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On vector equilibrium problem

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Abstract. This paper presents some existence results of a vector equilibrium problem. The several important special cases of the vector equilibrium problem are also discussed.

Keywords. Vector equilibrium problem; KKM–Fan lemma; P-monotone and P-convex mappings; P-maximum monotonicity.

1. Introduction

Let X be a real topological vector space; $K \subset X$ be a nonempty closed convex set; (Y, P) be a real ordered topological vector space with a partial order (or vector order) $\leq_P$ induced by a solid pointed closed convex cone P, thus $x \leq_P y \Longleftrightarrow y - x \in P$, $\forall x, y \in Y$; $f : X \times X \longrightarrow Y$ with $f(x, x) = 0$ for all $x \in X$. The vector equilibrium problem is to find $x \in K$, such that

(VEP) $\quad f(x, y) \notin -\operatorname{int} P, \quad \forall y \in K,$

where $\operatorname{int} P$ denotes interior of P. This problem includes as special cases, vector optimization problems, vector complementarity problems, fixed points problems, vector variational inequality problems etc. If $Y = \mathbb{R}$, $P = \mathbb{R}_+$ then (VEP) reduces to the equilibrium problem of finding $x \in K$ such that

$$f(x, y) \geq 0, \quad \forall y \in K. \tag{1}$$

This problem was considered and studied by Blum and Oettli [B–R]. In this paper, by making use of KKM–Fan lemma [F1], we prove some existence results for the vector equilibrium problem (VEP) in the case where

$$f(x, y) = g(x, y) + f(x, y). \tag{2}$$

Also, we review some special cases of (VEP).

2. Special cases

In this section we review some of the important examples for vector equilibrium problem (VEP). In the examples below X^*, the topological dual of X, should be topologized in such a way that the cannonical bilinear form $\langle \cdot, \cdot \rangle$ is continuous on $X^* \times X^*$.

DEFINITION 1

A function $f(\cdot, \cdot) : K \times K \longrightarrow Y$ is called P-monotone if and only if

$$f(x, y) + f(y, x) \in -P, \quad \forall x, y \in K.$$

(i) *Vector optimization*

Let $\phi : K \longrightarrow Y$, then to find $x \in K$ such that

$$\phi(y) - \phi(x) \notin -\operatorname{int} P, \quad \forall y \in K. \tag{3}$$

Equation (3) is equivalent to the following:

$$W_{\min}\phi(x) \text{ subject to } x \in K, \tag{4}$$

where $W_{\min}$ denotes weak minima, see for instance [C] and [C–C2]. Setting

$$f(x, y) := \phi(y) - \phi(x).$$

Then problem (3) coincides with (VEP) provided that f is P-monotone.

(ii) *Convex differentiable vector optimization*

Besides the significant connection between vector optima and vector equilibrium given in preceding example, there is a more subtle connection in the convex and differentiable case. Let $\phi : X \longrightarrow Y$ be P-convex and linear Gateaux differentiable. Then the problem (4) and the vector variational inequality of finding

$$x \in K, \quad \text{such that } \langle \phi'(x), y - x\rangle \notin -\operatorname{int} P, \quad \forall y \in K, \tag{5}$$

have the same set of solutions, see for instance [C] and [C–C2].

Upon setting $f(x, y) := \langle \phi'(x), y - x\rangle$ this becomes an example of (VEP). The function f is P-monotone in this case since $\phi'(\cdot)$ is P-monotone.

(iii) *Nonconvex differentiable vector optimization*

We can replace convexity by invexity in the preceding example as follows:

DEFINITION 2

A set $K \subseteq X$ is called *invex* if there is a mapping $\eta : K \times K \longrightarrow K$ such that, for every $x, y \in K$ and $\lambda \in [0, 1]$, there holds $y + \lambda\eta(x, y) \in K$.

