

Inequalities for a polynomial and its derivative

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Abstract. For an arbitrary entire function f and any $r > 0$, let $M(f, r) := \max_{|z|=r} |f(z)|$. It is known that if p is a polynomial of degree n having no zeros in the open unit disc, and $m := \min_{|z|=1} |p(z)|$, then

$$M(p', 1) \leq \frac{n}{2} \{M(p, 1) - m\},$$
$$M(p, R) \leq \left(\frac{R^n + 1}{2}\right) M(p, 1) - m \left(\frac{R^n - 1}{2}\right), \quad R > 1.$$

It is also known that if p has all its zeros in the closed unit disc, then

$$M(p', 1) \geq \frac{n}{2} \{M(p, 1) + m\}.$$

The present paper contains certain generalizations of these inequalities.

Keywords. Inequalities; zeros; polynomial.

1. Introduction and statement of results

Let $p(z)$ be a polynomial of degree n . Concerning the estimate of $|p'(z)|$ on the disc $|z| \leq 1$, we have the following famous result known as Bernstein's inequality [11].

Theorem A. *If $p(z)$ is a polynomial of degree n , then*

$$M(p', 1) \leq nM(p, 1), \tag{1.1}$$

with equality only for $p(z) = \alpha z^n$.

For polynomials having no zeros in $|z| < 1$, Erdős conjectured and Lax [5] proved

Theorem B. *If $p(z)$ is a polynomial of degree n , having no zeros in $|z| < 1$, then*

$$M(p', 1) \leq \frac{n}{2} M(p, 1), \tag{1.2}$$

with equality for those polynomials, which have all their zeros on $|z| = 1$.

For polynomials having all their zeros in $|z| \leq 1$, Turan [12] proved

Theorem C. *If $p(z)$ is a polynomial of degree n , having all its zeros in $|z| \leq 1$, then*

$$M(p', 1) \geq \frac{n}{2} M(p, 1), \tag{1.3}$$

with equality for those polynomials, which have all their zeros on $|z| = 1$.

On the other hand, concerning the estimate of $|p(z)|$ on the disc $|z| \leq R, R > 1$, we have, as a simple consequence of maximum modulus principle [7].

Theorem D. *If $p(z)$ is a polynomial of degree n , then*

$$M(p, R) \leq R^n M(p, 1), \quad R > 1, \quad (1.4)$$

with equality for $p(z) = \alpha z^n$.

For polynomials not vanishing in $|z| < 1$, Ankeny and Rivlin [1] proved

Theorem E. *If $p(z)$ is a polynomial of degree n , having no zeros in $|z| < 1$, then*

$$M(p, R) \leq \frac{R^n + 1}{2} M(p, 1), \quad R > 1, \quad (1.5)$$

with equality for $p(z) = \alpha + \beta z^n, |\alpha| = |\beta|$.

In [3], we had used a parameter β and obtained the following generalizations of inequalities (1.2), (1.5) and (1.3).

Theorem F. *Let $p(z)$ be a polynomial of degree n , having no zeros in $|z| < 1$. If $M(p, 1) = 1$, then for $|\beta| \leq 1$*

$$\left| zp'(z) + \frac{n\beta}{2} p(z) \right| \leq \frac{n}{2} \left\{ \frac{|\beta|}{2} + \left| 1 + \frac{\beta}{2} \right| \right\}, \quad |z| = 1, \quad (1.6)$$

$$\left| p(Rz) + \beta \left(\frac{R+1}{2} \right)^n p(z) \right| \leq \frac{1}{2} \left\{ \left| +\beta \left(\frac{R+1}{2} \right)^n \right| + \left| R^n + \beta \left(\frac{R+1}{2} \right)^n \right| \right\},$$

$$R \geq 1, \quad |z| = 1. \quad (1.7)$$

The result is best possible and equality holds in (1.6) and (1.7) for $p(z) = \alpha + \gamma z^n$, with $|\alpha| = |\gamma|$.

