1, \ldots, \overline{m}(H_0) + 1\} \times (H_0\backslash N(H_0))$ *and an element* $g \in G(\overline{m} + 1_{H_0})$; *suppose that*

$$g = ((g_H)), \text{where } g_H = ((\omega^H_1, \ldots, \omega^H_{(m+1_{H_0})(H)}); \kappa_H).$$

Then, the equations

$$g\cdot(s,f)=(s_1,f_1)\Leftrightarrow s_1=\kappa_{H_0}(s), f_1=\omega_{s_1}^{H_0}f$$

define a transitive action of $G(\overline{m}+1_{H_0})$ *on* $\{1,\ldots,\overline{m}(H_0)+1\}\times(H_0\backslash N(H_0))$. *In fact,* $g\cdot(s,f)=(s_1,f_1)\Leftrightarrow\sigma(s_1,f_1)^{-1}g\sigma(s,f))\in G(\overline{m})$; *equivalently, this action may be identified with that of* $G(\overline{m}+1_{H_0})$ *on* $G(\overline{m}+1_{H_0})/G(\overline{m})$.

Lemma 29. *Fix* $\overline{n}\in N_{[n]}, H_0\in\mathcal{C}$ *such that* $\overline{n}(H_0)>0$; *also fix* $1\le s_0\le\overline{n}(H_0), f\in H_0\backslash N(H_0)$ *and* $\sigma\in\{\sigma_\kappa^{H_0}:1\le\kappa\le[G:N(H_0)]\}$.

(1) *If* $\mathbf{Q}\in S_n(\overline{n})$, *define* $\gamma_{H_0,(\sigma,s_0,f)}(\mathbf{Q})=\mathbf{S}$, *thus*:

(a) (i) *the* R^S*-equivalence classes are* $C=C_{H_0,s_0}(\mathbf{Q})\cup\{n+1\}$ *and all the* R^Q*-equivalence classes other than* $C_{H_0,s_0}(\mathbf{Q})$;

(ii) $H_C^S=H_{C\cap[n]}^Q$;

(iii) $\phi^S(i)=\begin{cases}\phi^Q(i) & \text{if } i\le n\\ f\sigma & \text{if } i=n+1\end{cases}.$

(b) *Define* $C_{H_0,s_0}(\mathbf{S})$ *to be the* C *defined in* (1) (a) (i), *and for* $H\in\mathcal{C}, 1\le s\le\overline{n}(H)$, $(H,s)\ne(H_0,s_0)$, *define* $C_{H,s}(\mathbf{S})=C_{H,s}(\mathbf{Q})$.

Then $\gamma_{H_0,(\sigma,s_0,f)}$ *is a* 1–1 *map of* $S_n(\overline{n})$ *into* $S_{n+1}(\overline{n})$.

(2) *Conversely, if* $\mathbf{S}\in S_{n+1}(\overline{n})$, *and if* $[n+1]_{R^S}$ *is not a singleton set, and if this* R^S*-equivalence class is a 'distinguished class' – meaning that* $[n+1]_{R^S}=C_{H_0,s_0}(\mathbf{S})$ *for a necessarily unique* (H_0,s_0) *– then there exists a unique* $f\in H_0\backslash N(H_0)$, $\sigma\in\{\sigma_\kappa^{H_0}:1\le\kappa\le[G:N(H_0)]\}$ *and a* $\mathbf{Q}\in S_n(\overline{n})$ *such that* $\gamma_{H_0,(\sigma,s_0,f)}(\mathbf{Q})=\mathbf{S}$.

(3) *Suppose* $\pi,\chi\in\widehat{G(\overline{n})}$, *and suppose* $L:V_\pi\to V_\chi$ *is a non-zero* $G(\overline{n}-1_{H_0})$*-linear map. Also suppose that* $\sigma\in\{\sigma_\kappa^{H_0}:1\le\kappa\le[G:N(H_0)]\}$. *Then the equation*

$$\begin{aligned}&(C_{H_0,\sigma}(L))(\mathbf{T}\otimes\xi)\\&\quad=\sum_{(s,f)}(\gamma_{H_0,(\sigma,s,f)}(\mathbf{T})\otimes\chi(\sigma(s,f))L\pi(\sigma(s,f)^{-1})\xi)+W_{n+1}^0(\overline{n},\chi),\end{aligned}$$

where the sum ranges over $1\le s\le\overline{n}(H_0)$ *and* $f\in H_0\backslash N(H_0)$, *defines a non-zero* $A_n^G(d)$*-linear map* $C_{H_0,\sigma}(L):V_n(\overline{n},\pi)\to(V_{n+1}(\overline{n},\chi)/W_{n+1}^0(\overline{n},\chi))$.

Proof. The statements (1) and (2) are established exactly like their counterparts in Lemma 25 after observing that every element of $H_0\backslash G$ is uniquely expressible as $f\sigma$ for $f\in H_0\backslash N(H_0)$ and $\sigma\in\{\sigma_\kappa^{H_0}:1\le\kappa\le[G:N(H_0)]\}$. For (3), again as in Lemma 25, it will suffice to verify that $C_{H_0,\sigma}(L)$ is $J_n(\overline{n})$-linear. Thus we need to verify that

$$(C_{H_0,\sigma}(L))(P\cdot((\mathbf{T}\otimes\xi))=\tilde{P}\cdot((C_{H_0,\sigma}(L))(\mathbf{T}\otimes\xi))\ \forall\mathbf{T}\in S_n(\overline{n}),\xi\in V_\pi,$$

whenever either (i) $P\in I_n(\overline{m})$, where $\overline{m}\in N_{[n]}, t(\overline{m})<t(\overline{n})$, or (ii) $P=(\mathbf{Q},\rho,\mathbf{R})\in I_n(\overline{n})$.

