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Construction of 'Wachspress type' rational basis functions over rectangles

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Abstract. In the present paper, we have constructed rational basis functions of C^0 class over rectangular elements with wider choice of denominator function. This construction yields additional number of interior nodes. Hence, extra nodal points and the flexibility of denominator function suggest better approximation.

Keywords. Rational basis functions; C^0 approximation.

1. Introduction

The welknown wedge construction over rectangular elements of C^0 class in R^2 has been discussed by Ciarlet [1] (see also [4]) where $(n+1)^2$ monomials $x^iy^j, i, j \leq n$ are used as basis functions. Rational basis functions over convex quadrilaterals that are not rectangles were introduced by Wachspress [8]. This concept was used in [5] to achieve C^1-approximation of degree two and three over the triangular mesh of planar region (see also [7]).

Let P_n be the space of polynomials of degree n and Q_n be the space generated by the monomials then $P_n \subset Q_n \subset P_{2n}$ (see [1], p. 56). Here we construct rational basis functions of C^0 class over the same element which are ratios of polynomials of degree $2n$ and $n-1$, respectively. If Q_n^* denotes the space spanned by the rational basis functions then, $P_n \subset Q_n^* \subset P_{n+2}$ which is precisely a restricted class than that of P_{2n} for $n \geq 3$. The extra flexibility of the denominator polynomials and the additional number of interior nodes suggest favourable approximation properties of these functions (see [6]).

We first demonstrate the wedge construction and then show that our rational basis functions satisfy all the properties of wedges specified in [8]. In particular, we have also illustrated an example for $n = 3$ which gives the clear idea of our construction.

2. Notations and some preliminary results

The vertices a_i of a closed convex rectangle K in R^2 are labeled so that a_i and a_{i+1} are consecutive for $i = 1, 2, 3, 4$. For each subset $\mathcal{A}$ of $R^2, P_n^*(\mathcal{A})$ is the $\mathcal{A}$-restriction of the vector space of bivariate polynomial functions of degree $\leq n$ in each of the two variables. Let d_i be the straight line passing through the points a_i and a_{i+1} given by the equation $l_i(x, y) = l_i(\text{say}) = 0, i = 1, 2, 3, 4$. Let $a_{ij}, j = 1, 2 \ldots, n-1$ be any distinct points on the edge $[a_i, a_{i+1}]$. Suppose A and B are points on the lines d_1 and d_4 that do not coincide with a_1. Furthermore, suppose that the line through A and B given by $l(x, y) = l(\text{say}) = 0$ intersects d_3 and d_2 at points C and D, respectively (cf. figure 1). For arbitrary but fixed

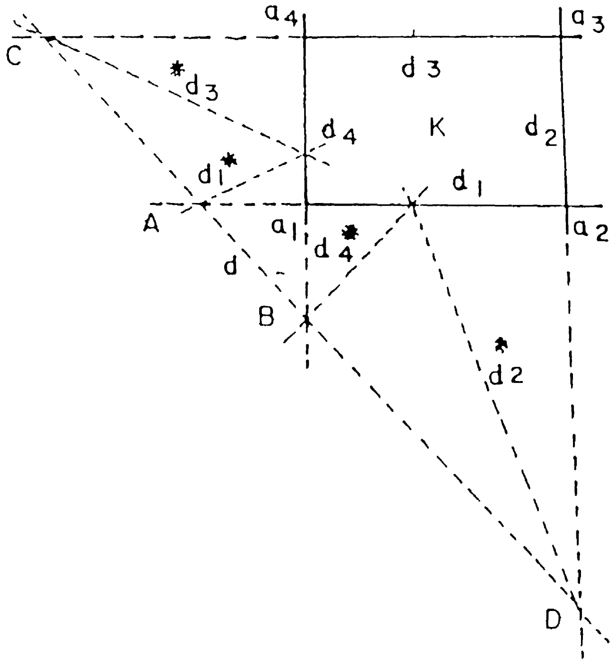

Figure 1. Rectangular element with rational basis.

$j = 1, 2, \ldots, n-1$, let d_1^*, d_2^*, d_3^* and d_4^* denote the straight lines passing through the respective pairs of points $(a_{4j}, A), (a_{1j}, D), (a_{4j}, C)$ and (a_{1j}, B), and let $l_i^*(x, y) = l_i^*(\text{say}) = 0$ be their corresponding defining equations. Let $\alpha_{ij}(x, y) = \alpha_{ij}(\text{say})$ be irreducible conics passing through the points a_{i-1j}, a_{ij}, p and q. The points p and q depend on i. For $i = 1, 2; i = 2, 3$; $i = 3, 4$; and $i = 4, 1$ we choose $p = A$; $q = D$; $p = C$ and $q = B$, respectively (see figure 1). We use the usual convention regarding the subscripts and throughout this paper, we shall use the notations and definitions given in [8] unless stated otherwise.

