Proc. Indian Acad. Sci. (Math. Sci.), Vol. 110, No. 1, February 2000, pp. 117–120.

Slow rotation of a sphere with source at its centre in a viscous fluid

SUNIL DATTA and DEEPAK KUMAR SRIVASTAVA

Department of Mathematics and Astronomy, Lucknow University, Lucknow 226 007, India

MS received 10 March 1999; revised 12 October 1999

Abstract. In this note, the problem of a sphere carrying a fluid source at its centre and rotating with slow uniform angular velocity about a diameter is studied. The analysis reveals that only the azimuthal component of velocity exists and is seen that the effect of the source is to decrease it. Also, the couple on the sphere is found to decrease on account of the source.

Keywords. Slow rotation; viscous fluid.

1. Introduction

The problem of determining the couple experienced by axially symmetric bodies, rotating steadily in a viscous and incompressible fluid has engaged the attention of many workers like Jeffery [2], Kanwal [3], Smith [6], Watson [7], and Ram Kissoon [5]. The purpose of this paper is to study slow rotation of a sphere, assumed to be pervious, with a source at its centre. If the strength Q of the source were of the same order as the angular velocity Ω of rotating sphere, the inertia terms could still be neglected and the total flow consists of only the source solution superimposed on the Stokes solution; thus in this case the Stokes drag and couple are not affected by the source. On the other hand if Q is large enough so that the $Q\Omega$ is not negligible, the inertia terms, being non-linear, cannot be altogether omitted; the equation, however, can still be linearized by assuming that the velocity perturbation in the source flow on account of the Stokes flow is small so that the terms containing square of angular velocity can be neglected. This assumption is justifiable at least in the vicinity of the sphere where the Stokes approximation is valid too. The problem corresponds to the problem of Stokes flow past a sphere with source at its centre investigated by Datta [1], the results of which have found application in investigating the diffusiophoresis target efficiency for an evaporating or condensing drop [4].

2. Formulation of the problem

Let us consider a pervious sphere of radius a with source of strength Q at its center generating radial flow field around it in an infinite expanse of incompressible fluid of density ρ and kinematic viscosity ν. The sphere is also made to rotate with small steady angular velocity Ω so that terms of an $O(\Omega^2)$ may be neglected but terms of $O(Q\Omega)$ is retained.

The motion is governed by Navier–Stokes equations and the continuity equation together with no-slip boundary condition

$$\boldsymbol{u} = a\Omega \hat{e}_x x \hat{e}_r, \tag{2.1}$$

on the surface $r = a$, and the condition of vanishing of velocity at far off points,

$$\boldsymbol{u} = 0, \quad \text{at infinity as } r \to \infty. \tag{2.2}$$

It will be convenient to work in spherical polar coordinates (r, θ, ϕ) with x-axis as the polar axis. We non-dimensionalize the space variables by a, velocity $\boldsymbol{u}(v_r, v_\theta, v_\phi)$ by $a\Omega$ and pressure by $\rho\nu\Omega$. Moreover, the symmetry of the problem and the boundary conditions ensure that velocity components $v_r = v_\theta = 0$, and then we may express the velocity vector $\boldsymbol{u}$ as

$$\boldsymbol{u} = \frac{Q}{a^2 r^2} r + a\Omega v_\phi(r, \theta)\hat{e}_\phi \tag{2.3}$$

and pressure as

$$p = \rho\nu\Omega[p_0(r) + p_1(r, \theta)]. \tag{2.4}$$

By using the value (2.3) and (2.4) in Navier–Stokes equation, the azimuthal componant v_ϕ is seen to satisfy the equation

$$\nabla^2 v_\phi - \frac{v_\phi}{r^2 \sin^2\theta} = \frac{s}{r^3}\frac{\partial}{\partial r}(r v_\phi), \tag{2.5}$$

where $s = Q\Omega/\nu a$ is the source parameter.