We first show that in case (i), both sides of the desired equation reduce to zero. Since $t(P\cdot(\mathbf{T},1,\mathbf{S}_0))\le t(P)<t(\overline{n})$, it is seen that the left side of the above equation is, indeed, zero. To evaluate the right side, we have to examine such products as $\tilde{P}\cdot(\gamma_{H_0,(\sigma,s,f)}(\mathbf{T}),1,\mathbf{S}_0)$, and it will suffice to show, therefore, that in this case, either such a product has less than $t(\overline{n})$ through classes, or such a product has exactly $t(\overline{n})$ through classes, in which case its 'top' belongs to $W_{n+1}^0(\overline{n})$. In any case, since the second term of the product has

exactly $t(\overline{n})$ through classes, the product can have at most $t(\overline{n})$ through classes. So, we may assume without loss of generality that the product has exactly $t(\overline{n})$ through classes. By the last line of Remark 27(1), it suffices therefore to show that if the 'top' of our product is $\mathbf{S}$, and if $\{n+1\}$ is not an R^S-equivalence class, then the product cannot have $t(\overline{n})$ through classes; but this is an easy consequence of the assumption that $t(P) < t(\overline{n})$.

To discuss the second (and less trivial) case, we will again find it convenient to introduce an auxiliary element $\tilde{\gamma}_{H_0,(\sigma,s,f)}$ which enables us to regard the mapping $\gamma_{H_0,(\sigma,s,f)}$ as a sort of right-multiplication. Thus, define $\tilde{\gamma}_{H_0,(\sigma,s,f)} \in I_{n+1}(\overline{n})$ by

$$\tilde{\gamma}_{H_0,(\sigma,s,f)} = (\gamma_{H_0,(\sigma,s,f)}(\mathbf{S}_0), 1, \mathbf{S}_0).$$

We can now state the desired analogue of eq. (4.23), namely:

$$(\mathbf{S}, g, \mathbf{S}_0)\tilde{} \cdot \tilde{\gamma}_{H_0,(\sigma,s,f)} = (\gamma_{H_0,(\sigma,g\cdot(s,f))}(\mathbf{S}), g, \mathbf{S}_0) \tag{4.26}$$

for all $\mathbf{S} \in S_n(\overline{n} - 1_{H_0}), g \in G(\overline{n} - 1_{H_0})$, where the g on the right is g thought of as an element of $G(\overline{n})$ (via the natural inclusion) and $g \cdot (s,f)$ refers to the action of $G(\overline{n})$ as in Lemma 28(2) applied to $\overline{m} = \overline{n} - 1_{H_0}$. As in the case of eq. (4.23), this equation is also verified by looking at the picture represented by the product on the left side, noting that it does belong to $I_{n+1}(\overline{n})$, and checking that its three ingredients are indeed as given by the right side of (4.26).

Suppose now that $P = (\mathbf{Q}, \rho, \mathbf{R}) \in I_n(\overline{n})$. If we let $g = \rho\beta^{\mathbf{R}}_{\mathbf{T}}$, we find that $(\mathbf{Q}, \rho, \mathbf{R}) \cdot (\mathbf{T}, 1, \mathbf{S}_0) = D(\mathbf{R}, \mathbf{T})(\mathbf{Q}, g, \mathbf{S}_0)$, and hence

$$\begin{aligned}(C_{H_0,\sigma}(L))&(P \cdot (\mathbf{T} \otimes \xi)) \\ &= (C_{H_0,\sigma}(L))(D(\mathbf{R}, \mathbf{T})(\mathbf{Q} \otimes \pi(g)\xi)) \\ &= D(\mathbf{R}, \mathbf{T}) \sum_{(s_1,f_1)} (\gamma_{H_0,(\sigma,s_1,f_1)}(\mathbf{Q}) \otimes \chi(\sigma(s_1,f_1))L\pi(\sigma(s_1,f_1)^{-1})\pi(g)\xi) \\ &\qquad + W^0_{n+1}(\overline{n}, \chi);\end{aligned} \tag{4.27}$$

writing $(s_1, f_1) = g \cdot (s, f)$, and noting that

$$\begin{aligned} L\pi(\sigma(s_1,f_1)^{-1})\pi(g) &= L\pi(\sigma(s_1,f_1)^{-1}g) \\ &= L\pi(\sigma(s_1,f_1)^{-1}g\sigma(s,f))\pi(\sigma(s,f)^{-1}) \\ &= \chi(\sigma(s_1,f_1)^{-1}g\sigma(s,f))L\pi(\sigma(s,f)^{-1}) \end{aligned}$$

(where we have used the fact that $\sigma(s_1,f_1)^{-1}g\sigma(s,f) \in G(\overline{n} - 1_{H_0})$), we see thus that the right side of eq. (4.27) may be rewritten (after a change of variable) as

$$D(\mathbf{R}, \mathbf{T}) \sum_{(s,f)} (\gamma_{H_0,(\sigma,g\cdot(s,f))}(\mathbf{Q}) \otimes \chi(g\sigma(s,f))L\pi(\sigma(s,f)^{-1})\xi) + W^0_{n+1}(\overline{n}, \chi). \tag{4.28}$$

