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Oscillations of first order difference equations

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Abstract. The oscillatory and asymptotic behaviour of solutions of first order difference equations is studied.

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1. Introduction

First order difference equations occur as mathematical models of some real world problems (see [1, 4]). Although books on difference equations devote some space on such equations (see [3, 4]), it seems that the qualitative behaviour of their solutions is not yet studied systematically. In ([4], (see p. 64)), a geometric approach is adopted to study asymptotic behaviour of solutions of a class of nonlinear homogeneous first order difference equations of the form

$$y_{n+1} = f(y_n).$$

The general solution of

$$z' + p(t)z = 0, \quad t \geq 0, \tag{1}$$

is given by

$$z(t) = A\exp\left(-\int_0^t p(s)\mathrm{d}s\right)$$

which has no zero in $[0, \infty)$ if $A \neq 0$. The discrete analogue of (1) is

$$y_{n+1} + p_n y_n = 0, \quad n \in [0, \infty) = \{0, 1, 2, \ldots\}, \tag{2}$$

which is oscillatory if $p_n > 0,\ n \geq 0$ (see Theorem 1 below). This is one of the properties which distinguishes difference equations from differential equations. In this paper we study oscillation of (2), the associated nonhomogeneous equation

$$y_{n+1} + p_n y_n = b_n \tag{3}$$

and the nonlinear equations

$$y_{n+1} + p_n G(y_n) = 0 \tag{4}$$

and

$$y_{n+1} + p_n G(y_n) = b_n. \tag{5}$$

By a solution of (2) on $[0,\infty)$ we mean a sequence $\{y_n\}$ of real numbers which satisfies (2) for $n \geq 0$. A solution $\{y_n\}$ of (2) is said to be nontrivial if for every $N \geq 0$ there exists $n \geq N$ such that $y_n \neq 0$. We may note that (3) (or (5)) does not admit a trivial solution if b_n is not the trivial sequence and (4) admits a trivial solution if $G(0)=0$. By a solution of (2) (or (4)) we mean a nontrivial solution. Each of the equations (2)–(5) admits a unique solution if y_0 is given. A solution $\{y_n\}$ of (2) is said to be oscillatory if for every $N \geq 0$ there exists $n \geq N$ such that $y_{n-1}y_n \leq 0$; otherwise, $\{y_n\}$ is said to be non-oscillatory. Equation (2) is said to be oscillatory if every solution of (2) is oscillatory. It is said to be non-oscillatory if it admits a non-oscillatory solution. A sequence $\{y_n\}, n \geq 0$, of real numbers is said to have a generalized zero or node at $n = n_0$ provided that $y_{n_0} = 0$ if $n_0 = 0$ and if $n_0 > 0$, then $y_{n_0} = 0$ or $y_{n_0-1}y_{n_0} < 0$. Thus a solution $\{y_n\}$ of (2) is oscillatory if and only if the generalized zeros of $\{y_n\}$ are unbounded. Equation (2) is said to be disconjugate on $[0,\infty)$ if no solution of (2) has a generalized zero in $[0,\infty)$. The above definitions apply to eqs (3)–(5). The concept of generalized zeros or nodes was first introduced by Hartman in his work [2].

2. Oscillatory behaviour of solutions

In this section we study oscillatory/non-oscillatory behaviour of solutions of eqs (2)–(5). We assume

(H_1) $\quad G \in C(R,R) \quad$ with $\quad uG(u) > 0 \quad$ for $\quad u \neq 0.$

Remark. If $\{y_n\}$ is a solution of (2) with $y_0 = 0$, then it is a trivial solution. If $p_n = 0$ for some $n \geq 0$, then $y_m = 0$ for $m \geq n+1$ and hence $\{y_n\}$ is a trivial solution of (2). Thus we assume $y_0 \neq 0$ and $p_n \neq 0$ for $n \geq 0$ when we consider (2). From (H_1) it follows that $G(0) = 0$. Hence the above observation holds for eq. (4).