Theorem G. *If $p(z)$ is a polynomial of degree n , having all its zeros in the closed unit disc, then for $|\beta| \leq 1$*

$$\max_{|z|=1} \left| zp'(z) + \frac{n\beta}{2} p(z) \right| \geq \frac{n}{2} \{1 + \operatorname{Re}(\beta)\} M(p, 1). \quad (1.8)$$

Aziz and Dawood [2] used

$$m = \min_{|z|=1} |p(z)| \quad (1.9)$$

to obtain certain refinements of inequalities (1.2), (1.5) and (1.3) and proved

Theorem H. *If $p(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then*

$$M(p', 1) \leq \frac{n}{2} [M(p, 1) - m], \quad (1.10)$$

$$M(p, R) \leq \left(\frac{R^n + 1}{2} \right) M(p, 1) - \left(\frac{R^n - 1}{2} \right) m, \quad R > 1. \quad (1.11)$$

The result is best possible and equality holds in (1.10) and (1.11) for $p(z) = \alpha z^n + \gamma$ with $|\alpha| \leq |\gamma|$.

Theorem I. If $p(z)$ is a polynomial of degree n which has all its zeros in $|z| \leq 1$, then

$$M(p', 1) \geq \frac{n}{2} \{M(p, 1) + m\}. \tag{1.12}$$

The result is best possible and equality in (1.12) holds for $p(z) = \alpha z^n + \gamma$, $|\gamma| \leq |\alpha|$.

In this paper, we have used a parameter β , to obtain generalizations of inequalities (1.10), (1.11) and (1.12), similar to the generalizations – namely Theorems F and G, of inequalities (1.2), (1.5) and (1.3), obtained earlier by us. More precisely, we prove

Theorem 1. If $p(z)$ is a polynomial of degree n , having no zeros in $|z| < 1$, then for β with $|\beta| \leq 1$

$$\max_{|z|=1} \left| zp'(z) + \frac{n\beta}{2} p(z) \right| \leq \frac{n}{2} \left\{ \left(\left| 1 + \frac{\beta}{2} \right| + \left| \frac{\beta}{2} \right| \right) M(p, 1) - m \left(\left| 1 + \frac{\beta}{2} \right| - \left| \frac{\beta}{2} \right| \right) \right\}, \tag{1.13}$$

$$\begin{aligned} \max_{|z|=1} \left| p(Rz) + \beta \left(\frac{R+1}{2} \right)^n p(z) \right| &\leq \frac{1}{2} \left\{ \left| R^n + \beta \left(\frac{R+1}{2} \right)^n \right| \right. \\ &\quad \left. + \left| 1 + \beta \left(\frac{R+1}{2} \right)^n \right| \right\} M(p, 1) \\ &\quad - \frac{m}{2} \left\{ \left| R^n + \beta \left(\frac{R+1}{2} \right)^n \right| - \left| 1 + \beta \left(\frac{R+1}{2} \right)^n \right| \right\}, \quad R > 1. \end{aligned} \tag{1.14}$$

Equality holds in (1.13) and (1.14) for $p(z) = \lambda + \mu z^n$ with $|\lambda| \geq |\mu|$.

Theorem 2. If $p(z)$ is a polynomial of degree n , having all its zeros in $|z| \leq 1$, then for β with $|\beta| \leq 1$

$$\max_{|z|=1} \left| zp'(z) + \frac{n\beta}{2} p(z) \right| \geq \frac{n}{2} [\{1 + \operatorname{Re}(\beta)\} M(p, 1) + m \{1 + \operatorname{Re}(\beta)\} - |\beta|]. \tag{1.15}$$

Equality holds in (1.15) for $p(z) = Ce^{i\alpha} z^n$, $C > 0$ and $\beta \geq 0$.

Remark 1. Theorem 1 is a refinement of Theorem F, it can be easily seen by observing that

$$\left| 1 + \frac{\beta}{2} \right| \geq \left| \frac{\beta}{2} \right|, \quad |\beta| \leq 1,$$

and

$$\left| R^n + \beta \left(\frac{R+1}{2} \right)^n \right| \geq \left| 1 + \beta \left(\frac{R+1}{2} \right)^n \right|, \quad |\beta| \leq 1 \quad \text{and} \quad R > 1.$$

Remark 2. Theorem 2 is a refinement of Theorem G.