On the other hand, notice (by two applications of eq. (4.26)) that

$$\begin{aligned}(\mathbf{Q}, \rho, \mathbf{R})\tilde{} \cdot (\gamma_{H_0,(\sigma,s,f)}(\mathbf{T}), 1, \mathbf{S}_0) &= (\mathbf{Q}, \rho, \mathbf{R})\tilde{} \cdot (\mathbf{T}, 1, \mathbf{S}_0)\tilde{} \cdot \tilde{\gamma}_{H_0,(\sigma,s,f)} \\ &= ((\mathbf{Q}, \rho, \mathbf{R}) \cdot (\mathbf{T}, 1, \mathbf{S}_0))\tilde{} \cdot \tilde{\gamma}_{H_0,(\sigma,s,f)} \\ &= D(\mathbf{R}, \mathbf{T})(\mathbf{Q}, g, S_0)\tilde{} \cdot \tilde{\gamma}_{H_0,(\sigma,s,f)} \\ &= D(\mathbf{R}, \mathbf{T})(\gamma_{H_0,(\sigma,g\cdot(s,f))}(\mathbf{Q}), g, \mathbf{S}_0).\end{aligned}$$

Hence, we see that

$$\begin{aligned}\widetilde{P}\cdot(C_{H_0,\sigma}(L))(\mathbf{T}\otimes\xi))\\ &=(\mathbf{Q},\rho,\mathbf{R})\tilde{}\cdot\sum_{(s,f)}(\gamma_{H_0,(\sigma,s,f)}(\mathbf{T})\otimes\chi(\sigma(s,f))L\pi(\sigma(s,f)^{-1})\xi)+W^0_{n+1}(\overline{n},\chi)\\ &=D(\mathbf{R},\mathbf{T})\sum_{(s,f)}(\gamma_{H_0,(\sigma,g\cdot(s,f))}(\mathbf{Q})\otimes\chi(g\sigma(s,f))L\pi(\sigma(s,f)^{-1})\xi)+W^0_{n+1}(\overline{n},\chi),\end{aligned}$$

and the lemma is proved. □

Remark 30. (1) Fix n and $\overline{n}\in N_{[n+1]}$. Consider two cases now:

Case 1: $\overline{n}\in N_{[n]}$. Fix $H\in\mathcal{C}$ such that $\overline{n}(H)>0$ and define $\overline{W}^1_{n+1}(\overline{n};H)$ to be the subspace of $\mathbb{C}S_{n+1}(\overline{n})/W^0_{n+1}(\overline{n})$ spanned by $\{\gamma_{H,(\sigma,s,f)}(\mathbf{T})+W^0_{n+1}(\overline{n})\colon \mathbf{T}\in S_n(\overline{n}),\sigma\in\{\sigma^H_\kappa\colon 1\leq\kappa\leq[G:N(H)]\},1\leq s\leq\overline{n}(H),\ f\in H\backslash N(H)\}$. By Lemma 29(2), this spanning set is a basis, which can alternatively be described as $\{\mathbf{S}+W^0_{n+1}(\overline{n})\colon \mathbf{S}\in S_{n+1}(\overline{n}),[n+1]_{R_S}$ is a distinguished class which is not a singleton, and $H^S_{[n+1]}=H\}$.

Also set $\overline{W}^1_{n+1}(\overline{n})=\sum_{H\in\mathcal{C},\overline{n}(H)>0}\overline{W}^1_{n+1}(\overline{n};H)$, and note (again, by Lemma 29(2)) that this sum of subspaces is a direct sum.

Case 2: $\overline{n}\notin N_{[n]}$. In this case, define $\overline{W}^1_{n+1}(\overline{n})=\{0\}\subseteq\mathbb{C}S_{n+1}(\overline{n})/W^0_{n+1}(\overline{n})$.

In either case, set $W^1_{n+1}(\overline{n})$ be the inverse image, in $\mathbb{C}S_{n+1}(\overline{n})$, under the natural quotient map, of $\overline{W}^1_{n+1}(\overline{n})$, and observe that a basis for $W^1_{n+1}(\overline{n})$ is furnished by $\{\mathbf{S}\in S_{n+1}(\overline{n})\colon [n+1]_{R^S}$ is either a singleton or is a distinguished R^S-class$\}$.

(2) We will need to use the following elementary fact about induced representations. Let $\mathcal{G}_0$ be a subgroup of a finite group $\mathcal{G}$ and $\mathcal{G}=\coprod_{k=1}^n g_k\mathcal{G}_0$ with $g_1=1$. For $\chi\in\hat{\mathcal{G}}$ and $\tilde{\chi}=\mathrm{Ind}_{\mathcal{G}_0\uparrow\mathcal{G}}\mathrm{Res}_{\mathcal{G}\downarrow\mathcal{G}_0}(\chi)$, we may regard $V_{\tilde{\chi}}$ as the space $V_\chi\otimes\mathbb{C}(\mathcal{G}/\mathcal{G}_0)$ with $\mathcal{G}$-action defined by $g(v\otimes g_i\mathcal{G}_0)=\chi(h(g,i))(v)\otimes g_{g(i)}\mathcal{G}_0$ where $gg_i=g_{g(i)}h(g,i)$ with $g_{g(i)}\in\{g_1,\ldots,g_k\}$ and $h(g,i)\in\mathcal{G}_0$. Furthermore, for $\pi\in\hat{\mathcal{G}}$, there is a natural bijection between $\mathcal{G}_0$-linear maps $L: V_\pi\to V_\chi$ and $\mathcal{G}$-linear maps $\tilde{L}: V_\pi\to V_{\tilde{\chi}}$ given by $\tilde{L}(\xi)=\sum_{k=1}^n L\pi(g_k^{-1})(\xi)\otimes g_k\mathcal{G}_0$.