DEFINITION 3

Let K be an invex set then $\phi : K \longrightarrow Y$ is called *P-preinvex* if, for every $x, y \in K$ and $\lambda \in [0, 1]$,

$$\lambda\phi(y) - (1 - \lambda)\phi(x) - \phi(x + \lambda\eta(y, x)) \in P, \quad \text{see [W–J]}.$$

Now, if $\phi : K \longrightarrow Y$ be P-preinvex and Fréchet (or linear Gateaux) differentiable then problem (4) and the vector variational-like inequality problem of finding $x \in K$ such that

$$\langle \phi'(x), \eta(y, x)\rangle \notin -\operatorname{int} P, \quad \forall y \in K \tag{6}$$

have the same set of solutions, see for instance [K2]. For related work, see [K1, K3, K–A].

Upon setting $f(x, y) := \langle \phi'(x), \eta(y, x)\rangle$ this becomes an example of (VEP). The function f is $P - \eta$-monotone in the case, since the mapping $\phi'(\cdot)$ is $P - \eta$-monotone with $\eta(x, y) = -\eta(y, x)$.

(iv) *Vector variational inequalities*

$L(X,Y)$ denotes the space of all linear bounded operators from X into Y. Let $T: K \longrightarrow L(X,Y)$, find $x \in X$ such that

$$x \in K, \quad \langle Tx, y-x \rangle \notin -\operatorname{int} P \quad y \in K. \tag{7}$$

Vector variational inequalities were first introduced by [G]. Set $f(x,y) := \langle Tx, y-x \rangle$. Then (7) $\Longleftrightarrow$ (VEP).

(v) *Vector complementarity problems*

This is special case of the previous example. Let K be a closed convex cone in X. The weak P-dual cone $K_P^{w^+}$ of K is defined by

$$K_P^{w^+} = \{l \in L(X,Y) : \langle l, x \rangle \notin -\operatorname{int} P, \quad \forall x \in K\}.$$

The strong P-dual cone $K_P^{s^+}$ of K is defined by

$$K_P^{s^+} = \{l \in L(X,Y) : \langle l, x \rangle \in P, \quad \forall x \in K\}.$$

Let $T: X \longrightarrow L(X,Y)$ be a given mapping. Then the vector complementarity problems:

$$\text{Find } x \in X \quad \text{such that} \quad x \in K, \quad Tx \in K_P^{w^+}, \quad \langle Tx, x \rangle \notin \operatorname{int} P \tag{8}$$

and

$$\text{Find } x \in X \quad \text{such that} \quad x \in K, \quad Tx \in K_P^{s^+}, \quad \langle Tx, x \rangle \notin \operatorname{int} P. \tag{9}$$

Problem (9) $\Rightarrow$ problem (7) $\Rightarrow$ problem (8), see [Y]. But we have seen that problem (7) is equivalent to (VEP).

(vi) *Fixed-point problem*

For each $x \in K$ let

$$F(x) := \{z \in K : \langle T(x), y-z \rangle \notin -\operatorname{int} P, \quad \forall y \in K\}.$$

Then the fixed point problem:

$$\text{Find } x \in K \text{ such that } x \in F(x). \tag{10}$$

Problem (10) $\Longleftrightarrow$ problem (7), see for instance [Y].

3. Existence results

In this section, we prove some existence results for (VEP) in the case where

$$f(x,y) = g(x,y) + h(x,y).$$

We need the following definitions and result.

DEFINITION 4

Let K and C be convex sets with $C \subset K$. Then $\operatorname{core}_K C$, the core of C relative to K, is defined as

$$a \in \operatorname{core}_K C \Longleftrightarrow (a \in C, \text{ and } C \cap (a,y) \neq 0 \quad \forall y \in K \setminus C).$$

Note that $\operatorname{core}_K K = K$.