2. Lemmas

For the proofs of the theorems, we require the following lemmas.

Lemma 1. If $p(z)$ is a polynomial of degree n , having all its zeros in $|z| \leq 1$, then

$$|p'(z)| \geq \frac{n}{2}|p(z)|, \quad |z| = 1.$$

This lemma is due to Malik and Vong [6]. It suffices to observe that if $p(z) = c\prod_{\nu=1}^n(z - z_\nu)$, then for $|z| = 1$, we have

$$R\left(\frac{zp'(z)}{p(z)}\right) = \sum_{\nu=1}^n R\left(\frac{z}{z - z_\nu}\right) \geq \frac{n}{2}.$$

Lemma 2. If $p(z)$ is a polynomial of degree n , having all its zeros in $|z| \leq 1$, then

$$|p(\operatorname{Re}^{i\theta})| \geq \left(\frac{R+1}{2}\right)^n |p(e^{i\theta})|, \quad R > 1 \quad \text{and} \quad 0 \leq \theta < 2\pi.$$

This lemma is due to Jain [4]. It was observed by Rivlin [10] that if f is a polynomial of degree at most n such that $f(z) \neq 0$ in $|z| < 1$, then

$$|f(\rho e^{i\theta})| \geq \left(\frac{1+\rho}{2}\right)^n |f(e^{i\theta})|, \quad (0 \leq \rho < 1, 0 \leq \theta < 2\pi).$$

Applying this result to the polynomial $f(z) := z^n \overline{p(1/\bar{z})}$ with $\rho := 1/R$ we obtain the desired estimate.

Lemma 3. If $p(z)$ is a polynomial of degree n , having all its zeros in $|z| \leq 1$, then for β with $|\beta| \leq 1$

$$\min_{|z|=1} \left| z\rho'(z) + \frac{n\beta}{z}p(z) \right| \geq mn \left| 1 + \frac{\beta}{2} \right|, \tag{2.1}$$

$$\min_{|z|=1} \left| p(Rz) + \beta \left(\frac{R+1}{2}\right)^n \right| \geq m \left| R^n + \beta \left(\frac{R+1}{2}\right)^n \right|, \quad R > 1. \tag{2.2}$$

Equality holds in (2.1) and (2.2) for $p(z) = me^{i\gamma} z^n, m > 0$.

Proof of Lemma 3. If $p(z)$ has a zero on $|z| = 1$, then inequalities (2.1) and (2.2) are trivial. Therefore we assume that $p(z)$ has all its zeros in $|z| < 1$. Then $m > 0$ and for α with $|\alpha| < 1$, we have

$$|\alpha m z^n| < m \leq |p(z)|, \quad |z| = 1, \quad (\text{by(1.9)}),$$

thereby implying by Rouché's theorem that the polynomial

$$p_1(z) = p(z) - \alpha m z^n$$

has all its zeros in $|z| < 1$. On applying Lemma 1, we get

$$|z\{p'(z) - \alpha mnz^{n-1}\}| \geq \frac{n}{2}|p(z) - \alpha m z^n|, \quad |z| = 1 \quad \text{and} \quad |\alpha| < 1.$$

Therefore for $|\beta| < 1$ and $|\alpha| < 1$, the polynomial

$$z\{p'(z) - \alpha mnz^{n-1}\} + \beta \frac{n}{2}\{p(z) - \alpha m z^n\}$$

i.e.

$$\left\{ zp'(z) + \frac{n\beta}{2}p(z) \right\} - \alpha nmz^n \left\{ 1 + \frac{\beta}{2} \right\}$$

will have no zeros on $|z| = 1$. As $|\alpha| < 1$, we have for β with $|\beta| < 1$

$$\left| zp'(z) + \frac{n\beta}{2}p(z) \right| \geq \left| nmz^n \left(1 + \frac{\beta}{2} \right) \right|, \quad |z| = 1,$$

i.e.

$$\left| zp'(z) + \frac{n\beta}{2}p(z) \right| \geq mn \left| 1 + \frac{\beta}{2} \right|, \quad |z| = 1. \tag{2.3}$$

For β with $|\beta| = 1$, (2.3) follows by continuity. And now, the inequality (2.1) follows.