(3) Fix $\overline{n}\in N_{[n]}$ and $H\in\mathcal{C}$ such that $\overline{n}(H)>0$. Also fix $\chi\in\widehat{G(\overline{n})}$. Let $\tilde{\chi}=\mathrm{Ind}_{G(\overline{n}-1_H)\uparrow G(\overline{n})}\mathrm{Res}_{G(\overline{n})\downarrow G(\overline{n}-1_H)}(\chi)$, and for appropriate $\pi\in\widehat{G(\overline{n})}$, choose non-zero $G(\overline{n})$-linear maps $\tilde{L}_{\pi,j}: V_\pi\to V_{\tilde{\chi}}$ so that the ranges of all these maps yield a direct sum decomposition of $V_{\tilde{\chi}}$. Let $L_{\pi,j}: V_\pi\to V_\chi$ be the $G(\overline{n}-1_H)$-linear map which is related to $\tilde{L}_{\pi,j}$ as in the above paragraph. We wish now to assert that

$$\oplus_{\sigma,j,\pi}\ \text{range}\ C_{H,\sigma}(L_{\pi,j})=\overline{W}^1_{n+1}(\overline{n};H)\otimes V_\chi; \tag{4.29}$$

hence the right side represents an $A^G_n(d)$-submodule of $V_{n+1}(\overline{n},\chi)/W^0_{n+1}(\overline{n},\chi)$.

Reason: Identify $V_{n+1}(\overline{n},\chi)/W^0_{n+1}(\overline{n},\chi)$ with $(\mathbb{C}S_{n+1}(\overline{n})/W^0_{n+1}(\overline{n}))\otimes V_\chi$; the definition of $C_{H,\sigma}(L)$ shows that every summand on the left (of eq. (4.29)) is contained in the space on the right. To see that the sum is direct, define $\Phi:\overline{W}^1_{n+1}(\overline{n};H)\otimes V_\chi\to\mathrm{Hom}(\mathbb{C}S_n(\overline{n}),\mathbb{C}(N(H)\backslash G)\otimes V_{\tilde{\chi}})$ as follows: an arbitrary element $Z\in\overline{W}^1_{n+1}(\overline{n};H)\otimes V_\chi$ can be expressed uniquely as $Z=\sum_{\mathbf{T}\in S_n(\overline{n})}\sum_{(\sigma,s,f)}\gamma_{H,(\sigma,s,f)}(\mathbf{T})\otimes\xi^{\mathbf{T}}_{(\sigma,s,f)}+W^0_{n+1}(\overline{n},\chi)$, where $\xi^{\mathbf{T}}_{(\sigma,s,f)}\in V_\chi$ for all (σ,s,f); define

$$\Phi(Z)(\mathbf{T})\in\mathbb{C}(N(H)\backslash G)\otimes V_{\tilde{\chi}}=\mathbb{C}(N(H)\backslash G)\otimes V_\chi\otimes\mathbb{C}(G(\overline{n})/G(\overline{n}-1_H))$$

by

$$\Phi(Z)(\mathbf{T}) = \sum_{(\sigma,s,f)} N(H)\sigma \otimes \chi(\sigma(s,f)^{-1})(\xi^{\mathbf{T}}_{(\sigma,s,f)}) \otimes \sigma(s,f)G(\overline{n}-1_H).$$

The map Φ is clearly injective since knowledge of all $\Phi(Z)(\mathbf{T})$ determines all the $\xi^{\mathbf{T}}_{(\sigma,s,f)}$ and hence $\Phi(Z)$ determines Z; i.e., Φ is an injective (clearly linear) map. Further, it is easy to see that if $Z \in$ range $C_{H,\sigma}(L)$ then range $\Phi(Z) \subseteq \mathbb{C}(N(H)\sigma) \otimes$ range $\tilde{L}$. Together with the injectivity of Φ and the choice of $L_{\pi,j}$, this implies that the ranges of the $C_{H,\sigma}(L_{\pi,j})$'s form a direct sum. Finally, a dimension count – using Frobenius reciprocity for the dimension of the left side (of eq. (4.29)), and the explicitly listed basis for the first factor of the tensor product on the right – shows that both sides of eq. (4.29) have dimension $|S_n(\overline{n})|d_\chi[G(\overline{n}) : G(\overline{n}-1_H)][G : N(H)]$; and therefore the direct sum on the left exhausts the space on the right.

(4) Define $\overline{W}^1_{n+1}(\overline{n},\chi) = \overline{W}^1_{n+1}(\overline{n}) \otimes V_\chi$, and as before, let $W^1_{n+1}(\overline{n},\chi)$ be the inverse image, in $V_{n+1}(\overline{n},\chi)$, of $\overline{W}^1_{n+1}(\overline{n},\chi)$, under the natural quotient mapping. If $\overline{n} \in N_{[n]}$ (as in (3) above), it then follows that

$$\overline{W}^1_{n+1}(\overline{n},\chi) = \oplus_{H\in\mathcal{C},\overline{n}(H)>0} \oplus_{\sigma,\pi,j} \text{ range } C_{H,\sigma}(L_{\pi,j});$$

since $\overline{W}^1_{n+1}(\overline{n},\chi) = \{0\}$ if $\overline{n} \notin N_{[n]}$, we find thus, in any case, that $W^1_{n+1}(\overline{n},\chi)$ is an $A^G_n(d)$-submodule of $V_{n+1}(\overline{n},\chi)$.

