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An approximate solution for spherical and cylindrical piston problem

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Abstract. A new theory of shock dynamics (NTSD) has been derived in the form of a finite number of compatibility conditions along shock rays. It has been used to study the growth and decay of shock strengths for spherical and cylindrical pistons starting from a non-zero velocity. Further a weak shock theory has been derived using a simple perturbation method which admits an exact solution and also agrees with the classical decay laws for weak spherical and cylindrical shocks.

Keywords. Shock dynamics; compatibility conditions; blast wave; weak shock propagation.

1. Introduction

Though the occurrence of a shock discontinuity in compressible flows and the jump conditions across it are known for more than a century, the idea of deriving an infinite system of compatibility conditions along shock rays (Prasad [9]) was discovered only recently (Grinfeld [2], Maslov [7]). By truncating the infinite system of compatibility conditions at an appropriate level, a new theory of shock dynamics (NTSD) has been proposed (Ravindran and Prasad [11]) which enables one to compute the position and strength of a shock front and also to determine the flow behind the shock up to a short distance. Lazarev, Prasad and Singh [6] used NTSD to study the growth and decay of a plane shock originating due to an accelerating or decelerating piston. They also compared the results from NTSD with those from Harten's total variation diminishing (TVD) finite difference scheme (FDM) and found good agreement. It was also noted that NTSD consumes only 0.5% of the computational time taken by FDM, while giving almost the same (and in some cases even better) accuracy for the solution.

The problem of blast wave propagation originating from the detonation of an explosive has been modeled as that of a symmetrically expanding spherical or cylindrical piston by several authors (see Stanyukovich [15], Courant and Friedrichs [1]). This problem presents an example of a flow field in which the flow behind the shock front is highly non-uniform due to a rapid decay of the flow behind the shock, which makes the use of Whitham's shock dynamics [18, 19] (which ignores the effects of the flow behind the shock) inapplicable for such problems.

Another conventional approach for solving blast problem has been either the use of self-similar solutions, valid only for a short time and a short distance from the site of explosion (Taylor [16], Sedov [12]) or resorting to specially devised finite difference schemes such as those due to Glimm or Godunov (see Holt [3], Peyret and Taylor [8]).

The relative efficiency of NTSD over a self-similar solution and a finite difference solution has been examined by Singh and Singh [13] in the case of a single conservation law.

In this paper, we apply NTSD to obtain an approximate solution to spherical and cylindrical piston problem. A general approach for both accelerating and decelerating cases has been presented, in particular the latter can model closely the phenomenon of rapid decay of the flow behind the shock occurring in a blast. Further, a theory for the propagation of weak shocks has been derived by using a simple perturbation of the ordinary differential equations appearing in the NTSD. This solution reduces to the well known classical results for the decay of weak spherical and cylindrical shocks.

2. Dynamical compatibility conditions

The unsteady flow of an ideal gas with constant specific heats for spherical or cylindrical symmetry is given by the following system

$$\begin{aligned}&\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial r}+\rho\frac{\partial u}{\partial r}+j\,\frac{\rho u}{r}=0,\\&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r}=0,\\&\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial r}+\gamma p\frac{\partial u}{\partial r}+j\frac{\gamma p u}{r}=0,\end{aligned}\tag{2.1}$$

where ρ, u, p are the density, velocity and the pressure of the gas, γ is the ratio of specific heats; t, r are the time and radial co-ordinates respectively and $j = 1, 2$ for cylindrical and spherical cases respectively.

Let $r = R(t)$ be the position of the shock front propagating into the gas at uniform state and at rest ahead of the shock front. We introduce the following notations

$$D_0 = [\rho], \quad H_0 = [u], \quad S_0 = [p],\tag{2.2}$$

where [ ] denotes the jump across the shock front: $[G] = G_+ - G_-$, $+$ denotes the state ahead and $-$ that behind the shock front. The expressions for H_0, S_0 and the shock velocity C are given by the well known Rankine–Hugoniot relations

$$\begin{aligned}&H_0 = CD_0(\rho_+ - D_0)^{-1}, \quad S_0 = \rho_+ D_0 C^2(\rho_+ - D_0)^{-1},\\&C = a_+(2(\rho_+ - D_0)(2\rho_+ + (\gamma-1)D_0)^{-1})^{1/2},\end{aligned}\tag{2.3}$$

where a_+ is the local sound velocity ahead of the shock. It is assumed that the flow variables $\rho(r,t), u(r,t)$ and $p(r,t) \in C^\infty$ behind the shock front. The following relations can be written for the derivatives of the flow variables on the shock front

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \frac{\partial Z}{\partial t} + C\frac{\partial Z}{\partial r}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial^N Z}{\partial r^N}\right) = \frac{\partial^{N+1} Z}{\partial t\partial r^N} + C\,\frac{\partial^{N+1} Z}{\partial r^{N+1}}\tag{2.4}$$

for $N = 1, 2, 3, \ldots$, and $Z(r,t)$ can be any of the flow variables ρ, u or p. Taking jump on both sides in (2.4), we obtain

$$\left[\frac{\partial Z}{\partial t}\right] = \frac{\mathrm{d}}{\mathrm{d}t}[Z] - C\left[\frac{\partial Z}{\partial r}\right], \quad \left[\frac{\partial^{N+1} Z}{\partial t\partial r^N}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial^N Z}{\partial r^N}\right] - C\left[\frac{\partial^{N+1} Z}{\partial r^{N+1}}\right].\tag{2.5}$$