On applying Lemma 2 to the polynomial $p_1(z)$, we get for $R > 1$ and $|\alpha| < 1$

$$|p(Rz) - \alpha mR^n z^n| \geq \left(\frac{R+1}{2} \right)^n |p(z) - \alpha mz^n|, \quad |z| = 1.$$

Therefore for $|\beta| < 1$ and $|\alpha| < 1$, the polynomial

$$p(Rz) - \alpha mR^n z^n + \beta \left(\frac{R+1}{2} \right)^n \{p(z) - \alpha mz^n\},$$

i.e.

$$\left\{ p(Rz) + \beta \left(\frac{R+1}{2} \right)^n p(z) \right\} - \alpha mz^n \left\{ R^n + \beta \left(\frac{R+1}{2} \right)^n \right\}$$

will have no zeros on $|z| = 1$. As $|\alpha| < 1$, we have for β with $|\beta| < 1$

$$\left| p(Rz) + \beta \left(\frac{R+1}{2} \right)^n p(z) \right| \geq \left| mz^n \left\{ R^n + \beta \left(\frac{R+1}{2} \right)^n \right\} \right|, \quad |z| = 1,$$

i.e.

$$\left| p(Rz) + \beta \left(\frac{R+1}{2} \right)^n p(z) \right| \geq m \left| R^n + \beta \left(\frac{R+1}{2} \right)^n \right|, \quad |z| = 1,$$

and the inequality (2.2) follows. This completes the proof of Lemma 3.

Lemma 4. Let $Q(z)$ be a polynomial of degree n , having all its zeros in $|z| \leq 1$ and $S(z)$ be a polynomial of degree not exceeding that of $Q(z)$. If

$$|S(z)| \leq |Q(z)| \tag{2.4}$$

for $|z| = 1$, then for any $|\beta| \leq 1$,

$$\left| \frac{zS'(z)}{n} + \beta \frac{S(z)}{2} \right| \leq \left| \frac{zQ'(z)}{n} + \beta \frac{Q(z)}{2} \right| \tag{2.5}$$

for $|z| = 1$.

This lemma is due to Malik and Vong [6]. However, this result is contained in ([9], Theorem 3.4) where it is shown that under the conditions of lemma 4,

$$|B_n S(z)| \leq |B_n Q(z)|, \quad (|z| = 1),$$

for every B_n -operator. It may be added that a linear operator T , which carries polynomials of degree at most n into polynomials of degree at most n , is called a B_n -operator provided that $T[f]$ has all its zeros in the open unit disc if f is of exact degree n and has all its zeros in the open unit disc.

Lemma 5. If $p(z)$ is a polynomial of degree n , with $M(p, 1) = 1$, then for $|\beta| \leq 1$ and $|z| = 1$

$$\left| zp'(z) + \frac{n\beta}{2}p(z) \right| + \left| zq'(z) + \frac{n\beta}{2}q(z) \right| \leq n \left(\left| \frac{\beta}{2} \right| + \left| 1 + \frac{\beta}{2} \right| \right),$$

where

$$q(z) = z^n \overline{p(1/\bar{z})}. \tag{2.6}$$

This lemma is due to Rahman ([8], inequality (5.3)).

Lemma 6. Let $Q(z)$ be a polynomial of degree n , having all its zeros in $|z| \leq 1$. If $S(z)$ is a polynomial of degree at most n such that

$$|S(z)| \leq |Q(z)|, \quad \text{for } |z| = 1, \tag{2.7}$$

then for β with $|\beta| \leq 1$ and $R \geq 1$, we have

$$\left| S(Rz) + \beta \left(\frac{R+1}{2} \right)^n S(z) \right| \leq \left| Q(Rz) + \beta \left(\frac{R+1}{2} \right)^n Q(z) \right|, \quad |z| = 1. \tag{2.8}$$

This lemma is due to Jain [4].