The above equation is to be solved under the boundary conditions

$$\left(\begin{array}{ll} v_\phi = a\Omega\sin\theta & \text{at } r = 1 \\ \text{and} & \\ v_\phi \to 0 & \text{as } r \to \infty. \end{array}\right. \tag{2.6}$$

3. Solution

We substitute

$$v_\phi = r\omega(r)\sin\theta \tag{3.1}$$

in equation (2.5) and solve the resulting differential equation in $\omega(r)$, using boundary conditions corresponding to (2.6) to get

$$v_\phi = r\sin\theta\left[\frac{s^2}{r^2} - 2\frac{s}{r} + 2 - 2\mathrm{e}^{-g/r}\right][s^2 - 2s + -2\mathrm{e}^{-g}]^{-1}. \tag{3.2}$$

Following limiting values of v_ϕ emerge from (3.2)

Case I. When parameter $s = Q\Omega/\nu a$ is small, we have

$$v_\phi \approx \frac{\sin\theta}{r^2}\left[1 + \frac{s}{4}\left(1 - \frac{1}{r}\right)\right]. \tag{3.3}$$

Case II. When parameter $s = Q\Omega/\nu a$ is large and r is finite, we get

$$v_\phi \approx \frac{\sin\theta}{r^2}\left[1 - \frac{2}{s}(r - 1)\right]. \tag{3.4}$$

Since v_ϕ can not be negative the above result is to be true for $1 \le r < 1 + s/2$.

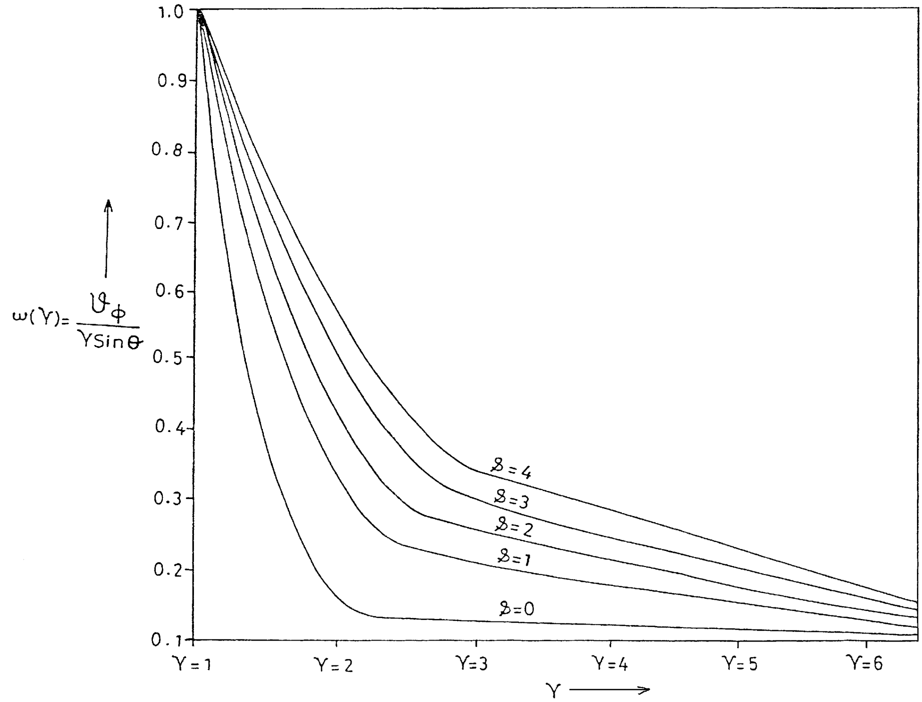

Figure 1. Variation of angular velocity $\omega(r)$ with respect to r and for varions values of parameter s.