We notice that the first equation in (2.5) is the first kinematical compatibility condition of Thomas [17]. Replacing Z by ρ, u, p in (2.5), we get the following relations

$$\left[\frac{\partial \rho}{\partial t}\right] = \frac{\mathrm{d}}{\mathrm{d}t}D_0 - CD_1, \quad \left[\frac{\partial^{N+1}\rho}{\partial t \partial r^N}\right] = \frac{\mathrm{d}}{\mathrm{d}t}D_N - CD_{N+1},$$
$$\left[\frac{\partial u}{\partial t}\right] = \frac{\mathrm{d}}{\mathrm{d}t}H_0 - CH_1, \quad \left[\frac{\partial^{N+1}u}{\partial t \partial r^N}\right] = \frac{\mathrm{d}}{\mathrm{d}t}H_N - CH_{N+1},$$
$$\left[\frac{\partial p}{\partial t}\right] = \frac{\mathrm{d}}{\mathrm{d}t}S_0 - CS_1, \quad \left[\frac{\partial^{N+1}p}{\partial t \partial r^N}\right] = \frac{\mathrm{d}}{\mathrm{d}t}S_N - CS_{N+1}, \tag{2.6}$$

where

$$D_N = \left[\frac{\partial^N \rho}{\partial r^N}\right], \quad H_N = \left[\frac{\partial^N u}{\partial r^N}\right], \quad S_N = \left[\frac{\partial^N p}{\partial r^N}\right]. \tag{2.7}$$

We consider the jump of the left hand side of (2.1) across the shock front. Using (2.6), we obtain the first set of dynamic compatibility conditions

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{U}_0 + \mathbf{P}\cdot\mathbf{U}_1 = \mathbf{f}_0, \tag{2.8}$$

where

$$\mathbf{U}_N = \begin{pmatrix} D_N \\ H_N \\ S_N \end{pmatrix}, \quad N = 1, 2, \ldots \tag{2.9}$$

and

$$\mathbf{P} = \begin{pmatrix} -(C+H_0) & \rho_+ - D_0 & 0 \\ 0 & -(C+H_0) & (\rho_+ - D_0)^{-1} \\ 0 & \gamma(p_+ - S_0) & -(C+H_0) \end{pmatrix} \tag{2.10}$$

and

$$\mathbf{f}_0 = -jr^{-1}\begin{pmatrix} H_0(\rho_+ - D_0) \\ 0 \\ \gamma H_0(p_+ - S_0) \end{pmatrix}. \tag{2.11}$$

To derive the second set of compatibility conditions, we differentiate (2.1) with respect to r and take the jump of the left hand side of the resulting equation. Using (2.6), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{U}_1 + \mathbf{P}\cdot\mathbf{U}_2 = \mathbf{f}_1, \tag{2.12}$$

where

$$\mathbf{f}_1 = \begin{pmatrix} 2D_1H_1 + jr^{-1}(H_0D_1 - (\rho_+ - D_0)H_1) + jr^{-2}(\rho_+ - D_0)H_0 \\ H_1^2 - S_1D_1(\rho_+ - D_0)^{-2} \\ (\gamma+1)H_1S_1 + jr^{-1}\gamma(H_0S_1 - (p_+ - S_1)H_1) + jr^{-2}\gamma(p_+ - S_0)H_0 \end{pmatrix}. \tag{2.13}$$

(It may be noted that for $j = 0$, the system of compatibility conditions (2.8) and (2.12) reduces to that for plane shocks, see [6] for details.)

Repeating the same procedure, an infinite system of compatibility conditions can be derived in the following general form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{U}_N + \mathbf{P}\cdot\mathbf{U}_{N+1} = \mathbf{f}_N, \quad N = 0, 1, 2\ldots \tag{2.14}$$

The equations (2.8) and (2.11) are the first two members of the system (2.14).

It is obvious that for computational purposes, it is more convenient to work with a scalar system of equations. We now describe a procedure to reduce the above system of vector compatibility conditions into an equivalent system of scalar compatibility conditions. We note that the eigenvalues of the matrix **P** are

$$\lambda_1 = -(C + H_0), \quad \lambda_{2,3} = -(C + H_0) \pm a_-, \tag{2.15}$$

where a_- is the sound velocity behind the shock front. As the shock strength H_0 tends to zero, the shock velocity and the local sound velocity tend to a common value, say a_0, hence λ_2 tends to zero. In this limit the first set of compatibility conditions (2.8) must lead to the characteristic compatibility condition in which the derivative terms must be zero. Hence, we choose the left eigenvector **L** corresponding to λ_2:

$$\mathbf{L} = (0, (\rho_+ - D_0)/2, (2a_-)^{-1})$$

and introduce the following notations

$$\begin{aligned}\pi_0 &= D_0,\\ \pi_N &= \mathbf{L}\cdot\mathbf{U}_N = (\rho_+ - D_0)H_N/2 + S_N(2a_-)^{-1}, \quad N = 1, 2, \ldots\end{aligned} \tag{2.16}$$