DEFINITION 5

Let (Y, P) be an ordered topological vector space. $T : X \longrightarrow Y$ is called *P-convex* iff for each pair $x, y \in X$ and $\lambda \in [0, 1]$,

$$T(\lambda y + (1-\lambda)x) \leq_P \lambda T(y) + (1-\lambda)T(x).$$

Lemma 6. *See* [C]. *Let (Y, P) be an ordered topological vector space with a solid pointed closed convex cone P. Then, $\forall x,\ y \in X$, we have*

(i) $y - x \in \operatorname{int} P$ *and* $y \notin \operatorname{int} P$ imply $x \notin \operatorname{int} P$;
(ii) $y - x \in P$ *and* $y \notin \operatorname{int} P$ imply $x \notin \operatorname{int} P$;
(iii) $y - x \in -\operatorname{int} P$ *and* $y \notin -\operatorname{int} P$ imply $x \notin -\operatorname{int} P$;
(iv) $y - x \in -P$ *and* $y \notin -\operatorname{int} P$ imply $x \notin -\operatorname{int} P$.

Theorem 7. *Let the following assumptions hold:*

(i) *X is a real topological vector space; $K \subset X$ is a closed convex nonempty set; (Y, P) is a real ordered topological vector space with a solid pointed closed convex cone P in Y.*
(ii) *$g : X \times X \longrightarrow Y$ has the following properties: $g(x, x) = 0, \quad \forall x \in K$; g is P-monotone; $\forall x, y \in K$, the function $t \in [0, 1] \longrightarrow g(ty + (1-t)x, y)$ is continuous at 0_+; g is a P-convex and continuous in the second argument.*
(iii) *$h : X \times X \longrightarrow Y$ has the following properties: $h(x, x) = 0,\ \forall x \in K$; h is continuous in the first argument; h is P-convex in the second argument.*
(iv) *There exists a nonempty compact convex C of K such that for every $x \in C \backslash \operatorname{core}_K C$ there exists $a \in \operatorname{core}_K C$ such that*

$$g(x, a) + h(x, a) \in -P.$$

Then there exists $x \in C$ such that

$$g(x, y) + h(x, y) \notin -\operatorname{int} P, \quad \forall y \in K.$$

First we shall prove the following three lemmas, for which the hypotheses remain the same as for theorem 7.

Lemma 8. *There exists $x \in C$ such that*

$$h(x, y) - g(y, x) \notin -\operatorname{int} P, \quad \forall y \in C.$$

Proof. Let, for each fixed $y \in C$,

$$S(y) := \{x \in C : h(x, y) - g(y, x) \notin -\operatorname{int} P\}.$$

Claim that $\cap_{y \in C} S(y) \neq \phi$. Indeed, let $\{y_1, \ldots, y_n\}$ be a finite subset of C. Let $I \subset N$ be nonempty; let $z \in \operatorname{conv}\{y_i : i \in I\}$ be arbitrary. Then $z = \sum_{i \in I} \lambda_i y_i$ with $\lambda_i \geq 0$ and $\sum_{i \in I} \lambda_i = 1$. Suppose, if possible, $z \notin \cup_{i \in I} S(y_i)$. Then

$$h(z, y_i) - g(y_i, z) \in -\operatorname{int} P, \quad \forall i \in I.$$

From this follows

$$\sum_{i \in I} \lambda_i h(z, y_i) - \sum_{i \in I} \lambda_i g(y_i, z) \in -\operatorname{int} P. \tag{11}$$

Now, since g is P-convex and p-monotone, then we have

$$\begin{aligned}\sum_{i\in I}\lambda_i h(z,y_i) &\leq_P \sum_{i\in I}\sum_{j\in I}\lambda_i\lambda_j g(y_i,y_j)\\ &=\frac{1}{2}\sum_{i,j\in I}\lambda_i\lambda_j(g(y_i,y_j)+g(y_i,y_j))\\ &\leq_P 0,\end{aligned} \tag{12}$$