Remark 2.1. It is essential to construct α_{ij} irreducible. If it degenerates into the product of two linear forms the denominator function may become the factor of numerator and the basis functions will not be of the desired class. Since points A, B, C, D and the edge nodes are arbitrary, it is always possible to choose four points such that no three of them are collinear (see [2] and also [3]) in order to get α_{ij} irreducible.

For the demonstration of the wedge properties of the basis functions which would be defined in the next section, we need the following results of [8].

Lemma 2.1. *If three lines intersect at a point then the ratio of the linear forms which vanish on any two of these lines is constant on the third line.*

Lemma 2.2. *Let P_n, Q_m and L_1 have s distinct triple points then*

$$\frac{P_n(x, y)}{Q_m(x, y)} \equiv \frac{P^1_{n-s}(v)}{Q^1_{m-s}(v)} \bmod L_1,$$

where polynomials P^1 and Q^1 are derived from P_n and Q_m by elimination of x or y on line L_1.

Lemma 2.3. *Let Q be a polynomial in x and y which is a product of distinct irreducible factors and let P be a polynomial which is not identically zero. If* $P \equiv 0 \bmod Q$, *then* $Q(x, y)$ *must be a factor of* $P(x, y)$.

3. Construction of wedges and their properties

With the notations described in § 2, we define functions $W_i(x, y)$ and $W_{ij}(x, y)$ for $i = 1, 2, 3, 4$ and $j = 1, 2, \ldots, n-1$ for the vertices a_i and edge nodes a_{ij} respectively. For each $(x, y) \in K$ and $n \geq 1$

$$W_i(x, y) = \frac{H_i l_{i+1} l_{i+2} \prod_{j=1}^{n-1} \alpha_{ij}}{l^{n-1}}, \tag{3.1}$$

$$W_{ij}(x, y) = \frac{H_{ij} l_i^* l_{i+1} l_{i+2} l_{i+3} \prod_{k=1\, k\neq j}^{n-1} \alpha_{ik}}{l^{n-1}}, \tag{3.2}$$

where H_i and H_{ij} are suitable normalizing constants to ensure that $W_i(a_i) = 1$, $W_{ij}(a_{ij}) = 1$, $i = 1, 2, 3, 4$; $j = 1, 2, \ldots, n-1$.

With the applications of Lemmas 2.1–2.3, we now show that properties described in article 1.5 of [8] are satisfied by the functions (3.1) and (3.2). It is clear from the construction that $W_i(x, y)$ vanishes on the edges opposite of the vertex a_i and at all the nodes a_{ij} and at vertices a_k $(k = 1, 2, 3, 4)$ $k \neq i$. Similarly $W_{ij}(x, y)$ vanishes on the edges opposite of the nodes a_{ij} and at all the vertices a_i and at nodes a_{is} $(s = 1, 2, \ldots, n-1)$ $s \neq j$. Another important property may be formulated as follows.

Theorem 3.1. *The restriction of the wedges* $W_i(x, y)$ and $W_{ij}(x, y)$ *to their respective adjacent edges are polynomials of degree n.*

Proof. Edges d_{i-1} and d_i are adjacent to the vertex a_i and the edge d_i is adjacent to the edge nodes a_{ij}. Consider $R_s \in P_s^*(R^2)$ defined in § 2.

For convenience, we introduce the following notations,

$$\left.\begin{array}{c} \displaystyle\prod_{j=1}^{n-1} \alpha_{ij}(x, y) = R_{2n-2}, \prod_{k=1\, k\neq j}^{n-1} \alpha_{ik}(x, y) = R_{2n-4} \\ l^{n-1}(x, y) = R_{n-1}, l^{n-2}(x, y) = R_{n-2} \end{array}\right\}. \tag{3.3}$$