Multiplying (2.8) by **L**, we get after some simplifications

$$\frac{\mathrm{d}\pi_0}{\mathrm{d}t} = -g\left(\lambda_2\pi_1 + \frac{j}{2r}Ca\pi_0\right), \tag{2.17}$$

where

$$g = ((\rho_+ - \pi_0)a/2 + b(2a_-)^{-1})^{-1}, \tag{2.18}$$

$$a = \frac{\partial H_0}{\partial \pi_0} = \frac{C\rho_+(4\rho_+ + (\gamma - 3)\pi_0)}{2(\rho_+ - \pi_0)^2(2\rho_+ + (\gamma - 1)\pi_0)}, \tag{2.19}$$

$$b = \frac{\partial S_0}{\partial \pi_0} = 4\gamma p_+\rho_+(2\rho_+ + (\gamma - 1)\pi_0)^2)^{-2}. \tag{2.20}$$

Next, to express D_1, H_1, S_1 in terms of π_0 and π_1, we multiply (2.8) by $\mathbf{P}^{-1}$ to get

$$\mathbf{U}_1 = \mathbf{P}^{-1}\left(\mathbf{f}_0 - \frac{\mathrm{d}\mathbf{U}_0}{\mathrm{d}t}\right). \tag{2.21}$$

Using (2.17), we get from (2.21)

$$\begin{aligned}D_1 &= jr^{-1}Cd_{10}\pi_0 + gd_{11}\pi_1, \quad H_1 = jr^{-1}Ch_{10}\pi_0 + h_{11}\pi_1,\\ S_1 &= jr^{-1}Cs_{10}\pi_0 + s_{10}\pi_1,\end{aligned} \tag{2.22}$$

where

$$\begin{aligned}d_{10} = \lambda_1^{-1}(-1 + ga_-/2) - aga_-(2\lambda_2\lambda_3)^{-1}(\rho_+ - \pi_0)\\ + (\lambda_1\lambda_2\lambda_3)^{-1}(-a_-^2 + bga_-/2),\end{aligned}$$

$$
\begin{aligned}
d_{11} &= \lambda_2\lambda_1^{-1} - a\lambda_3^{-1}(\rho_+ - \pi_0) + b(\lambda_1\lambda_3)^{-1},\\
h_{10} &= (\lambda_2\lambda_3)^{-1}(aga_-\lambda_1/2 - (-a_-^2 + bga_-/2)(\rho_+ - \pi_0)^{-1}),\\
h_{11} &= \lambda_3^{-1}(\lambda_1 a - b(\rho_+ - \pi_0)^{-1}),\\
s_{10} &= (\lambda_2\lambda_3)^{-1}(-\gamma(p_+ - S_0)aga_-/2 + \lambda_1(-a_-^2 + bga_-/2)),\\
s_{11} &= \lambda_3^{-1}(-\gamma(p_+ - S_0)a + \lambda_1 b).
\end{aligned}
\tag{2.23}
$$

To reduce the second compatibility condition into scalar form, we multiply the equation (2.12) by **L** to obtain

$$
\frac{\mathrm{d}\pi_1}{\mathrm{d}t} = \delta_1\pi_0^2 + \delta_2\pi_1^2 + \delta_3\pi_0\pi_1 + \delta_4\pi_0 + \delta_5\pi_1 - \lambda_2\pi_2,
\tag{2.24}
$$

where

$$
\begin{aligned}
\delta_1 &= \left(\frac{jC}{r}\right)^2\left\{\frac{1}{2}(\rho_+ - \pi_0)(h_{10}^2 - (\rho_+ - \pi_0)^{-2}s_{10}d_{10}) + \frac{s_{10}}{2a_-}((\gamma+1)h_{10}\right.\\
&\qquad\left. + \gamma(\rho_+ - \pi_0)^{-1}) + \frac{1}{4}ga_-(h_{10} - es_{10})\right\},\\
\delta_2 &= \frac{1}{2}g^2\left\{(\rho_+ - \pi_0)(h_{11}^2 - (\rho_+ - \pi_0)^{-2}) + (\gamma+1)a_-^{-1}h_{11}s_{11} + \lambda_2(h_{11} - es_{11})\right\},\\
\delta_3 &= \frac{j}{2r}Cg\left\{(\rho_+ - \pi_0)(2h_{10}h_{11} - (\rho_+ - \pi_0)^{-2}(s_{10}d_{11} + s_{11}d_{10}) + (\gamma+1)a_-^{-1}\right.\\
&\qquad\left.\times(h_{10}s_{11} + h_{11}s_{10}) + \gamma s_{11}a_-^{-1}(\rho_+ - \pi_0)^{-1} + \frac{2}{g}\lambda_2(h_{10} - es_{10}) + a_-(h_{11} - es_{11})\right\},\\
\delta_4 &= \frac{j}{2r^2}Ca_-(1 - jh_{10}(\rho_+ - \pi_0)), \quad \delta_5 = -\frac{j}{2r}ga_-h_{11}(\rho_+ - \pi_0)
\end{aligned}
\tag{2.25}
$$

and

$$
e = -\frac{1}{a_-^2}\frac{\partial a_-}{\partial \pi_0} = (2a_-(\rho_+ - \pi_0))^{-1}\left(\frac{\gamma\rho_+C^4}{a_-^2a_+^2(\rho_+ - \pi_0)^2} - 1\right).
\tag{2.26}
$$

We add the equation of the shock path to the above system of compatibility conditions

$$
\frac{\mathrm{d}R}{\mathrm{d}t} = C,
\tag{2.27}
$$

where C is the shock velocity given by (2.3).