Theorem 1. (i) *If $p_n < 0$, $n \geq 0$, then eq.* (2) *is disconjugate on* $[0,\infty)$. (ii) *If $p_n > 0$, $n \geq 0$, then eq.* (2) *is oscillatory.* (iii) *If $\{p_n\}$ is oscillatory in the sense that for every $N \geq 0$ there exists $n \geq N$ such that $p_{n-1}\,p_n < 0$, then eq.* (2) *is oscillatory.*

Proof. Any solution of (2) is given by

$$y_n = (-1)^n y_0 \prod_{i=0}^{n-1} p_i. \tag{6}$$

(i) If $p_n < 0$ for $n \geq 0$, then writing (6) as

$$y_n = (-1)^{2n} y_0 \prod_{i=0}^{n-1} (-p_i),$$

we may observe that $y_n > 0$ or < 0 for $n \geq 0$ as $y_0 > 0$ or < 0. Thus (2) is disconjugate on $[0,\infty)$.

(ii) Since $p_n > 0$ for $n \geq 0$, we obtain from (6) that

$$y_{n-1}y_n = (-1)^{2n-1} y_0^2 p_{n-1} \prod_{i=0}^{n-2} p_i^2$$

for $n \geq 1$. Hence (2) is oscillatory.

(iii) Suppose that $\{p_n\}$ is oscillatory. Let $N \geq 0$. Choose $N^* \geq N+1$. Since $\{p_n\}$ is oscillatory, there exists $m \geq N^*$ such that $p_{m-1}\ p_m < 0$. If $p_m > 0$, for $m \geq N^*$, then from (6) we have

$$y_m y_{m+1} = (-1)^{2m+1} p_m\, y_0^2 \prod_{i=0}^{m-1} p_i^2 < 0.$$

If $p_m < 0$ for $m \geq N^*$, then $p_{m-1} > 0$ and hence

$$y_{m-1} y_m = (-1)^{2m-1} p_{m-1}\, y_0^2 \prod_{i=0}^{m-2} p_i^2 < 0.$$

Thus $\{y_n\}$ is oscillatory. Since it is an arbitrary solution of (2), eq. (2) is oscillatory.

Hence the theorem is proved.

Theorem 2. *Let* $(1+p_n) \geq 0$ *for* $n \geq 0$. *Suppose there exists a sequence* $\{c_n\}$ *with the following property:*

(H_2) *for every* $n \geq 0$ *there exists* $m \geq n$ *such that*

$$c_{m-1}c_m < 0 \quad \textit{and} \quad b_n = c_{n+1} - c_n.$$

If

$$\sum_{n=0}^{\infty}(1+p_n)c_n^+ = +\infty \quad \textit{and} \quad \sum_{n=0}^{\infty}(1+p_n)c_n^- = +\infty,$$

then eq. (3) *is oscillatory, where* $c_n^+ = \max\{c_n, 0\}$ *and* $c_n^- = \max\{-c_n, 0\}$.

Proof. Let $\{y_n\}, n \geq 0$, be a non-oscillatory solution of (3). Hence there exists $N \geq 0$ such that $y_n > 0$ or < 0 for $n \geq N$. Let $y_n > 0$ for $n \geq N$. Writing eq. (3) as

$$(y_{n+1} - c_{n+1}) - (y_n - c_n) + (1+p_n)y_n = 0, \tag{7}$$

we notice that the sequence $\{y_n - c_n\}$ is monotonic decreasing for $n \geq N$. We claim that $y_n - c_n > 0$ for $n \geq N$. If not, then $y_l - c_l \leq 0$ for some $l \geq N$. Then $k \geq l$ implies that $y_k - c_k \leq y_l - c_l \leq 0$ and hence $0 < y_k \leq c_k$ for $k \geq l$, a contradiction to (H_2). Hence our claim holds. From (7) we get

$$\begin{aligned}\sum_{n=N}^{N+i}(1+p_n)y_n &= \sum_{n=N}^{N+i}[(y_n - c_n) - (y_{n+1} - c_{n+1})]\\ &= (y_N - c_N) - (y_{N+i+1} - c_{N+i+1})\\ &< y_N - c_N\end{aligned}$$

which implies that

$$\sum_{n=N}^{\infty}(1+p_n)y_n \leq y_N - c_N < +\infty.$$

On the other hand, $y_n \geq c_n^+$ for $n \geq N$ implies that

$$\sum_{n=N}^{\infty}(1+p_n)y_n \geq \sum_{n=N}^{\infty}(1+p_n)c_n^+ = +\infty,$$