Lemma 7. If $p(z)$ is a polynomial of degree at most n such that $M(p, 1) = 1$, then for β with $|\beta| \leq 1, R \geq 1$ and $|z| = 1$

$$\begin{aligned} & \left| p(Rz) + \beta \left(\frac{R+1}{2} \right)^n p(z) \right| + \left| q(Rz) + \beta \left(\frac{R+1}{2} \right)^n q(z) \right| \\ & \leq \left| 1 + \beta \left(\frac{R+1}{2} \right)^n \right| + \left| R^n + \beta \left(\frac{R+1}{2} \right)^n \right|. \end{aligned}$$

where $q(z)$ is, as in lemma 5.

This lemma is due to Jain [4].

Lemma 8. If $p(z)$ is a polynomial of degree n , having all its zeros in $|z| \leq 1$, then for β with $|\beta| \leq 1$ and $|z| = 1$,

$$\left| zp'(z) + \frac{n\beta}{2}p(z) \right| \geq \frac{n}{2} \{1 + \operatorname{Re}(\beta)\} |p(z)|.$$

This lemma is due to Jain ([3], Remark 2).

3. Proofs of the theorems

Proof of Theorem 1. If $p(z)$ has a zero on $|z| = 1$, then Theorem 1 reduces to ([3], Theorem 1). Therefore we assume that $p(z)$ has all its zeros in $|z| > 1$ (i.e. $m > 0$). Now for α with $|\alpha| < 1$, we have

$$|\alpha m| < m \leq |p(z)|, \quad |z| = 1, \quad (\text{by (1.9)}),$$

thereby implying by Rouché's theorem that the polynomial

$$p_2(z) = p(z) - \alpha m$$

has no zeros in $|z| < 1$. Therefore the polynomial

$$\begin{aligned} q_2(z) &= z^n \overline{p_2(1/\bar{z})} \\ &= q(z) - \bar{\alpha} m z^n, \quad (\text{by (2.6)}) \end{aligned}$$

will have all its zeros in $|z| \leq 1$. Also

$$|p_2(z)| = |q_2(z)|, \quad |z| = 1.$$

On applying Lemma 4, we get for $|z| = 1$,

$$\left| z p'_2(z) + \frac{n\beta}{2} p_2(z) \right| \leq \left| z q'_2(z) + \frac{n\beta}{2} q_2(z) \right|,$$

i.e.

$$\begin{aligned} \left| \left\{ z p'(z) + \frac{n\beta}{2} p(z) \right\} - \frac{n\beta}{2} \alpha m \right| &\leq \left| \left\{ z q'(z) + \frac{n\beta}{2} q(z) \right\} - \bar{\alpha} m z^n \left(1 + \frac{\beta}{2} \right) \right|, \\ &|\alpha| < 1, \end{aligned}$$

i.e.

$$\begin{aligned} \left| z p'(z) + \frac{n\beta}{2} p(z) \right| - n m |\alpha| \frac{|\beta|}{2} &\leq \left| \left| z q'(z) + \frac{n\beta}{2} q(z) \right| - |\alpha| m n \left| 1 + \frac{\beta}{2} \right| \right|, \\ &|\alpha| < 1. \end{aligned} \quad (3.1)$$

The polynomial $q(z)$, given by (2.6) has all its zeros in $|z| \leq 1$ and

$$\min_{|z|=1} |q(z)| = \min_{|z|=1} |p(z)| = m, \quad (\text{by (1.9)}).$$

And so, by Lemma 3 (inequality (2.1))

$$\min_{|z|=1} \left| z q'(z) + \frac{n\beta}{2} q(z) \right| \geq m n \left| 1 + \frac{\beta}{2} \right|,$$

thereby allowing us to rewrite (3.1) as

$$\begin{aligned} \left| z p'(z) + \frac{n\beta}{2} p(z) \right| - m n |\alpha| \frac{|\beta|}{2} &\leq \left| z q'(z) + \frac{n\beta}{2} q(z) \right| - |\alpha| m n \left| 1 + \frac{\beta}{2} \right|, \\ &|z| = 1 \quad \text{and} \quad |\alpha| < 1. \end{aligned}$$