The new theory of shock dynamics (NTSD) using compatibility conditions up to the second order is obtained by putting $\pi_2 = 0$ in (2.24). The NTSD is valid for a shock of arbitrary strength.

3. Initial conditions for accelerating or decelerating piston problem

In this section we derive the initial conditions to solve the set of ordinary differential equations (2.17) and (2.24) (with π_2 set equal to zero) to obtain an approximate solution for an expanding spherical or cylindrical piston. We consider the flow produced by a spherical or cylindrical piston expanding with a non-zero positive velocity and a non-zero positive or negative acceleration into a gas at rest ahead of the piston. Let the piston position at time t be $R_p(t)$. Mathematically, the problem consists in solving the system of equations (2.1) with the following initial and boundary conditions

$$u(r,0) = 0, \quad p(r,0) = p_+, \quad \rho(r,0) = \rho_+ \quad \text{for } r > 0 \tag{3.1}$$

and

$$u(R_p(t),t) = R_p'(t)\,(= \text{piston velocity}), \quad \text{for } t > 0. \tag{3.2}$$

The flow variables are non-dimensionalized as follows

$$r = r_0\bar{r}, \quad t = \frac{r_0}{a_+}\bar{t}, \quad \rho = \rho_+\bar{\rho}, \quad p = \gamma p_+\bar{p}, \quad C = a_+\bar{C}, \tag{3.3}$$

where the overhead bar denotes the non-dimensional variable, r_0 is a characteristic length which has been chosen as the initial radius of the piston at $t = 0$.

We take the piston path as a power series in $\bar{t}$

$$\bar{R}_p(\bar{t}) = \bar{r}_0 + R_{p_1}\bar{t} + R_{p_2}\bar{t}^2 + R_{p_3}\bar{t}^3 + \cdots, \tag{3.4}$$

where $R_{p_j} = 0$ for $j > 2$ if the piston acceleration (or deceleration) is constant. We assume that the shock path is also given by a power series

$$\bar{R}(\bar{t}) = \bar{r}_0 + C_1\bar{t} + C_2\bar{t}^2 + C_3\bar{t}^3 + \cdots \tag{3.5}$$

and also that the solution in a small neighbourhood of $\bar{t} = 0$ can be expanded in a Taylor's series of the form

$$\bar{\rho} = \rho_0 + \rho_{11}(\bar{r} - \bar{r}_0) + \rho_{12}\bar{t} + \cdots, \quad \bar{u} = u_0 + u_{11}(\bar{r} - \bar{r}_0) + u_{12}\bar{t} + \cdots,$$
$$\bar{p} = p_0 + p_{11}(\bar{r} - \bar{r}_0) + p_{12}\bar{t} + \cdots, \tag{3.6}$$

where $\bar{r}_0$ is the non-dimensional value of r_0, and ρ_0, u_0, p_0 are the limiting values of the variables $\bar{\rho}, \bar{u}, \bar{p}$ as we approach the shock front from the piston at $\bar{t} = 0$.

Differentiating (3.4) and (3.5) with respect to $\bar{t}$, we get the series expansion for the piston velocity and the shock velocity respectively. Given the coefficients $R_{p_j}, j = 1, 2 \ldots,$ we need to find the coefficients in (3.6) and those for the shock path in (3.5). This is quite straightforward, but involves complex algebraic manipulations. Substituting the expansions (3.6) into the gas-dynamic equations and equating the coefficients of various powers of $\bar{r}$ and $\bar{t}$, we get an undetermined system of linear algebraic equations for the coefficients appearing in (3.6).

To complete this system, we use the series expansion of various quantities appearing in the boundary condition at the piston, namely the equation (3.2) and use the Rankine–Hugoniot condition on the piston path (3.4) to obtain the following set of linear equations for the determination of ρ_{11}, u_{11} and C_2,

$$L_1u_{11} + M_1\rho_{11} + N_1C_2 + P_1 = 0, \tag{3.7}$$
$$L_2u_{11} + M_2\rho_{11} + N_2C_2 + P_2 = 0, \tag{3.8}$$
$$L_3u_{11} + M_3\rho_{11} + N_3C_2 + P_3 = 0, \tag{3.9}$$

where

$$L_1 = 2\rho_0(R_{p_1} - C_1), \; M_1 = (R_{p_1} - C_1)^2, \; N_1 = 2(\rho_0 - 1),$$
$$P_1 = (j/\bar{r}_0)\rho_0R_{p_1}(R_{p_1} - C_1) + 2\rho_0R_{p_2},$$

$$L_2 = \gamma p_0 + 3\rho_0(R_{p1} - C_1)^2,\ \ M_2 = (R_{p1} - C_1)^3,\ \ N_2 = 4\rho_0(R_{p1} - C_1) + 4C_1,$$
$$P_2 = 4(R_{p1} - C_1)\rho_0 R_{p2} + (j\gamma/\bar{r}_0)p_0 R_{p1} + (j/\bar{r}_0)\rho_0 R_{p1}(R_{p1} - C_1)^2,$$
$$L_3 = \gamma^2 p_0/(\gamma - 1) + 3\rho_0(R_{p1} - C_1)^2/2 - \rho_0(1/(\gamma - 1) + C_1^2/2),$$
$$M_3 = (R_{p1} - C_1)^3/2 - (R_{p1} - C_1 - 1)(1/(\gamma - 1) + C_1^2/2),\ \ N_3 = 2\rho_0 R_{p1},$$
$$P_3 = 2\rho_0 R_{p2}(2\gamma - 1)(C_1 - R_{p1})/(\gamma - 1) + (j\gamma/\bar{r}_0)p_0 R_{p1} + (j\rho_0 R_{p1}/\bar{r}_0)\{1/(\gamma - 1) + C_1^2/2 - (C_1 - R_{p1})^2)/2\}. \tag{3.10}$$