a contradiction. Hence $y_n < 0$ for $n \geq N$. From (7) it follows that $\{y_n - c_n\}$ is monotonic increasing for $n \geq N$. If $y_l - c_l \geq 0$ for some $l \geq N$, then $k \geq l$ implies that $y_k - c_k \geq y_l - c_l \geq 0$, that is, $0 > y_k \geq c_k$ for $k \geq l$, which contradicts (H_2). Thus $y_n - c_n < 0$ for $n \geq N$. Since $-y_n \geq c_n^-$, then

$$\sum_{n=N}^{\infty}(1+p_n)y_n \leq -\sum_{n=N}^{\infty}(1+p_n)c_n^- = -\infty.$$

However, (7) yields

$$\sum_{n=N}^{\infty}(1+p_n)y_n \geq y_N - c_N > -\infty,$$

a contradiction. Hence $\{y_n\}$ is oscillatory. This completes the proof of the theorem.

Remark. We may note that $\{b_n\}$ changes sign, that is, for every $n \geq 0$ there exists $l \geq n$ such that $b_{l-1}b_l < 0$ if and only if (H_2) holds. Indeed, if $\{b_n\}$ changes sign, then defining $c_0 = 0, c_n = \sum_{i=0}^{n-1} b_i,\ n \geq 1$, we obtain $c_{n+1} - c_n = b_n$. Let $l \geq n+1$. If $b_l < 0$, then taking $m = l+1$, we get

$$c_{m-1}c_m = c_l c_{l+1} = \left(\sum_{i=0}^{l-1} b_i\right)^2 b_l < 0.$$

If $b_l > 0$, then $b_{l-1} < 0$. For $m = l$, we get

$$c_{m-1}c_m = c_{l-1}c_l = \left(\sum_{i=0}^{l-2} b_i\right)^2 b_{l-1} < 0.$$

Thus (H_2) is satisfied. Further, if (H_2) holds, then there exists $m \geq n+1$ such that $c_{m-1}c_m < 0$ and $c_m c_{m+1} < 0$. Hence $b_m = c_{m+1} - c_m$ implies that $b_{m-1}b_m < 0$, that is, $\{b_n\}$ is changing sign.

Theorem 3. *Let* $(1+p_n) \leq 0$ *for* $n \geq 0$. *Suppose there exists a sequence* $\{c_n\}$ *satisfying* (H_2). *If* $\sum_{n=0}^{\infty}(1+p_n)c_n^+ = -\infty$, $\sum_{n=0}^{\infty}(1+p_n)c_n^- = -\infty$ *and* $-\infty < \liminf_{n\to\infty} c_n < \limsup_{n\to\infty} c_n < \infty$, *then every solution of* (3) *oscillates or tends to* $\pm\infty$ *as* $n \to \infty$.

Proof. If possible, let $\{y_n\}$ be a non-oscillatory solution of (3). Hence $y_n > 0$ or < 0 for $n \geq N \geq 0$. Let $y_n > 0$ for $n \geq N$. From (7) it follows that the sequence $\{y_n - c_n\}$ is monotonic increasing for $n \geq N$. If possible, let there exist an $l \geq N$ such that $y_l - c_l = 0$. If $y_k - c_k = y_l - c_l$ for every $k \geq l$, then $c_k = y_k > 0$ for $k \geq l$, a contradiction to (H_2). Hence there exists a $k_1 > l$ such that $y_{k_1} - c_{k_1} > y_l - c_l = 0$. Then $n \geq k_1$ implies that $y_n - c_n \geq y_{k_1} - c_{k_1} > 0$. If such an l does not exist, then $y_n - c_n < 0$ or > 0 for $n \geq N$. In the former case, $0 < y_n < c_n$ for $n \geq N$, which contradicts (H_2). Hence $y_n - c_n > 0$ for $n \geq N$. Thus, in any case, there exists $k^* \geq N$ such that $y_n - c_n > 0$ for $n \geq k^*$. If $\lambda = \lim_{n\to\infty}(y_n - c_n)$, then $0 < \lambda \leq +\infty$. Suppose that $0 < \lambda < +\infty$. From (7) it follows that

$$-\sum_{n=k^*}^{\infty}(1+p_n)y_n \leq \lim_{i\to\infty}(y_{k^*+i} - c_{k^*+i}) = \lambda.$$