and from the properties of h follows

$$0=h(z,z)\leq_P \sum_{i\in I}\lambda_i h(z,y_i). \tag{13}$$

From (12) and (13) and, using the properties of P, we have

$$\sum_{i\in I}\lambda_i h(z,y_i)-\sum_{i\in I}\lambda_i g(y_i,z)\in P,$$

which is a contradiction to (11). Hence, our supposition is false. Thus

$$\mathrm{conv}\{y_i : i\in I\}\subset\bigcup_{i\in I}S(y_i).$$

Also, this is true for every nonempty subset I of N. The sets $S(y_i)$ are closed for every i. Indeed, let $\{x_n^i\}$ be a sequence in $S(y_i)$ such that $x_n^i\longrightarrow x^i$ then

$$\begin{aligned}&h(x_n^i,y_i)-g(y_i,x_n^i)\notin-\operatorname{int}P\\ &\Longrightarrow h(x_n^i,y_i)-g(y_i,x_n^i)\in W:=Y\backslash(-\operatorname{int}P).\end{aligned}$$

Since W is closed and h and g are continuous in the first and second argument respectively, then we have

$$h(x^i,y_i)-g(y_i,x^i)\in W.$$

This implies that the sets $S(y_i)$ are closed for every i. Hence it follows from the KKM–Fan lemma that

$$\cap_{i\in N}S(y_i)\neq\phi.$$

In other words, any finite subfamily $S(y)$ for each $y\in C$, has nonempty intersection. Since these sets are closed subsets of the compact set C, it follows that the entire family has nonempty intersection.

Hence

$$\cap_{y\in C}S(y)\neq\phi,$$

i.e. there exists atleast one $x\in C$ such that

$$h(x,y)-g(y,x)\notin-\operatorname{int}P,\quad\forall y\in C.$$

Lemma 9. *The following statements are equivalent:*

(A) $x\in C,\quad h(x,y)-g(y,x)\notin-\operatorname{int}P,\quad\forall y\in C;$

(B) $x\in C,\quad h(x,y)+g(y,x)\notin-\operatorname{int}P,\quad\forall y\in C.$

Proof. Let (B) hold. Since g is P-monotone,

$$g(x,y) \leq_P -g(y,x).$$

Also,

$$h(x,y) + g(x,y) \leq_P h(x,y) - g(y,x). \tag{14}$$

Since $h(x,y) + g(x,y) \notin -\operatorname{int} P$, using (iv) of lemma 6, (14) implies (A).

Let (A) hold. Let $y \in C$ be arbitrary, and let $x_\lambda := \lambda y + (1-\lambda)x$, $0 < \lambda \leq 1$. Then $x_\lambda \in C$, and hence from (A)

$$(1-\lambda)h(x,x_\lambda) - (1-\lambda)g(x_\lambda,x) \notin -\operatorname{int} P. \tag{15}$$

Since g is P-convex in the second argument and $g(x,x) = 0$, $\forall x \in C$ then for all $0 < \lambda \leq 1$,

$$0 = g(x_\lambda,x_\lambda) \leq_P \lambda g(x_\lambda,y) + (1-\lambda)g(x_\lambda,x)$$

or,

$$-(1-\lambda)g(x_\lambda,x) \leq_P \lambda g(x_\lambda,y).$$

Since $(1-\lambda)h(x,x_\lambda) \in Y$,

$$(1-\lambda)h(x,x_\lambda) - (1-\lambda)g(x_\lambda,x) \leq_P (1-\lambda)h(x,x_\lambda) + \lambda g(x_\lambda,y). \tag{16}$$

From (15) and (16) and using (iv) of lemma 6, we have

$$(1-\lambda)h(x,x_\lambda) + \lambda g(x_\lambda,y) \notin -\operatorname{int} P. \tag{17}$$

Since h is P-convex in the second argument and $h(x,x) = 0$, $\forall x \in C$, then

$$\lambda(1-\lambda)h(x,y) - \lambda g(x_\lambda,y)\ {}_P{\geq}\ (1-\lambda)h(x,x_\lambda) + \lambda g(x_\lambda,y) \notin -\operatorname{int} P$$

and hence by (iv) of lemma 6,

$$\lambda(1-\lambda)h(x,y) + \lambda g(x_\lambda,y) \notin -\operatorname{int} P.$$

Dividing by $\lambda > 0$ we obtain

$$g(x_\lambda,y) + (1-\lambda)h(x,y) \notin -\operatorname{int} P.$$

Since g is hemicontinuous in the first argument, it follows that

$$g(x,y) + h(x,y) \in -\operatorname{int} P$$

as $\lambda \longrightarrow 0_+$. Hence (B) holds.