Solving the above system of algebraic equations, we get the values of the required coefficients in (3.6). Hence, finally the initial conditions at $\bar{t} = 0$, for π_0 and π_1 in non-dimensional form is obtained as

$$\bar{\pi}_0 = R_{p1}/(R_{p1} - C_1), \quad \bar{\pi}_1 = (\bar{g}d_{11})^{-1}(-\rho_{11} + jC_1\bar{d}_{10}\bar{\pi}_0/\bar{r}_0), \tag{3.11}$$

where

$$C_1 = R_{p1}(2 - 2\mu^2)^{-1} + (1 + R_{p1}^2(2 - 2\mu^2)^{-2})^{1/2} \tag{3.12}$$

which is obtained using Prandtl's relation in the present case of purely radial flow (see [1], p. 425); $\bar{d}_{10}$ and $\bar{d}_{11}$ are the non-dimensional forms of d_{10} and d_{11} respectively, and $\mu^2 = (\gamma - 1)/(\gamma + 1)$. The detailed derivations are available in [14]. We note that the additional term $jC_1\bar{d}_{10}\pi_0/\bar{r}_0$ on the right hand side of (3.12) arising purely due to the geometry of the shock front causes an additional deceleration which accounts for the usual geometric attenuation for the curved shock fronts. By putting $j = 0$ in our equations, it is easily seen that the problem reduces to that of a plane shock (see [6]). In this case it is obvious that the terms $P_j,\ j = 1, 2, 3$ vanish if $R_{p2} = 0$, which in turn implies that ρ_{11} and consequently π_1 at $t = 0$ also vanish. Physically, it corresponds to the case of a plane piston moving with a uniform velocity giving rise to a shock of uniform strength.

Thus, it is seen that the initial conditions for the equations (2.17), (2.24) and (2.27) are completely determined in terms of coefficients appearing in the power series expansion of the piston path (3.4). It is also observed that the initial condition for π_0 depends on the piston velocity and the effects of any perturbation (i.e. acceleration or deceleration) in the uniform piston velocity are contained in the initial value for π_1.

4. Approximate solution for weak shock propagation

Some interesting well known results for weak shock propagation can be obtained by assuming that the shock strength is of the order of a small quantity ϵ, i.e. we assume that

$$\pi_0 = \sum_{j=1}^{\infty} \epsilon^j \pi_0^{(j)}(t), \quad \pi_1 = \sum_{j=0}^{\infty} \epsilon^j \pi_1^{(j)}(t). \tag{4.1}$$

It is to be noted that the expansion for π_0 starts with the first power of ϵ whereas that for π_1 starts with a constant term. We further assume that $R(t)$ can also be expanded as

$$R(t) = \sum_{j=0}^{\infty} \epsilon^j R^{(j)}. \tag{4.2}$$

Substituting the above expansions into the equations of NTSD, and retaining terms only up to first order, we obtain the following set of equations for $\pi_0^{(1)}, \pi_1^{(0)}, R^{(0)}$ and $R^{(1)}$

$$\frac{\mathrm{d}\pi_0^{(1)}}{\mathrm{d}t} = -\pi_0^{(1)}\left(-\frac{\gamma+1}{4\rho_+}\pi_1^{(0)} + \frac{ja_+}{2r}\right), \tag{4.3}$$

$$\frac{\mathrm{d}\pi_1^{(0)}}{\mathrm{d}t} = \frac{\gamma+1}{2\rho_+}(\pi_1^{(0)})^2 - \frac{ja_+}{2r}\pi_1^{(0)}, \tag{4.4}$$

$$\frac{\mathrm{d}R^{(0)}}{\mathrm{d}t} = a_+, \quad \frac{\mathrm{d}R^{(1)}}{\mathrm{d}t} = -a_+\left(\frac{\gamma+1}{4}\right)\pi_0^{(1)}. \tag{4.5}$$

The equations (4.3) and (4.4) can be exactly integrated subject to initial conditions for $\pi_0^{(0)}$ and $\pi_1^{(0)}$ at $t = 0$, say

$$\pi_0^{(1)} = \pi_{00}, \quad \pi_1^{(0)} = \pi_{10}. \tag{4.6}$$

We note that the expansion (4.2) does not hold near the centre as $r \to 0$. Hence we assume that $R^{(0)} \neq 0$ (i.e. the shock front has a finite radius at $t = 0$ which indeed is the case with conventional explosive charges), say $R^{(0)} = r_0$ at the initial instant. Then from (4.5), we have

$$r \approx R^{(0)} = r_0 + a_+ t. \tag{4.7}$$

Plane case: In this case, $j = 0$ and the equations (4.5) are not required. The solution to the equations (4.3) and (4.4) when integrated with the initial conditions (4.6) are