The algebraic curves R_{2n-2}, R_{n-1} defined in (3.3) and a straight line d_i intersect at a point and a point of intersection is a multiple point of R_{2n-2} and R_{n-1} with multiplicity $(n-1)$. In view of Lemma 2.2, we have

$$\frac{R_{2n-2}}{R_{n-1}} \equiv R_{n-1} \bmod d_i. \tag{3.4}$$

Since K is rectangle,

$$l_{i+2} \equiv C_1 \bmod d_i, \tag{3.5}$$

where C_1 is some constant. Relation (3.4) together with (3.5) gives the following:

$$W_i \equiv R_{n-1} l_{i+1} \mod d_i$$

or

$$W_i \equiv R_n \mod d_i.$$

By the similar argument, we may show that

$$W_i \equiv R_n \mod d_{i-1}.$$

For the other case, the algebraic curves R_{2n-4}, R_{n-2} given by (3.3) and a straight line d_i intersect at a point and a point of intersection is a multiple point of R_{2n-4}, R_{n-2} with multiplicity $(n-2)$. Again we get

$$\frac{R_{2n-4}}{R_{n-2}} \equiv R_{n-2} \mod d_i \tag{3.6}$$

when we appeal to lemma 2.2. The three straight lines d_i^*, d and d_i intersect at a point, in view of lemma 2.1, we have

$$\frac{l_i^*}{l} \equiv C_2 \mod d_i, \tag{3.7}$$

$$l_{i+2} \equiv C_3 \mod d_i (\because K \text{ is rectangle}), \tag{3.8}$$

where C_2 and C_3 are some constants. The following congruence relation follows from (3.6), (3.7) and (3.8).

$$W_{ij} \equiv R_{n-2} l_{i+1} l_{i+3} \mod d_i$$

or

$$W_{ij} \equiv R_n \mod d_i.$$

This completes the proof of Theorem 3.1.

DEFINITION 3.1

Let $Q_n^*(K)$ be the vector space generated by the function $W_i\,(i = 1, 2, 3, 4)$ and $W_{ij}\,(i = 1, 2, 3, 4;\ j = 1, 2, \ldots, n-1)$ defined by (3.1) and (3.2).

DEFINITION 3.2

Let Σ_n be the set of linearly independent linear forms defined over the space Q_n^* and is given by

$$\sum_n = (v \to v(a_i);\ \ v \to v(a_{ij}) : i = 1, 2, 3, 4;\ j = 1, 2, \ldots, n-1). \tag{3.9}$$

Denoting a finite element of our construction by $(K,\ Q_n^*, \Sigma_n)$, we are now set to prove our next results.

Theorem 3.2. *The finite element* (K, Q_n^*, Σ_n) *given by definitions* 3.1 *and* 3.2 *is of* C^0*-class.*

Proof. Let D^* be an open bounded subset polygon in R^2 and let D_h^* be a triangulation of D^* by rectangles K defined in § 2. Let V_h be a finite element space whose generic element

(K, Q_n^*, Σ_n) is given by (3.1), (3.2) and (3.9). Considering two adjacent rectangles K_i and K_j with common side $K' = [a_i, a_{i+1}]$ and $v \in V_h$, we have the following,

$$v|_{K_i} = p_i \in Q^*(K_i); \quad v|_{K_j} = p_j \in Q^*(K_j).$$

It follows from Theorem 3.1 that the restriction to K' of the basis functions are elements of $P_n^*(K')$ and we get

$$(p_i - p_j)|_{K'} \in P_n^*(K'). \tag{3.10}$$

Since $(n+1)$ data points are prescribed on the rectangular edges, in view of (3.10), we have

$$(p_i - p_j)|_{K'} = 0.$$

This is the desired result.

Theorem 3.3. *Assuming respectively the definitions* 3.1 *and* 3.2 *of* Q_n^* *and* Σ_n, (K, Q_n^*, Σ_n) *is a finite element.*

Proof. Excluding the interior nodes, it may be observed that

$$\dim Q_n^* = \dim \Sigma_n = 4n.$$

It is sufficient, if we prove that the set Σ_n is Q_n^*-unisolvent. In fact, we show that the functions W_i and W_{ij} defined by (3.1) and (3.2) are basis functions of Q_n^* with respect to Σ_n. For i, $k = 1, 2, 3, 4$,

$$W_i(a_k) = \delta_{ik} = [1 \text{ if } i = k; \ 0 \text{ otherwise}]. \tag{3.11}$$

For $i, r = 1, 2, 3, 4; \ j, s = 1, 2, \ldots, n-1$, we have

$$W_{ij}(a_{rs)} = \delta_{ir}\delta_{js} = [1 \text{ if } i = r, \ j = s; \ 0 \text{ otherwise}], \tag{3.12}$$

$$W_i(a_{rs}) = W_{ij}(a_k) = 0. \tag{3.13}$$

The linear independence of the functions defined in (3.1) and (3.2) follows by the relations (3.11), (3.12) and (3.13). Hence, these are the basis functions of the space Q_n^* and the proof is completed.