Lemma 10. *Assume that* $\phi : K \longrightarrow Y$ *is* P*-convex,* $x_0 \in \operatorname{core}_K C, \phi(x_0) \notin \operatorname{int} P$, *and* $\phi(y) \notin \operatorname{int} P$, $\forall y \in C$. *Then* $\phi(y) \notin -\operatorname{int} P$, $\forall y \in K$.

Proof of lemma 10 is omitted because it can be obtained by using the same arguments as used in the proof of lemma 4 of [B–R].

Proof of theorem 7. By lemma 8, it follows that there exists atleast one $x \in C$ such that

$$h(x,y) - g(y,x) \notin -\operatorname{int} P, \quad \forall y \in C.$$

By lemma 9, it follows that the above assertion is equivalent to

$$h(x, y) - g(x, y) \notin -\operatorname{int} P, \quad \forall y \in C.$$

Set

$$\phi(\cdot) := h(x, \cdot) + g(x, \cdot).$$

Clearly $\phi(\cdot)$ is P-convex and

$$\phi(y) \notin -\operatorname{int} P, \quad \forall y \in C.$$

If $x \in \operatorname{core}_K C$, then set $x_0 := x$. If $x \in C \setminus \operatorname{core}_K C$, then set $x_0 := a$, where a is as in assumption (iv). In both cases $x_0 \in \operatorname{core}_K C$, and $\phi(x_0) \notin \operatorname{int} P$. Hence, by lemma 10, it follows that

$$\phi(y) \notin -\operatorname{int} P, \quad \forall y \in K,$$

i.e.

$$g(x, y) + h(x, y) \notin -\operatorname{int} P, \quad \forall y \in K.$$

Thus, there exists atleast one $x \in C$ such that

$$g(x, y) + h(x, y) \notin -\operatorname{int} P, \quad \forall y \in K.$$

Let $Y = \mathbb{R}, P = \mathbb{R}_+$. If $g = 0$ then theorem 7 becomes a variant of Ky Fan's minimax theorem [F2], whereas for $h = 0$ it becomes a variant of the Browder–Minty theorem for variational inequalities.

Remark. Assumption (iv) in theorem 7 can be replaced by the following assumption:

(iv)* There exists a nonempty compact convex set B in K such that for every $x \in K \setminus B$, there exists $a \in B$ with

$$g(x, a) + h(x, a) \in -\operatorname{int} P. \tag{18}$$

Theorem 11. *Let the assumptions* (i)–(iii) *of theorem* 7 *and* (iv)* *hold. Then there exists* $x \in B$ *such that*

$$g(x, y) + h(x, y) \notin -\operatorname{int} P, \quad \forall y \in K.$$

Proof. Let $\{y_i : i \in N\}$ be a finite subset of K. Let

$$C := \operatorname{conv}\{B, \cup_{i \in N} y_i\}.$$

C is convex and compact. Hence, by lemmas 8 and 9, it follows that there exists at least one $x \in C$ such that

$$h(x, y) + g(x, y) \notin -\operatorname{int} P, \quad \forall y \in C. \tag{19}$$

For choosing $y := a \in B$, we obtain from (19) and (18) that $x \in B$. The P-monotonicity of g follows

$$h(x, y) - g(y, x) \notin -\operatorname{int} P, \quad \forall y \in C.$$

In particular

$$h(x, y_i) - g(y_i, x) \notin -\operatorname{int} P, \quad \forall i \in N.$$

As in the proof of lemma 8, it can be easily seen that every finite subfamily of the family of closed sets

$$S(y) := \{x \in B : h(x,y) - g(y,x) \notin -\operatorname{int} P\} \text{ for each fixed, } y \in K$$

has nonempty intersection, and since B is compact then

$$\cap_{y\in K} S(y) \neq \phi.$$

Hence, there exists atleast one $x \in B$ such that

$$h(x,y) - g(y,x) \notin -\operatorname{int} P, \quad \forall y \in K.$$

From lemma 9, it follows that

$$h(x,y) - g(y,x) \notin -\operatorname{int} P, \quad \forall y \in K.$$

Finally we extend the notion of maximal monotonicity of a scalar multivalued mapping to a vector multivalued mapping.