$$\pi_1^{(0)} = \pi_{10}(1 - (\gamma+1)\pi_{10}t/(2\rho_+))^{-1}, \tag{4.8}$$

$$\pi_0^{(1)} = \pi_{00}(1 - (\gamma+1)\pi_{10}t/(2\rho_+))^{-1/2}. \tag{4.9}$$

Cylindrical case: In this case (i.e. when $j = 1$) the solutions to the equations (4.3) and (4.4) assume the following form

$$\pi_1^{(0)}(t) = \frac{a_+\rho_+\pi_{10}r_0^{1/2}}{(\rho_+ + (\gamma+1)r_0\pi_{10})(r_0 + a_+t)^{1/2} - (\gamma+1)(r_0 + a_+t)\pi_{10}r_0^{1/2}}, \tag{4.10}$$

$$\pi_0^{(1)}(t) = \pi_{00}\{(r_0 + a_+t)((\rho_+ + (\gamma+1)r_0\pi_{10}) - (\gamma+1)\pi_{10}r_0^{1/2}(r_0 + a_+t)^{1/2}\}^{-1/2}. \tag{4.11}$$

Spherical case: In this case (i.e. $j = 2$), the solutions for (4.3) and (4.4) are given by

$$\pi_1^{(0)}(t) = \frac{2a_+\rho_+\pi_{10}r_0}{(r_0 + a_+t)(2a_+\rho_+ + (\gamma+1)\pi_{10}r_0\log r_0 - (\gamma+1)\pi_{10}r_0\log(r_0 + a_+t))}, \tag{4.12}$$

$$\pi_0^{(1)}(t) = \frac{\pi_{00}}{(r_0 + a_+t)}(2a_+\rho_+ + (\gamma+1)r_0\pi_{10}\log r_0 - (\gamma+1)r_0\pi_{10}\log(r_0 + a_+t))^{-1/2}. \tag{4.13}$$

Critical time: It is seen that if $\pi_{10} > 0$, then the solutions given by (4.8)–(4.13) cannot be continued beyond a time t_c (called the critical time). It is also seen that as $t \to t_c$, $\pi_1^{(0)}$ and $\pi_0^{(1)}$ approach infinitely large values in each of the above cases. In fact, the weak shock assumption breaks down before these quantities approach infinity and t_c is an indication of this. The critical time t_c in the above three cases are given by

$$\text{Plane case:} \quad t_c = 2\rho_+/((\gamma+1)\pi_{10}), \tag{4.14}$$

$$\text{Cylindrical case:} \quad t_c = a_+^{-1}\{(a_+\rho_+/\pi_{10} + (\gamma+1)r_0)^2/((\gamma+1)^2 r_0) - r_0\}, \tag{4.15}$$

$$\text{Spherical case:} \quad t_c = a_+^{-1}\{\exp(2a_+\rho_+/((\gamma+1)\pi_{10}r_0) + \log r_0) - r_0\}. \tag{4.16}$$

In all the above three cases, we observe that

(i) there is a positive value for t_c for all positive values of π_{10}, which corresponds to the case of accelerating piston,
(ii) there is no finite value for t_c when π_{10} is negative, (i.e. the solutions (4.8)–(4.13) can be continued for all times for all negative values of π_{10}).

We also note that the case $\pi_{10} < 0$ corresponds to the case when the slope of the density versus spatial coordinate curve is positive, which implies that no positive finite value for t_c exists and hence the solution can be continued for all times (see [10]).

Comparison with exact results for decay of weak shocks: Taking the limit as $t \to \infty$ in (4.9), (4.11) and (4.13), so that the terms independent of t can be ignored, it is seen that the shock strength decays as $t^{-1/2}$, $t^{-3/4}$ and $t^{-1}(\log t)^{-1/2}$ for the cases of plane, cylindrical and spherical shocks respectively. Thus, the decay rule for weak shocks from NTSD agrees with classical results for the asymptotic decay for the cylindrical and spherical waves (see Landau [5], Whitham [19] and also Grinfeld [2]).

5. Results and discussions

The system of ordinary differential equations (2.17) and (2.24) (with $\pi_2 = 0$) and (2.27) are solved using Runge–Kutta–Gill method. In figure 1, the case of accelerating

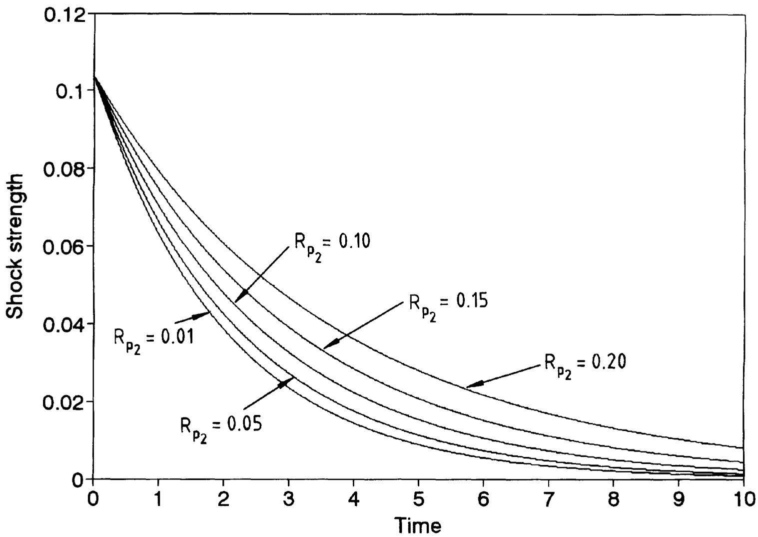

Figure 1. Decay of a cylindrical shock, originating from an initial piston velocity $R_{p1} = 0.10$ and with indicated values of accelerations R_{p2}.