Remark 3.1. Since our basis functions consists of polynomials of degrees $(2n, n-1)$, we may introduce $(2n-2)(2n-3)/2$ interior nodes to get degree n-approximation. The basis function at each interior node can be defined as the product of the four boundary linear functions with the algebraic curve of degree $2n-4$ which passes through $(2n-2)$ $(2n-3)/2 - 1$ interior nodes in the numerator and the denominator given in that of (3.1) and (3.2). In fact, it may be observed that the set S^* formed by the wedges at the interior nodes together with the wedges defined over the boundary of the rectangle is linearly independent and thus forms the basis for Q_n^* and the theorem 3.2 holds with respect to S^* also.

DEFINITION 3.3

A rational approximation is said to have degree n if every polynomial of greatest degree n can be written as linear combination of the functions (3.1) and (3.2) (see [4], p. 83).

Theorem 3.4. *The finite element given by the definitions* 3.1 *and* 3.2 *is of degree n.*

Proof. Let $\Phi_n(x, y)$ be an element of $P_n^*(K)$ and consider the function $V_n(x, y)$ defined over K by

$$V_n(x, y) = \Phi_n(x, y) - [\Sigma_i \Phi_n(a_i) W_i(x, y) + \Sigma_{i,j} \Phi_n(a_{ij}) W_{ij}(x, y)]. \tag{3.14}$$

The function $V_n(x, y)$ must be of the form

$$V_n(x, y) = \frac{R_{2n}}{R_{n-1}}$$

and

$$R_{2n} = B_4 \beta_{2n-4},$$

where B_4 is the boundary curve which vanishes on the rectangle boundary and in view of remark 3.1, the algebraic curve β_{2n-4} is identically zero at all the interior nodes. Therefore, R_{2n} is the zero polynomial when we appeal to Lemma 2.3. We thus have $V_n(x, y) = 0$. It follows from (3.14) that $\Phi_n(x, y) \in Q_n^*$ and the proof is completed.

4. Importance and relevance of the new construction

(a) We have already mentioned in our remark 2.1 that the conic α_{ij} should be irreducible. α_{ij} may also be expressed as the product of two straight lines in which the denominator of the wedge functions is one of the factors. With this choice of α_{ij}, our construction reduces to the standard basis which span monomials (tensor product polynomials). Hence, monomial functions are special case of our rational basis functions.

(b) In case of monomials the number of interior nodes is $(n-1)^2$ to achieve degree n-approximation whereas in our construction, we get $(2n-2)(2n-3)/2$ interior nodes to achieve the same degree of approximation. Therefore, better approximations are expected with these additional $(n-1)(n-2), (n \geq 3)$ interior nodes.

5. Illustration

In this section, we discuss the following example for special choice of n.

Example 5.1. Consider $n = 3$. Refer figure 2.

Let K be the square with the vertices $a_1(0, 0)$, $a_2(1, 0)$, $a_3(1, 1)$, $a_4(0, 1)$ and side nodes $a_{11}(1/3, 0)$, $a_{12}(2/3, 0)$, $a_{21}(1, 1/3)$, $a_{22}(1, 2/3)$, $a_{31}(2/3, 1)$, $a_{32}(1/3, 1)$, $a_{41}(0, 2/3)$ and $a_{42}(0, 1/3)$. Let $b_m (m = 1, 2, \ldots, 6)$ are distinct interior nodes (no three of them are collinear) situated at the interior of K and are specified by their respective cartesian coordinates, $b_1(1/3, 1/3)$, $b_2(1/2, 1/3)$, $b_3(2/3, 1/2)$, $b_4(1/2, 1/2)$, $b_5(1/3, 2/3)$, and $b_6(1/9, 2/3)$. The linear forms of d_i and d_i^* for $i = 1, 2, 3, 4$ are given by

$$l_1 \cong y = 0; \quad l_2 \cong x - 1 = 0; \quad l_3 \cong y - 1 = 0,$$

$$l_4 \cong x = 0; \quad l_1^* \cong 3y - x - 1 = 0; \quad l_2^* \cong y + 3x - 1 = 0,$$