4. Couple on the sphere

The couple on the sphere required to maintain the motion is obtained by integration of viscous stress

$$\sigma_{r\phi} = \mu\Omega\left(\frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r}\right)_{r=1}.$$

Thus, the moment of the required couple is given by

$$\begin{aligned} M &= -\int_{\theta=0}^{\pi} \sigma_{r\phi} R^3 2\pi \sin\theta \mathrm{d}\theta \\ &= \frac{16}{3}\pi a^3 \mu\Omega[s(1-\mathrm{e}^{-s}) - s^2][2 - 2s + s^2 - 2\mathrm{e}^{-s}]^{-1}. \end{aligned} \tag{4.1}$$

Further, when s is small, we get from (4.1)

$$M = 8\pi\mu a^3\Omega\left(1 - \frac{s}{12}\right), \tag{4.2}$$

which provides the classical value $M_0 = 8\pi\mu a^3\Omega$ for $s = 0$. Also for large values of s, we have the approximation as

$$M = \frac{16}{3}\pi\mu a^3\Omega\left(1 + \frac{1}{s}\right). \tag{4.3}$$

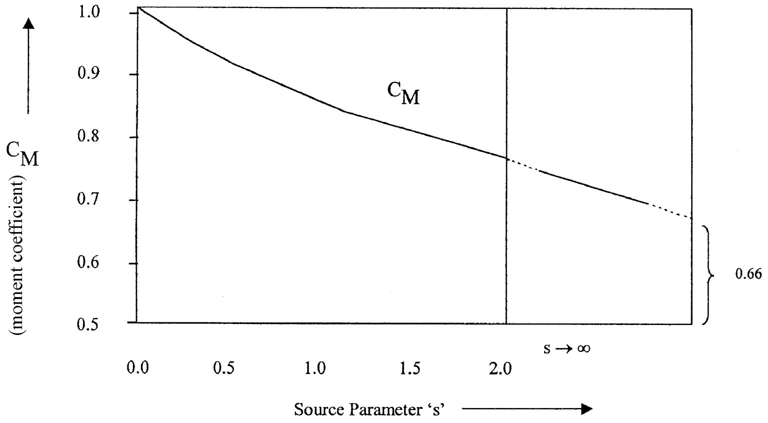

Figure 2. Variation of moment coefficient C_M with respect to source parameter s.

We find that M tends to 2/3 M_0, as $s \to \infty$. The expression for angular velocity $\omega(r) = v_\phi / r \sin\theta$ can be easily obtained from equation (3.2) as

$$\omega(r) = \left[\frac{s^2}{r^2} - 2\frac{s}{r} + 2 - 2\mathrm{e}^{-s/r}\right][s^2 - 2s + 2 - 2\mathrm{e}^{-s}]^{-1}. \tag{4.4}$$

The general behaviour of angular velocity $\omega(r)$ with respect to r and for various values of parameter s has been shown in figure 1. It may be concluded that flow gets dampened as the source strength increases.

The variation of moment coefficient $C_M = M/M_0$ with source parameter $s = Q\Omega/\nu a$, where M_0 is the moment for $s = 0$, has been shown in figure 2.

Figure (2) clearly shows that the effect of source is to reduce the moment ultimately to two third of its value in the absence of the source.

References

[1] Datta S, Stokes flow past a sphere with a source at its centre. *Math. Vesnik* **10(25)** (1973) 227–229

[2] Jeffry G B, Steady rotation of a solid of revolution in a viscous fluid. *Proc. London Math. Soc.* **14** (1955) 327–38

[3] Kanwal R P, Slow steady rotation of axially symmetric bodies in a viscous fluid. *J. Fluid Mech.* **10** (1960) 17–24

[4] Placek T D and Peters L K, A hydrodynamic approach to particle target efficiency in the presence of diffusiophoresis. *Aerospace J.* **11** (1980) 521–533

[5] Ram Kissoon H, A Slip flow problem. *J. Math. Sci (Calcutta)* **8(1)** (1997) 23–27

[6] Smith S H, The rotation of two circular cylinders in a viscous fluid. *Mathematika*, **38(1)** (1991) 63–66

[7] Watson E J, Slow viscous fluid past two rotating cylinders, *Q. J. Mech. Appl. Math.* **49(2)** (1996) 195–216