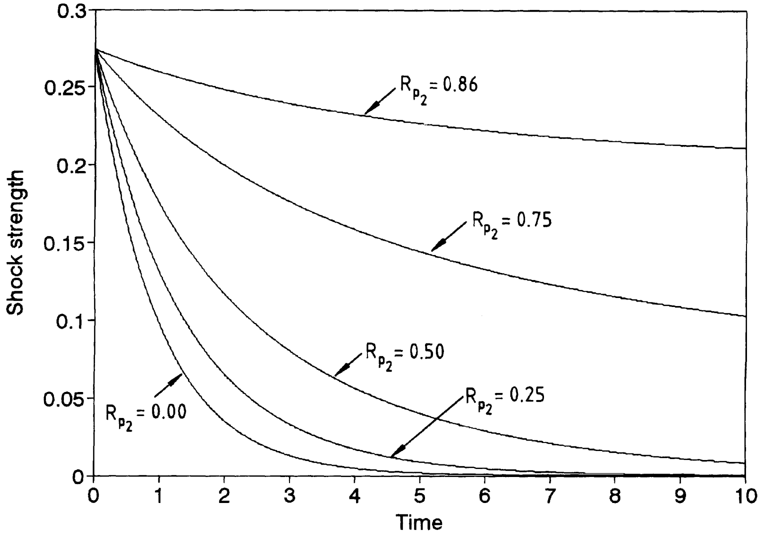

Figure 2. Decay of a spherical shock originating from an initial piston velocity $R_{p1} = 0.25$ and with indicated values of accelerations.

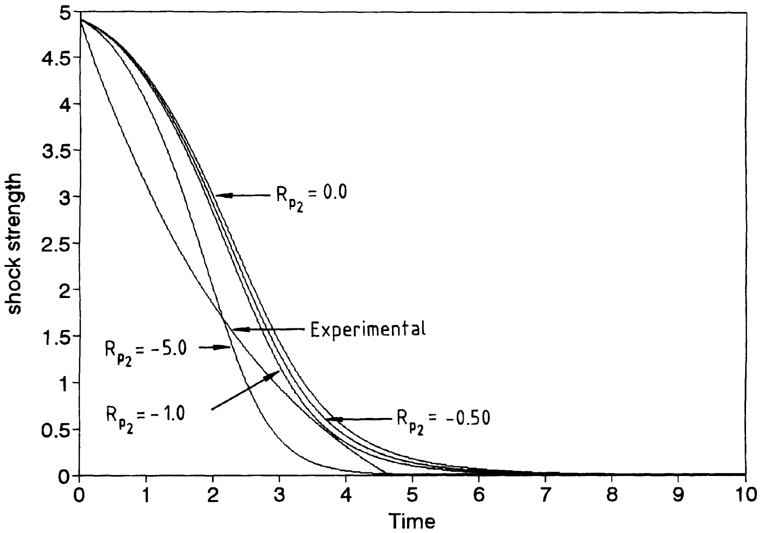

Figure 3. Decay of a strong spherical shock, originating from an initial velocity $R_{p1} = 15.0$.

cylindrical piston is shown; the initial piston velocity R_{p_1} is chosen as 0.10 and the values chosen for the acceleration R_{p_2} are 0.01, 0.05, 0.10, 0.15 and 0.20 respectively. In figure 2, the case of accelerating spherical piston is shown corresponding to $R_{p_1} = 0.25$ and $R_{p_2} =$ 0.00, 0.25, 0.50, 0.75 and 0.86 respectively. The attenuation of the shock solely due to the geometrical effects is obvious in these curves. It is also observed (as in figure 2) that to maintain the initial strength of the shock (i.e. to overcome the effects of geometrical attenuations), a considerable amount of constant acceleration is required.

In figure 3, attenuation of a very strong spherical shock corresponding to $R_{p1} = 15.0$ with varying amounts of decelerations $R_{p_2} = 0.00$, -0.50, -1.00 and -5.00 is shown.

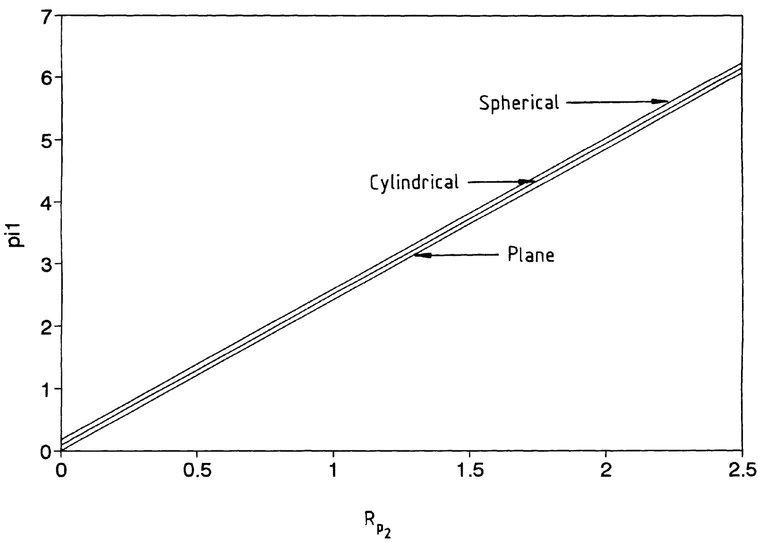

Figure 4. Initial values of π_1 at $t = 0$ as functions of the piston accelerations R_{p2} for the plane, cylindrical and spherical cases.