Lemma 31. *Fix* $\overline{n} \in N_{[n]}, H_0 \in \mathcal{C}, 1 \leq s_0 \leq \overline{n}(H_0)+1,\ f \in H_0\backslash N(H_0)$ *and* $\sigma \in \{\sigma^{H_0}_\kappa: 1 \leq \kappa \leq [G:N(H_0)]\}$.

(1) *If* $\mathbf{Q} \in S_n(\overline{n}+1_{H_0})$, *define* $\delta_{H_0,(\sigma,s_0,f)}(\mathbf{Q}) = \mathbf{S}$, *thus*:

(a) (i) R^S *is defined exactly as in Proposition* 29(1) (a) (i);
(ii) $H^S_C = H^Q_{C\cap[n]}$;
(iii) $\phi^S(i) = \begin{cases} \phi^Q(i) & \textit{if } i \leq n \\ f\sigma & \textit{if } i = n+1 \end{cases}.$

(b) *Define*

$$C_{H,s}(\mathbf{S}) = \begin{cases} C_{H,s}(\mathbf{Q}) & \textit{if } H \neq H_0 \textit{ or } H = H_0, 1 \leq s < s_0 \\ C_{H_0,s+1}(\mathbf{Q}) & \textit{if } H = H_0 \textit{ and } s_0 \leq s \leq \overline{n}(H_0). \end{cases}$$

Then $\delta_{H_0,(\sigma,s_0,f)}$ *defines a* 1–1 *map of* $S_n(\overline{n}+1_{H_0})$ *into* $S_{n+1}(\overline{n})$.

(2) *Conversely*, *if* $\mathbf{S} \in S_{n+1}(\overline{n})$, *and if* $[n+1]_{R^S}$ *is not a singleton set*, *and if this* R^S-*equivalence class is not a 'distinguished class' – then there exists a unique* $H_0 \in \mathcal{C}$, *a unique* $1 \leq s_0 \leq \overline{n}(H_0)+1$, *a unique* $f \in H_0\backslash N(H_0)$, *a unique* $\sigma \in \{\sigma^{H_0}_\kappa : 1 \leq \kappa \leq [G:N(H_0)]\}$ *and a unique* $\mathbf{Q} \in S_n(\widehat{\overline{n}+1_{H_0}})$ *such that* $\delta_{\mathcal{H}_0,(\sigma,s_0,f)}(\mathbf{Q}) = \mathbf{S}$.

(3) *Suppose* $\pi \in \widehat{G(\overline{n}+1_{H_0})}, \chi \in \widehat{G(\overline{n})}$, *and suppose* $L : V_\pi \to V_\chi$ *is a non-zero* $G(\overline{n})$-*linear map. Then the equation*

$$(D_{H_0,\sigma}(L))(\mathbf{T}\otimes\xi) = \sum_{(s,f)} (\delta_{H_0,(\sigma,s,f)}(\mathbf{T}) \otimes L\pi(\sigma(s,f)^{-1})\xi) + W^1_{n+1}(\overline{n},\chi), \tag{4.30}$$

where the sum ranges over all choices $1 \leq s \leq \overline{n}(H_0), f \in H_0\backslash N(H_0)$, (*with* $\sigma(s,f)$ *as in Lemma 28 applied with* $\overline{m} = \overline{n}+1_{H_0}$), *defines a non-zero* $A^G_n(d)$-*linear map* $D_{H_0,\sigma}(L)$: $V_n(\overline{n}+1_{H_0},\pi) \to (V_{n+1}(\overline{n},\chi)/W^1_{n+1}(\overline{n},\chi))$.

Proof. The statements (1) and (2) are established exactly like their counterparts in Lemma 25. For (3), exactly as in Lemma 25, it will suffice to verify that $D_{H_0,\sigma}(L)$ is $J_n(\overline{n}+1_{H_0})$-linear. Thus we need to verify that

$$(D_{H_0,\sigma}(L))(P\cdot(\mathbf{T}\otimes\xi)) = \tilde{P}\cdot(D_{H_0,\sigma}(L))(\mathbf{T}\otimes\xi))\ \forall\mathbf{T}\in S_n(\overline{n}+1_{H_0}), \xi\in V_\pi,$$

whenever either (i) $P\in I_n(\overline{m})$, where $\overline{m}\in N_{[n]}, t(\overline{m}) < t(\overline{n}+1_{H_0})$, or (ii) $P=(\mathbf{Q},\rho,\mathbf{R})\in I_n(\overline{n}+1_{H_0})$.