DEFINITION 12

A P-monotone multivalued mapping $T : K \longrightarrow 2^{L(X,Y)}$ is called P-maximal monotone, iff, for every pair

$$(u,x) \in L(X,Y) \times K : \langle v-u,\ y-x\rangle \notin -\operatorname{int} P, \quad \forall y \in K, \quad v \in Ty \Longrightarrow u \in Tx. \tag{20}$$

By analogy of definition 12, we can define the following:

DEFINITION 13

A mapping $g : K \times K \longrightarrow Y$ with $g(x,x) = 0$, $\forall x \in K$ is called P-maximal monotone in the wide sense iff, for every pair

$$\begin{aligned}(u,x) \in L(X,Y) \times K : \langle -u, y-x\rangle - g(y,x) \notin -\operatorname{int} P, \quad \forall y \in K \\ \Longrightarrow g(x,y) - \langle u, y-x\rangle \notin -\operatorname{int} P, \quad \forall y \in K.\end{aligned} \tag{21}$$

Remark. If $g(x,y) := \sup_{u\in Tx}\langle u, y-x\rangle$ and T is P-maximal monotone, then g would be P-maximal monotone according to this definition. However, to simplify matter, we adopt the following definition.

DEFINITION 14

A mapping $g : K \times K \longrightarrow Y$ with $g(x,x) = 0$, $\forall x \in K$ is called P-maximal monotone iff, for every $x \in K$ and for every P-convex mapping $\phi : K \longrightarrow Y$ with $\phi(x) = 0$:

$$\phi(y) - g(y,x) \notin -\operatorname{int} P, \quad \forall y \in K \Longrightarrow g(x,y) + \phi(y) \notin -\operatorname{int} P, \quad \forall y \in K. \tag{22}$$

The relation of definitions 13 and 14 is the following:

Lemma 15. *Let $g : K \times K \longrightarrow Y$ be P-monotone; P-convex and lower semicontinuous in the second argument, and linear Gateaux differentiable. Then definitions* 13 *and* 14 *both are equivalent.*

Proof. (21) $\Longrightarrow$ (22). Let $x \in K$, let $\phi : K \longrightarrow Y$ be P-convex with $\phi(x) = 0$, then

$$-\phi(y) - \langle \phi'(y), x - y \rangle \in P, \quad \forall y \in K. \tag{23}$$

where $\phi'(y)$ denotes linear Gateaux derivative of ϕ at y.

Assume that

$$\phi(y) - g(y, x) \notin -\operatorname{int} P, \quad \forall y \in K. \tag{24}$$

From (23), (24) and from (iv) of lemma 6, it follows that

$$-g(y, x) - \langle \phi'(y), x - y \rangle \notin -\operatorname{int} P, \quad \forall y \in K.$$

Let $y \in K$ be fixed, and set $x_\lambda := \lambda y + (1 - \lambda)x$, $\lambda \in (0, 1]$. Then $x_\lambda \in K$, and

$$-(1-\lambda) g(x_\lambda, x) - (1-\lambda)\langle \phi'(x_\lambda), x - x_\lambda \rangle \notin -\operatorname{int} P. \tag{25}$$