As $|\alpha| \rightarrow 1$, we get for $|z| = 1$,

$$\left| z p'(z) + \frac{n\beta}{2} p(z) \right| - \left| z q'(z) + \frac{n\beta}{2} q(z) \right| \leq -m n \left(\left| 1 + \frac{\beta}{2} \right| - \left| \frac{\beta}{2} \right| \right). \quad (3.2)$$

Now, by lemma 5, we have for $|z| = 1$,

$$\left| z p'(z) + \frac{n\beta}{2} p(z) \right| + \left| z q'(z) + \frac{n\beta}{2} q(z) \right| \leq n \left(\left| 1 + \frac{\beta}{2} \right| + \left| \frac{\beta}{2} \right| \right) M(p, 1). \quad (3.3)$$

Addition of inequalities (3.2) and (3.3) easily leads to inequality (1.13)

On applying lemma 6 to the polynomials $p_2(z)$ and $q_2(z)$, we get for $R > 1$ and $|z| = 1$,

$$\left| p_2(Rz) + \beta \left(\frac{R+1}{2} \right)^n p_2(z) \right| \leq \left| q_2(Rz) + \beta \left(\frac{R+1}{2} \right)^n q_2(z) \right|,$$

i.e.

$$\left| \left\{ p(Rz) + \beta \left(\frac{R+1}{2} \right)^n p(z) \right\} - \alpha m \left\{ 1 + \beta \left(\frac{R+1}{2} \right)^n \right\} \right| \\ \leq \left| \left\{ q(Rz) + \beta \left(\frac{R+1}{2} \right)^n q(z) \right\} - \bar{\alpha} m z^n \left\{ R^n + \beta \left(\frac{R+1}{2} \right)^n \right\} \right|, \quad |\alpha| < 1,$$

i.e.

$$\left| p(Rz) + \beta \left(\frac{R+1}{2} \right)^n p(z) \right| - m|\alpha| \left| 1 + \beta \left(\frac{R+1}{2} \right)^n \right| \\ \leq \left| \left| q(Rz) + \beta \left(\frac{R+1}{2} \right)^n q(z) \right| - |\alpha| m \left| R^n + \beta \left(\frac{R+1}{2} \right)^n \right| \right|, \\ |\alpha| < 1. \quad (3.4)$$

Further on applying lemma 3 (inequality (2.2)) to the polynomial $q(z)$, we get for $R > 1$

$$\min_{|z|=1} \left| q(Rz) + \beta \left(\frac{R+1}{2} \right)^n q(z) \right| \geq \left| R^n + \beta \left(\frac{R+1}{2} \right)^n \right| \min_{|z|=1} |q(z)|, \\ = m \left| R^n + \beta \left(\frac{R+1}{2} \right)^n \right|,$$

thereby allowing us to rewrite (3.4) as

$$\left| p(Rz) + \beta \left(\frac{R+1}{2} \right)^n p(z) \right| - m|\alpha| \left| 1 + \beta \left(\frac{R+1}{2} \right)^n \right| \\ \leq \left| q(Rz) + \beta \left(\frac{R+1}{2} \right)^n q(z) \right| - |\alpha| m \left| R^n + \beta \left(\frac{R+1}{2} \right)^n \right|, \\ |z| = 1, \quad R > 1 \quad \text{and} \quad |\alpha| < 1.$$

As $|\alpha| \rightarrow 1$, we get for $|z| = 1$ and $R > 1$,

$$\left| p(Rz) + \beta \left(\frac{R+1}{2} \right)^n p(z) \right| - \left| q(Rz) + \beta \left(\frac{R+1}{2} \right)^n q(z) \right| \\ \leq -m \left\{ \left| R^n + \beta \left(\frac{R+1}{2} \right)^n \right| - \left| 1 + \beta \left(\frac{R+1}{2} \right)^n \right| \right\}. \quad (3.5)$$

Now, by lemma 7, we have for $|z| = 1$ and $R > 1$,

$$\left| p(Rz) + \beta \left(\frac{R+1}{2} \right)^n p(z) \right| + \left| q(Rz) + \beta \left(\frac{R+1}{2} \right)^n q(z) \right| \\ \leq \left\{ \left| R^n + \beta \left(\frac{R+1}{2} \right)^n \right| + \left| 1 + \beta \left(\frac{R+1}{2} \right)^n \right| \right\} M(p, 1). \quad (3.6)$$

Addition of inequalities (3.5) and (3.6) easily leads to inequality (1.14). This also completes the proof of Theorem 1.