We first show that in case (i), both sides of the desired equation reduce to zero. Since $t(P\cdot(\mathbf{T},1,\mathbf{S}_0))\le t(P) < t(\overline{n}+1_{H_0})$, it is seen that the left side of the above equation is, indeed, zero. To evaluate the right side, we have to examine such products as $\tilde{P}\cdot(\delta_{H_0,(\sigma,s,f)}(\mathbf{T}),1,\mathbf{S}_0)$, and it will suffice to show, therefore, that in this case, either such a product has less than $t(\overline{n})$ through classes, or such a product has exactly $t(\overline{n})$ through classes, in which case its 'top' belongs to $W^1_{n+1}(\overline{n})$. In any case, since the second term of the product has exactly $t(\overline{n})$ through classes, the product can have at most $t(\overline{n})$ through classes. So, we may assume without loss of generality that the product has exactly $t(\overline{n})$ through classes. By the last line of Remark 30(1), it suffices therefore to show that if the 'top' of our product is $\mathbf{S}$, and if $[n+1]_{R^S}$ is neither a singleton nor a distinguished R^S-class, then the product cannot have $t(\overline{n})$ through classes; but this is an easy consequence of the assumption that $t(P) < t(\overline{n}+1_{H_0})$.

To discuss the second (and less trivial) case, we will again find it convenient to introduce an auxiliary element $\tilde{\delta}_{H_0,(\sigma,s,f)}$ which enables us to regard the mapping $\delta_{H_0,(\sigma,s,f)}$ as a sort of right-multiplication. Thus, define $\tilde{\delta}_{H_0,(\sigma,s,f)}\in I_{n+1}(\overline{n})$ by

$$\tilde{\delta}_{H_0,(\sigma,s,f)} = (\delta_{H_0,(\sigma,s,f)}(\mathbf{S}_0(n,\overline{n}+1_{H_0})),1,\mathbf{S}_0(n+1,\overline{n})).$$

We come now to the desired analogue of eq. (4.26), namely:

$$(\mathbf{S},g,\mathbf{S}_0)\tilde{}\cdot\tilde{\delta}_{H_0,(\sigma,s,f)} = (\delta_{H_0,(\sigma,g\cdot(s,f))}(\mathbf{S}),\tilde{g},\mathbf{S}_0) \tag{4.31}$$

for all $\mathbf{S}\in S_n(\overline{n}+1_{H_0}), g\in G(\overline{n}+1_{H_0})$, where $\tilde{g}=\sigma(g\cdot(s,f))^{-1}g\sigma(s,f)$, which is an element of $G(\overline{n})$. As in the case of eq. (4.26), this equation is also verified by looking at the picture represented by the product on the left side, noting that it does belong to $I_{n+1}(\overline{n})$, and checking that its three ingredients are indeed as given by the right side of (4.31).

Suppose now that $P=(\mathbf{Q},\rho,\mathbf{R})\in I_n(\overline{n}+1_{H_0})$. If we let $g=\rho\beta^{\mathbf{R}}_{\mathbf{T}}$, we find that $(\mathbf{Q},\rho,\mathbf{R})\cdot(\mathbf{T},1,\mathbf{S}_0) = D(\mathbf{R},\mathbf{T})(\mathbf{Q},g,\mathbf{S}_0)$, and hence

$$\begin{aligned}(D_{H_0,\sigma}(L))(P\cdot(\mathbf{T}\otimes\xi)) &= (D_{H_0,\sigma}(L))(D(\mathbf{R},\mathbf{T})(\mathbf{Q}\otimes\pi(g)\xi))\\ &= D(\mathbf{R},\mathbf{T})\sum_{(s_1,f_1)}(\delta_{H_0,(\sigma,s_1,f_1)}(\mathbf{Q})\otimes L\pi(\sigma(s_1,f_1)^{-1})\pi(g)\xi) + W^1_{n+1}(\overline{n},\chi);\end{aligned} \tag{4.32}$$

writing $(s_1,f_1)=g\cdot(s,f)$, and noting that

$$\begin{aligned}L\pi(\sigma(s_1,f_1)^{-1})\pi(g) &= L\pi(\sigma(s_1,f_1)^{-1}g)\\ &= L\pi(\sigma(s_1,f_1)^{-1}g\sigma(s,f))\pi(\sigma(s,f)^{-1})\\ &= \chi(\sigma(s_1,f_1)^{-1}g\sigma(s,f))L\pi(\sigma(s,f)^{-1}),\end{aligned}$$

(where we have used the fact that $\sigma(s_1, f_1)^{-1} g\sigma(s, f)) \in G(\overline{n})$) we see thus that the right side of eq. (4.32) may be rewritten (after a change of variable) as

$$D(\mathbf{R}, \mathbf{T}) \sum_{(s,f)} (\delta_{H_0,(\sigma,g\cdot(s,f))}(\mathbf{Q}) \otimes \chi(\tilde{g}) L\pi(\sigma(s,f)^{-1})\xi) + W^1_{n+1}(\overline{n}, \chi),$$

where $\tilde{g} = \sigma(g \cdot (s, f))^{-1} g\sigma(s, f)$, as before.

On the other hand, notice (by two applications of eq. (4.31)) that

$$\begin{aligned}(\mathbf{Q}, \rho, \mathbf{R})^{\sim} \cdot (\delta_{H_0,(\sigma,s,f)}(\mathbf{T}), 1, \mathbf{S}_0) &= (\mathbf{Q}, \rho, \mathbf{R})^{\sim} \cdot (\mathbf{T}, 1, \mathbf{S}_0)^{\sim} \cdot \tilde{\delta}_{H_0,(\sigma,s,f)} \\ &= ((\mathbf{Q}, \rho, \mathbf{R}) \cdot (\mathbf{T}, 1, \mathbf{S}_0))^{\sim} \cdot \tilde{\delta}_{H_0,(\sigma,s,f)} \\ &= D(\mathbf{R}, \mathbf{T})(\mathbf{Q}, g, S_0)^{\sim} \cdot \tilde{\delta}_{H_0,(\sigma,s,f)} \\ &= D(\mathbf{R}, \mathbf{T})(\delta_{H_0,(\sigma,g\cdot(s,f))}(\mathbf{Q}), \tilde{g}, \mathbf{S}_0).\end{aligned}$$