However, $y_n \geq c_n^+$ for $n \geq k^*$ implies that

$$-\sum_{n=k^*}^{\infty}(1+p_n)y_n \geq -\sum_{n=k^*}^{\infty}(1+p_n)c_n^+ = +\infty,$$

a contradiction. If $\lambda = +\infty$, then

$$\begin{aligned}\liminf_{n\to\infty} y_n &= \liminf_{n\to\infty}[(y_n - c_n) + c_n] \\ &\geq \liminf_{n\to\infty}(y_n - c_n) + \liminf_{n\to\infty} c_n \\ &= +\infty\end{aligned}$$

implies that $\lim_{n\to\infty} y_n = +\infty$. Similarly, if $y_n < 0$ for $n \geq N$, then we may show that $\lim_{n\to\infty} y_n = -\infty$. This completes the proof of the theorem.

The following examples illustrate the above results.

Example 1. Consider

$$y_{n+1} - \tfrac{1}{2}y_n = \tfrac{3}{2}(-1)^{n+1}, \quad n \geq 0.$$

Defining $c_0 = 1$ and

$$c_n = \begin{cases} 1, & n \text{ even} \\ -\frac{1}{2}, & n \text{ odd}, \end{cases}$$

we notice that $c_{n+1} - c_n = b_n$, $n \geq 0$ and $c_{n-1}c_n < 0$ for $n \geq 1$. As

$$c_n^+ = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \quad \text{and} \quad c_n^- = \begin{cases} 0, & n \text{ even} \\ \frac{1}{2}, & n \text{ odd} \end{cases}$$

then

$$\sum_{n=0}^{\infty}(1+p_n)c_n^+ = \sum_{i=0}^{\infty}(1+p_{2i})c_{2i}^+ = +\infty$$

and

$$\sum_{n=0}^{\infty}(1+p_n)c_n^- = \sum_{i=0}^{\infty}(1+p_{2i+1})c_{2i+1}^- = +\infty.$$

From Theorem 2 it follows that every solution of the equation oscillates. In particular, $\{y_n\} = \{(-1)^n\}$ is an oscillatory solution of the equation.

Example 2. Consider

$$y_{n+1} - 2y_n = 3(-1)^{n+1}, \quad n \geq 0.$$

Clearly, $b_n = c_{n+1} - c_n$, for $n \geq 0$, where

$$c_n = \begin{cases} 1, & n \text{ even} \\ -2, & n \text{ odd}, \end{cases}$$

$c_{n-1}c_n < 0$ for $n \geq 1$, and $-\infty < \liminf_{n\to\infty} c_n = -2 < \limsup_{n\to\infty} c_n = 1 < \infty$. Since

$$\sum_{n=0}^{\infty}(1+p_n)c_n^+ = \sum_{i=0}^{\infty}(1+p_{2i})c_{2i}^+ = -\infty$$

and

$$\sum_{n=0}^{\infty}(1+p_n)c_n^- = \sum_{i=0}^{\infty}(1+p_{2i+1})c_{2i+1}^- = -\infty,$$

then every solution of the equation oscillates or tends to $\pm\infty$ as $n \to \infty$ by Theorem 3. In particular, $\{y_n\} = \{(-1)^n\}$ is an oscillatory solution of the equation.

Remark. If $p_n \equiv -1$, then a solution of (3) is given by

$$y_n = y_0 + \sum_{i=0}^{n-1} b_i, \quad n \geq 1.$$

If $\{b_n\}$ changes sign, then it is not possible to conclude that eq. (3) is oscillatory. For example, we take $b_n = (-1)^n$, $n \geq 0$. Then $\{y_n\}$ is an oscillatory solution if $-1 \leq y_0 \leq 0$, a positive solution if $y_0 > 0$ and a negative solution if $y_0 < -1$. Thus the behaviour of solutions of $y_{n+1} - y_n = (-1)^n$, $n \geq 0$, is determind by the region of initial values of the solutions.

Remark. If $\{1 + p_n\}$ changes sign, then no information is available.