Proof of Theorem 2. If $p(z)$ has a zero on $|z| = 1$, then Theorem 2 reduces to ([3], Remark 2). Therefore we assume that $p(z)$ has all its zeros in $|z| < 1$. Now as in the proof of lemma 3, for α with $|\alpha| < 1$, the polynomial

$$p_1(z) = p(z) - \alpha m$$

will have all its zeros in $|z| < 1$. On applying lemma 8, we get for α with $|\alpha| < 1$ and $|z| = 1$,

$$\left| zp'(z) + \frac{n\beta}{2} \{p(z) - \alpha m\} \right| \geq \frac{n}{2} \{1 + \operatorname{Re}(\beta)\} |p(z) - \alpha m|, \tag{3.7}$$

i.e.

$$\left| zp'(z) + \frac{n\beta}{2} p(z) \right| - \left| \frac{n\beta}{2} \alpha m \right| \geq \frac{n}{2} \{1 + \operatorname{Re}(\beta)\} \{|p(z)| - |\alpha m|\}. \tag{3.8}$$

Further, by lemma 3 (inequality (2.1)), we have for $|z| = 1$.

$$\begin{aligned} \left| zp'(z) + \frac{n\beta}{2} p(z) \right| &\geq nm \left| 1 + \frac{\beta}{2} \right|, \\ &\geq nm \frac{|\beta|}{2}, \\ &\geq \left| \frac{n\beta}{2} \alpha m \right|, \quad \text{for } |\alpha| < 1, \end{aligned}$$

thereby allowing us to rewrite (3.8) as

$$\left| zp'(z) + \frac{n\beta}{2} p(z) \right| - \left| \frac{n\beta}{2} \alpha m \right| \geq \frac{n}{2} \{1 + \operatorname{Re}(\beta)\} \{|p(z)| - |\alpha m|\},$$

$|z| = 1 \quad \text{and} \quad |\alpha| < 1.$

As $|\alpha| \rightarrow 1$, we get for $|z| = 1$,

$$\left| zp'(z) + \frac{n\beta}{2} p(z) \right| \geq \frac{n}{2} \{1 + \operatorname{Re}(\beta)\} |p(z)| + \frac{nm}{2} [|\beta| - \{1 + \operatorname{Re}(\beta)\}],$$

thereby implying

$$\max_{|z|=1} \left| zp'(z) + \frac{n\beta}{2} p(z) \right| \geq \frac{n}{2} (\{1 + \operatorname{Re}(\beta)\} M(p, 1) + m[|\beta| - \{1 + \operatorname{Re}(\beta)\}]). \tag{3.9}$$

Again, by (3.7), we have for $|\alpha| < 1$ and $|z| = 1$,

$$\left| zp'(z) + \frac{n\beta}{2} p(z) \right| + \left| \frac{n\beta}{2} \alpha m \right| \geq \frac{n}{2} \{1 + \operatorname{Re}(\beta)\} \{|p(z)| + |\alpha m|\}.$$

As $|\alpha| \rightarrow 1$, we get for $|z| = 1$,

$$\left| zp'(z) + \frac{n\beta}{2} p(z) \right| \geq \frac{n}{2} \{1 + \operatorname{Re}(\beta)\} |p(z)| + \frac{nm}{2} [\{1 + \operatorname{Re}(\beta)\} - |\beta|],$$

thereby implying

$$\max_{|z|=1} \left| zp'(z) + \frac{n\beta}{2} p(z) \right| \geq \frac{n}{2} (\{1 + \operatorname{Re}(\beta)\} M(p, 1) + m[\{1 + \operatorname{Re}(\beta)\} - |\beta|]). \quad (3.10)$$

From (3.9) and (3.10), Theorem 2 follows.

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