Hence, we see that

$$\begin{aligned}&\tilde{P} \cdot (D_{H_0,\sigma}(L))(\mathbf{T} \otimes \xi)) \\ &\quad = (\mathbf{Q}, \rho, \mathbf{R})^{\sim} \cdot \sum_{(s,f)} (\delta_{H_0,(\sigma,s,f)}(\mathbf{T}) \otimes L\pi(\sigma(s,f)^{-1})\xi) + W^1_{n+1}(\overline{n}, \chi) \\ &\quad = D(\mathbf{R}, \mathbf{T}) \sum_{(s,f)} (\delta_{H_0,(\sigma,g\cdot(s,f))}(\mathbf{Q}) \otimes \chi(\tilde{g}) L\pi(\sigma(s,f)^{-1})\xi) + W^1_{n+1}(\overline{n}, \chi),\end{aligned}$$

and the lemma is proved. □

Remark 32. (1) Fix n, $\overline{n} \in N_{[n+1]}$ and $H \in \mathcal{C}$. Consider two cases now:

Case 1: $\overline{n} + 1_H \in N_{[n]}$. Define $\overline{W}^2_{n+1}(\overline{n}; H)$ to be the subspace of $\mathbb{C}S_{n+1}(\overline{n})/W^1_{n+1}(\overline{n})$ spanned by $\{\delta_{H,(\sigma,s,f)}(\mathbf{T}) + W^1_{n+1}(\overline{n}) : \mathbf{T} \in S_n(\overline{n} + 1_H), 1 \leq s \leq \overline{n}(H) + 1, f \in H\backslash N(H), \sigma \in \{\sigma^H_\kappa : 1 \leq \kappa \leq [G : N(H)]\}\}$. By Lemma 31(2), this spanning set is a basis which can also be characterized as $\{\mathbf{S} + W^1_{n+1}(\overline{n}) : \mathbf{S} \in S_{n+1}(\overline{n}), [n+1]_{R_S}$ is not distinguished and not a singleton, and $H^S_{[n+1]} = H\}$.

Case 2: $\overline{n} + 1_H \notin N_{[n]}$. In this case, define $\overline{W}^2_{n+1}(\overline{n}; H) = \{0\} \subseteq \mathbb{C}S_{n+1}(\overline{n})/W^1_{n+1}(\overline{n})$. Observe that Lemma 31(2) also implies that

$$\mathbb{C}S_{n+1}(\overline{n})/W^1_{n+1}(\overline{n}) = \oplus_{H\in\mathcal{C}, \overline{n}+1_H\in N_{[n]}} \overline{W}^2_{n+1}(\overline{n}; H).$$

(2) We will need the following slightly stronger version of Remark 30(2): Let $\mathcal{G}_0$ be a subgroup of a finite group $\mathcal{G}$ and $\mathcal{G} = \coprod_{k=1}^n g_k\mathcal{G}_0$ with $g_1 = 1$. For $\chi \in \hat{\mathcal{G}_0}$ and $\tilde{\chi} = \mathrm{Ind}_{\mathcal{G}_0\uparrow\mathcal{G}}(\chi)$, we may regard $V_{\tilde{\chi}}$ as the space $V_\chi \otimes \mathbb{C}(\mathcal{G}/\mathcal{G}_0)$ with $\mathcal{G}$-action defined by $g(v \otimes g_i\mathcal{G}_0) = \chi(h(g, i))(v) \otimes g_{g(i)}\mathcal{G}_0$ where $gg_i = g_{g(i)}h(g, i)$ with $g_{g(i)} \in \{g_1, \ldots, g_k\}$ and $h(g, i) \in \mathcal{G}_0$. Furthermore, for $\pi \in \hat{\mathcal{G}}$, there is a natural bijection between $\mathcal{G}_0$-linear maps $L : V_\pi \to V_\chi$ and $\mathcal{G}$-linear maps $\tilde{L} : V_\pi \to V_{\tilde{\chi}}$ given by $\tilde{L}(\xi) = \sum_{k=1}^n L\pi(g_k^{-1})(\xi) \otimes g_k\mathcal{G}_0$.

(3) Fix $H \in \mathcal{C}$ such that $\overline{n} + 1_H \in N_{[n]}$; also fix $\chi \in \widehat{G(\overline{n})}$, and let $\tilde{\chi}$ be the result of inducing χ up to $G(\overline{n} + 1_H)$. For appropriate $\pi \in \widehat{G(\overline{n} + 1_H)}$, pick non-zero $G(\overline{n} + 1_H)$-linear maps $\tilde{L}_{\pi,j} : V_\pi \to V_{\tilde{\chi}}$ such that their ranges afford a direct sum decomposition of $V_{\tilde{\chi}}$. Let $L_{\pi,j}$ be related to $\tilde{L}_{\pi,j}$ as in (2) above. Then, by an argument exactly analogous to the corresponding one used in Remark 30(3), it may be verified that:

$$\oplus_{\sigma,\pi,j} \text{ range } D_{H,\sigma}(L_{\pi,j}) = \overline{W}^2_{n+1}(\overline{n}; H) \otimes V_\chi. \tag{4.33}$$

(4) We conclude from (1) and (3) that

$$V_{n+1}(\overline{n},\chi)/W^1_{n+1}(\overline{n},\chi) = \oplus_{H\in\mathcal{C},\overline{n}+1_H\in N_{[n]}} \oplus_{\sigma,\pi,j} \text{ range } D_{H,\sigma}(L_{\pi,j})$$

which is a decomposition of the left hand side as a direct sum of irreducible $A_n^G(d)$-modules.