Hence

$$\begin{aligned} 0 = g(x_\lambda, x_\lambda) &\leq_P \lambda g(x_\lambda, y) + (1-\lambda) g(x_\lambda, x) \\ &\Longrightarrow -(1-\lambda) g(x_\lambda, x) \leq_P \lambda g(x_\lambda, x) \\ &\Longrightarrow -(1-\lambda) g(x_\lambda, x) - (1-\lambda)\langle \phi'(x_\lambda), x - x_\lambda \rangle \leq_P \lambda g(x_\lambda, y) \\ &\quad -(1-\lambda)\langle \phi'(x_\lambda), x - x_\lambda \rangle. \end{aligned} \tag{26}$$

From (25), (26) and (iv) of lemma 6, we have

$$\lambda g(x_\lambda, y) - (1-\lambda)\langle \phi'(x_\lambda), x - x_\lambda \rangle \notin -\operatorname{int} P.$$

Dividing by λ and using the P-monotonicity of g and (iv) of lemma 6, we get

$$-g(y, x_\lambda) + (1-\lambda)\langle \phi'(x_\lambda), y - x \rangle \notin -\operatorname{int} P.$$

Letting $\lambda \longrightarrow 0_+$ and using the lower semicontinuity of g in the second argument, we obtain

$$-g(y, x) + \langle \phi'(x), y - x \rangle \notin -\operatorname{int} P, \quad \forall y \in K.$$

Since g is P-maximal monotone in the wide sense, we obtain

$$g(x, y) + \langle \phi'(x), y - x \rangle \notin -\operatorname{int} P \quad \forall y \in K. \tag{27}$$

But P-convexity of ϕ gives

$$\langle \phi'(x), y - x \rangle + g(x, y) - (g(x, y) + \phi(y)) \in P. \tag{28}$$

From (27), (28) and (iv) of lemma 6, we have

$$g(x, y) + \phi(y) \notin -\operatorname{int} P, \quad \forall y \in K.$$

(22) $\Longrightarrow$ (21). It follows immediately.

Now, we have the following theorem.

Theorem 16. *Let assumptions* (i) *and* (iii) *of Theorem* 7 *hold, and assume that the following conditions are satisfied:*

(ii)* $g : K \times K \longrightarrow Y$ *has the following properties:* $g(x, x) = 0$, $\forall x \in K$; *g is P-monotone and P-maximal monotone* (*defined as* (22)); *g is convex and lower semicontinuous in the second argument.*

(iv)* *There exists a nonempty compact convex subset B of K, such that for every $x \in K \setminus B$, there exists $a \in B$ such that*

$$-g(a,x) + h(x,a) \in -\operatorname{int} P.$$

Then there exists atleast one $x \in B$ such that

$$g(x,y) + h(x,y) \notin -\operatorname{int} P, \quad \forall y \in K. \tag{29}$$

Proof. Let $\{y_i : i \in N\}$, be a finite subset of K. Let $C := \operatorname{conv}\{B, \cup_{i \in N} y_i\}$. Then C is convex and compact. By lemma 8, it follows that there exists atleast one $x \in C$ such that

$$h(x,y) - g(y,x) \notin -\operatorname{int} P, \quad \forall y \in C. \tag{30}$$

Choosing $y := a \in B$, then (29) and (30) implies that $x \in B$. The P-monotonicity of g gives

$$h(x,y_i) - g(y_i,x) \notin -\operatorname{int} P, \quad \forall i \in N.$$

As in the proof of lemma 8, we have that every finite subfamily of the family of closed sets

$$S(y) := \{x \in B : h(x,y) - g(y,x) \notin -\operatorname{int} P\} \quad \text{for each fixed} \quad y \in K,$$

has nonempty intersection, and since B is compact then

$$\cap_{y \in K} S(y) \neq \phi,$$

i.e. there exists atleast one $x \in B$ such that

$$h(x,y) - g(y,x) \notin -\operatorname{int} P, \quad \forall y \in K. \tag{31}$$

Since g is P-maximal monotone, (31) implies

$$g(x,y) + h(x,y) \notin -\operatorname{int} P, \quad \forall y \in K.$$

Remark. Some results of this paper are the generalization of the results of [B–R].

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