Remark. If $b_n \geq 0, p_n \leq 0$ with $p_n^2 + b_n^2 \neq 0, n \geq 0$, then eq. (3) admits a positive solution $\{y_n\}$ whenever $y_0 > 0$ and hence is non-oscillatory. However, if $b_n \geq 0, p_n \geq 0$ with $p_n^2 + b_n^2 \neq 0$, $n \geq 0$, then we cannot say that eq. (3) admits a non-oscillatory solution. Although several examples of the form (3) are given in [4] (see pp. 48–56), the qualitative behaviour of solutions is not studied.

The general solution of eq. (3) is given by (see p. 48, [4])

$$y_n = \left(\prod_{i=0}^{n-1}(-p_i)\right)\left[\sum_{r=0}^{n-1}\left(b_r \middle/ \prod_{i=0}^{r}(-p_i)\right) + A\right], \tag{8}$$

where $p_n \neq 0$ for $n \geq 0$ and A is an arbitrary constant. One may get different solutions of eq. (3) by changing A.

Theorem 4. *Let $p_n > 0$, $n \geq 0$. If*

$$\sum_{r=0}^{\infty}\left(b_r \middle/ \prod_{i=0}^{r}(-p_i)\right) = \pm\infty, \tag{9}$$

then every solution of eq. (3) *oscillates.*

Proof. It is possible to choose $N > 0$ sufficiently large such that

$$\sum_{r=0}^{n-1}\left(b_r \middle/ \prod_{i=0}^{r}(-p_i)\right) + A > 0 \text{ or } < 0$$

for $n \geq N$. Since

$$\prod_{i=0}^{n-1}(-p_i)\begin{cases} > 0, & \text{for } n \text{ odd} \\ < 0, & \text{for } n \text{ even} \end{cases}$$

then from (8) it follows that the generalized zeros of any solution $\{y_n\}$ of eq. (3) forms an unbounded set and hence is oscillatory. Thus the theorem is proved.

Example 3. Every solution of

$$y_{n+1} + y_n = b_n, \quad n \geq 0$$

oscillates, where

$$b_n = \begin{cases} 1, & n \text{ even} \\ 2^n, & n \text{ odd} \end{cases}$$

by Theorem 4, because

$$\sum_{r=0}^{\infty} \left(b_r \Big/ \prod_{i=0}^{r} (-p_i) \right)$$
$$= -1 + 2 - 1 + 2^3 - 1 + 2^5 - 1 + \cdots$$
$$\geq \sum_{r=0}^{\infty} 2^r = \infty.$$

Indeed, if $y_0 = A = 1$, then the generalized zeros of $\{y_n\}$ are at $n = 1, 2, 3, \ldots$.

Remark. If the series in (9) is oscillating, then eq. (3) may admit both oscillatory and non-oscillatory solutions.

Example 4. Consider

$$y_{n+1} + y_n = 1, n \geq 0.$$

Then the series

$$\sum_{r=0}^{\infty} \left(b_r \Big/ \prod_{i=0}^{r} (-p_i) \right) = \sum_{n=0}^{\infty} (-1)^{n+1}$$

is oscillating. Clearly, $y_0 = 1/2$,

$$y_n = (-1)^n \left[\sum_{r=0}^{n-1} (-1)^{r+1} + \frac{1}{2} \right], \quad n \geq 1,$$

is a positive solution of the equation and

$$u_n = (-1)^n \left[\sum_{r=0}^{n-1} (-1)^{r+1} + 2 \right], \quad n \geq 1 \quad \text{with} \quad u_0 = 2$$

is an oscillatory solution of the equation.

Remark. If the series in (9) is absolutely convergent, then eq. (3) may admit both oscillatory and non-oscillatory solutions.

Example 5. For the difference equation

$$y_{n+1} + y_n = \frac{1}{(n+1)^2}, \quad n \geq 0,$$

$$\sum_{r=0}^{\infty}\left|\left(b_r\Big/\prod_{i=0}^{r}(-p_i)\right)\right| = \sum_{r=0}^{\infty}\left|\frac{(-1)^{r+1}}{(r+1)^2}\right| = \sum_{r=1}^{\infty}\frac{1}{r^2} < \infty.$$

Using (8) we write

$$y_n = (-1)^n\left[\sum_{r=0}^{n-1}\frac{(-1)^{r+1}}{(r+1)^2} + A\right], n \geq 1.$$

If $A \geq 1$, then $\{y_n\}$ is an oscillatory solution of the equation with $y_0 = A$. If $A = 5/6$, then $\{y_n\}$ is a positive solution of the equation with $y_0 = 5/6$.