All the pieces are now in place for the required description of the Bratteli diagram for the inclusion $A_n^G(d) \subset A_{n+1}^G(d)$.

Theorem 33. *Fix $n \in \mathbb{N}$ and let d be any positive number satisfying the hypothesis of Theorem* 21. *Then*,

(a) *The set $\widehat{A_n^G(d)}$ of irreducible representations of $A_n^G(d)$ can be parametrised by the set $\{(n,\overline{n},\pi) : \overline{n} \in N_{[n]}, \pi \in \widehat{G(\overline{n})}\}$, in such a way that the associated module $V_n(\overline{n},\pi)$ has dimension equal to $|S_n(\overline{n})|d_\pi$.*

(b) *When viewed as an $A_n^G(d)$-module, the multiplicity with which the module $V_{n+1}(\overline{n},\chi)$ contains $V_n(\overline{m},\pi)$ is given by*:

  (i) $\langle\chi|_{G(\overline{m})},\pi\rangle[G:N(H)]$, *if* $\overline{n}=\overline{m}+1_H$ *for some* $H\in\mathcal{C}$;

  (ii) $\delta_{\pi,\chi}\sum_{H\in\mathcal{C}}[G:N(H)] + \sum_{H\in\mathcal{C},\overline{n}(H)>0}[G:N(H)]\langle\tilde{\chi},\pi\rangle$, *if* $\overline{n}=\overline{m}$, *where* $\tilde{\chi} = \mathrm{Ind}_{G(\overline{n}-1_H)\uparrow G(\overline{n})}\mathrm{Res}_{G(\overline{n})\downarrow G(\overline{n}-1_H)}\chi$;

  (iii) $\langle\pi|_{G(\overline{n})},\chi\rangle[G:N(H)]$, *if* $\overline{n}=\overline{m}-1_H$, *for some* $H\in\mathcal{C}$; *and*

  (iv) 0, *otherwise.*

Proof. (a) is an immediate consequence of Theorem 21, while (b) follows from Remark 27(4), Remark 30(4), Remark 32(4), and the simple fact that if W is a sub-module of a semi-simple module V, then $V \cong W \oplus (V/W)$. □

5. Concluding remarks

We wish to make a few remarks in two directions: first, we wish to discuss the special case of the algebra $A_1^G(d)$ and the question of whether, as a filtered algebra (with filtering given by the number of through classes), this determines the group G; and second, we wish to discuss the trivial specialization $|G| = 1$, which has already appeared in the literature.

The filtered algebra $A_1^G(d)$

The algebra $A_1^G(d)$ admits the filtration given by the ideals $I_k^G, 0 \le k \le |G|$. Further, it should be clear from the definitions that if $k \neq 0$, then $N_{1;k} \neq \emptyset$ only when $k|n$, in which case, $\overline{n} \in N_{1;k}$ if and only if there exists $H_0 \in \mathcal{C}$ such that $[G : H_0] = k$ and $\overline{n}(H) = \delta_{H,H_0}$. In particular, if we assume, further, that G is abelian, then, for any fixed divisor k of $|G|$, we see from Theorem 21 that

$$I_k^G/I_{k-1}^G \cong \oplus_H(\mathbb{C}^k \otimes M_k(\mathbb{C})),$$

where the direct sum is over all subgroups H of G of index k. In particular, the knowledge of the filtered algebra $A_1^G(d)$ amounts, in case G is abelian, to the knowledge of the number $s_G(l)$ of subgroups of G of any given order l. It is a curious fact that this knowledge of an abelian group G – i.e., of the function $s_G : \mathbb{N} \to \mathbb{N} \cup \{0\}$ defined by the previous sentence – completely determines the abelian group G (up to isomorphism).

It is natural to ask whether the filtered algebra $A_1^G(d)$ determines the isomorphism class of the (general, possibly non-abelian) group G. What we can see is that the filtered algebra $A_1^G(d)$ determines whether or not G is simple, and more generally, it can detect the set of orders of all quotients of G.

The case $|G| = 1$

When $G = \{1\}$ is the trivial group, then what we have called $A_n^G(d)$ is exactly the same as what we called $A_n(d)$; this algebra was originally discussed in [J], where it was shown that at least when $d = k$, the algebra $A_n(d)$ can be identified with the commutant of the natural representation (given by the diagonal action) of Σ_k on $\otimes^n V$, where V is a k-dimensional vector space, provided that $k \geq 2n$.

Algebras related to certain subalgebras of the general $A_n(d)$ have occurred in numerous contexts – for instance, see [B, J, W]; (see also Remark 2).

Recall that the Temperley–Lieb algebra is the subalgebra of $A_n(d)$ generated by those equivalence relations which have the property that all equivalence classes are two-element sets, and whose diagrams are planar. Another subalgebra, call it $B_n(d)$, is obtained by dropping this planarity requirement. The structure of the inclusion $B_n \subset A_n$ may be analysed using techniques similar to the ones discussed here, and involve the representation theory of the various symmetric groups and the 'induction-restriction' relations between several naturally arising subgroups thereof.

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