Since, in an actual blast phenomenon, the strong spherical or cylindrical shock front is attenuated due to geometrical effects as well as the rarefaction waves following it, the decelerating spherical or cylindrical piston problem may serve as a simplified and approximate model to represent the actual phenomenon. For comparison, we have also plotted the time decay curve for 1000 kg TNT at a distance of 1 m from the point of detonation (see Kinney and Graham [4]). We have chosen this particular example as the particle velocity in this case is comparable with the piston velocity under consideration at $t = 0$. For this problem $r_0 = 0.548$ m and $\bar{t} \approx 620t$. Since in an actual blast, the decay pattern depends on a number of factors such as the chemical composition of the explosive, its packing density etc. which are not included in our mathematical formulation, it may explain the deviation of NTSD results from the experimental curves.

As described in § 3, the value of $\pi_1|_{t=0}$ depends on the piston acceleration (or deceleration) R_{p_2}. For a typical case of $R_{p_1} = 0.25$, we have plotted $\pi_1|_{t=0}$ against R_{p_2} in figure 4 for the plane, cylindrical and spherical cases. The relationship is nearly linear. It is also seen that $\pi_1|_{t=0} = 0$ for the plane case whereas it has small nonzero positive values for the cases of curved pistons.

As indicated in §§ 3 and 4, the value of the critical time t_c would depend upon the piston acceleration. We have tabulated the values of t_c for a typical case of $R_{p_1} = 0.25$ and R_{p_2} varying from 0.25 to 5.0 in table 1. It is observed that with gradual increase in R_{p_2}, the plane shock reaches the strong shock limit (i.e. $\pi_0 \to 5.0$) and $\pi_1 \to \infty$ as $t \to t_c$ in almost all the cases. The curved shocks behave in a different way: they continue to decay in the presence of comparatively smaller values of piston acceleration (as they do in its absence). It is seen that there is a threshold value for R_{p_2} below which the curved shocks decay. There is another threshold value for R_{p_2} up to which the strength of a curved shock reaches a constant value in a finite time. The entry 'con' in the table 1 refers to this value and t is the time at which this constant shock strength is reached. Beyond this second threshold value for R_{p_2}, the shock strength grows until it attains the strong shock limit and $\pi_1 \to \infty$ at $t = t_c$ where the NTSD algorithm breaks down.

Table 1. Critical times t_c for $R_{p1} = 0.25$.

R_{p2}	Plane	Cylindrical	Spherical
0.25	con^1	sd	sd
0.50	26.3960	con^2	sd
0.75	17.5791	con^3	con^8
1.00	13.1743	con^4	con^9
1.25	10.5197	con^5	con^{10}
1.50	8.7600	con^6	con^{11}
1.75	7.5052	con^7	con^{12}
2.00	6.5601	175.4784	con^{13}
2.25	5.8251	25.4558	con^{14}
2.50	5.2401	15.7289	con^{15}
2.75	4.7601	11.6196	con^{16}
3.00	4.3551	9.2849	con^{17}
3.25	4.0201	7.7552	con^{18}
3.50	3.7301	6.6701	con^{19}
3.75	3.4751	5.8601	44.0306
4.00	3.2551	5.2301	17.3091
4.25	3.0601	4.7201	11.7045
4.50	2.8900	4.3051	9.0350
4.75	2.7350	3.9551	7.4201
5.00	2.5950	3.6601	6.3301

where sd: shock decays, con^1: $\pi_0 = 5.0000$ at $t = 52.8258$, con^2: $\pi_0 = 0.39028$ at $t = 54.1351$, con^3: $\pi_0 = 1.08578$ at $t = 37.6291$, con^4: $\pi_0 = 1.78331$ at $t = 35.5953$, con^5: $\pi_0 = 2.48500$ at $t = 41.0972$, con^6: $\pi_0 = 3.19524$ at $t = 49.4776$, con^7: $\pi_0 = 3.92700$ at $t = 71.6121$, con:^{8}: $\pi_0 = 0.04452$ at $t = 185.5282$, con^9: $\pi_0 = 0.39282$ at $t = 29.5517$, con^{10}: $\pi_0 = 0.74192$ at $t = 24.1956$, con^{11}: $\pi_0 = 1.09205$ at $t = 19.3595$, con^{12}: $\pi_0 = 1.44375$ at $t = 19.8696$, con^{13}: $\pi_0 = 1.79763$ at $t = 21.0199$, con^{14}: $\pi_0 = 2.15451$ at $t = 20.2747$, con^{15}: $\pi_0 = 2.51575$ at $t = 20.7798$, con^{16}: $\pi_0 = 2.88369$ at $t = 24.0205$, con^{17}: $\pi_0 = 3.26236$ at $t = 30.9670$, con^{18}: $\pi_0 = 3.65961$ at $t = 35.0955$, con^{19}: $\pi_0 = 4.10130$ at $t = 43.9806$.

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