The asympotic behaviour of solutions of (4) is studied geometrically in ([4], p. 64). We have the following result concerning eq. (4). We may note that a solution of (4) cannot be expressed in the form (6).

Theorem 5. (i) *If $p_n < 0$ for $n \geq 0$, then* (4) *is disconjugate on* $[0,\infty)$. (ii) *If $p_n > 0$ for $n \geq 0$, then* (4) *is oscillatory.* (iii) *If $\{p_n\}$ is oscillatory, then* (4) *is oscillatory.*

Proof. (i) If possible, let $k \in [0,\infty) = \{0,1,2,\ldots\}$ be a generalized zero of a solution $\{y_n\}$ of (4). If $k = 0$, then $y_0 = 0$ and hence $\{y_n\}$ is a trivial solution. Let $k \in (0,\infty)$. If $y_k = 0$, then $y_n = 0$ for $n \geq k$, that is, $\{y_n\}$ is a trivial solution. If $y_k \neq 0$, then $y_{k-1}y_k < 0$. However, $y_k > 0$ implies that $y_{k-1} < 0$ and hence

$$0 < y_k = -p_{k-1}G(y_{k-1}) < 0,$$

a contradiction and $y_k < 0$ implies that $y_{k-1} > 0$ and hence $0 > y_k = -p_{k-1}G(y_{k-1}) > 0$, a contradiction. Thus $\{y_n\}$ has no generalized zero in $[0,\infty)$, that is, eq. (4) is disconjugate on $[0,\infty)$.

(ii) If possible, let $\{y_n\}$ be a non-oscillatory solution of (4). Hence $y_n > 0$ or < 0 for $n \geq N > 0$. Let $y_n > 0$ for $n \geq N$. From (4) we obtain

$$0 < y_{n+1} = -p_nG(y_n) < 0,$$

a contradiction. A similar contradiction is obtained if $y_n < 0$ for $n \geq N$. Then $\{y_n\}$ is oscillatory.

(iii) Let $\{y_n\}$ be a non-oscillatory solution of (4) such that $y_n > 0$ for $n \geq N > 0$. Let $N^* \geq N+1$. Since $\{p_n\}$ changes sign, then there exists $k \geq N^*$ such that $p_{k-1}p_k < 0$. If $p_k > 0$, then we obtain

$$0 < y_{k+1} = -p_kG(y_k) < 0,$$

a contradiction. Suppose that $p_k < 0$. Then $p_{k-1} > 0$ and hence

$$0 < y_k = -p_{k-1}G(y_{k-1}) < 0,$$

a contradiction. We may obtain a similar contradiction if $y_n < 0$ for $n \geq N$. Thus eq. (4) is oscillatory.

This completes the proof of the theorem.

Theorem 6. (i) *If $p_n \geq 0$ and $\{b_n\}$ changes sign, then* (5) *is oscillatory.* (ii) *If $p_n \leq 0$ and $b_n \geq 0$ such that $p_n^2 + b_n^2 > 0$, then* (5) *is non-oscillatory.*

Proof. (i) If $\{y_n\}$ is a non-oscillatory solution of (5) with $y_n > 0$ for $n \geq N$, then (5) yields

$$0 < y_{n+1} + p_n G(y_n) = b_n,$$

a contradiction. A similar contradiction is obtained if $y_n < 0$ for $n \geq N$. Thus eq. (5) is oscillatory. (ii) If $y_0 > 0$, then $\{y_n\}$ is a positive solution of (5) and hence (5) is non-oscillatory.

Remark. The following examples suggest that if $\{b_n\}$ changes sign and either $p_n < 0$ or $\{p_n\}$ changes sign, then eq. (5) admits an oscillatory solution or is oscillatory.

Example 6. Clearly, $\{(-1)^n\}$ is an oscillatory solution of each of the following equations

$$y_{n+1} - 2y_n^3 = 3(-1)^{n+1}, \quad n \geq 0 \tag{10}$$

and

$$y_{n+1} + (-\tfrac{1}{2})^n y_n^3 = (-1)^{n+1}(1 + 2^{-n}), \quad n \geq 0.$$

We note that (10) admits a positive solution $\{y_n\}$ with $y_0 = 2$.

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