$$l_3^* \cong 3y + x - 1 = 0; \quad l_4^* \cong y - 3x + 1 = 0. \tag{5.1}$$

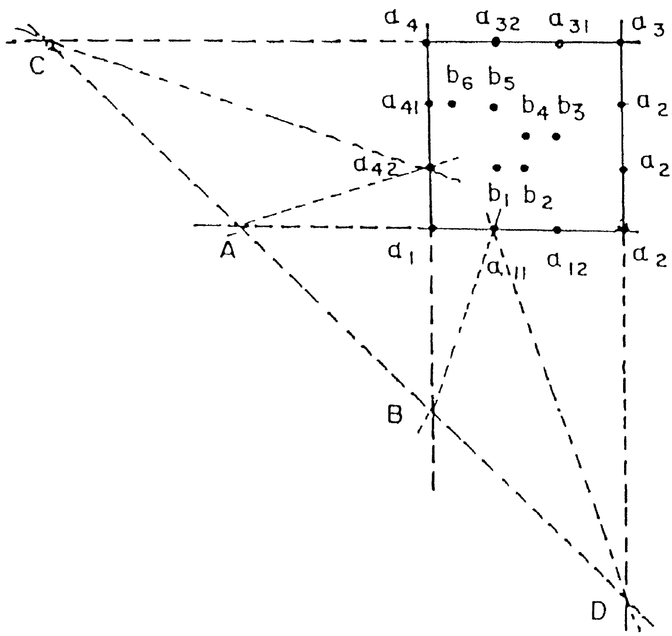

Figure 2. Case when $n = 3$.

If we choose $A(-1,0)$, $B(0,-1)$, the denominator function $l(x,y)$ reduces to

$$l(x,y) \cong x + y + 1 = 0. \tag{5.2}$$

Let $W_m^*(b_r)$ be the basis elements defined at the interior nodes b_r for $r = 1,2,\ldots,6$ such that

$$W_m^*(b_r) = \delta_{mr}; \quad m,r = 1,2,\ldots,6,$$

where δ_{mr} is usual Kronecker's delta. We determine conics α_{ij} which are defined in § 2. Suppose

$$\alpha_{ij}(x,\ y) \cong a_0x^2 + a_1y^2 + a_3x + a_4y + 1 = 0. \tag{5.3}$$

We get the following conics in view of (5.3)

$$\begin{aligned}
&\alpha_{11}(x,\ y) \cong 6x^2 + 3y^2 + 4x + y - 2 = 0,\\
&\alpha_{12}(x,\ y) \cong 3x^2 + 6y^2 + x + 4y - 2 = 0,\\
&\alpha_{21}(x,\ y) \cong 3x^2 - 6y^2 + 4x - 2y + 4 = 0,\\
&\alpha_{22}(x,\ y) \cong 6x^2 - 3y^2 + 2x - 4y - 4 = 0,\\
&\alpha_{31}(x,\ y) \cong 6x^2 + 3y^2 + 8x + 5y - 16 = 0,\\
&\alpha_{32}(x,\ y) \cong 3x^2 + 6y^2 + 5x + 8y - 16 = 0,\\
&\alpha_{41}(x,\ y) \cong 3x^2 - 6y^2 + 2x - 10y - 1 = 0,\\
&\alpha_{42}(x,\ y) \cong 6x^2 - 3y^2 + 10x - 2y + 1 = 0.
\end{aligned} \tag{5.4}$$

We now suppose that β_m be the irreducible conic passing through the points b_r (no three of b'_r s are collinear) such that $m \neq r$ for $m, r = 1, 2, \ldots, 6$. Let

$$\beta_m(x,\, y) \cong a_0x^2 + a_1y^2 + a_2xy + a_3x + a_4y + 1 = 0. \tag{5.5}$$

In view of (5.5), we obtain the following conics:

$$\begin{aligned}
&\beta_1(x, y) \cong 36x^2 + 42y^2 - 120x - 113y + 156xy + 58 = 0,\\
&\beta_2(x, y) \cong 9x^2 - 18y^2 - 30x + 5y + 39xy + 5 = 0,\\
&\beta_3(x, y) \cong 18x^2 + 21y^2 - 22x - 28y + 21xy + 10 = 0,\\
&\beta_4(x, y) \cong 54x^2 + 234y^2 - 66x - 255y + 63xy + 68 = 0,\\
&\beta_5(x, y) \cong -54x^2 + 336y^2 + 9x - 334y + 108xy + 65 = 0,\\
&\beta_6(x, y) \cong -6x^2 + 12y^2 + x - 16y + 12xy + 3 = 0.
\end{aligned} \tag{5.6}$$

In view of (3.1), (3.2) and remark 3.1, we are now set to write the basis functions when we appeal to relations (5.1), (5.2), (5.4) and (5.6).

$$\begin{aligned}
W_1(a_1) &= \frac{H_1l_2l_3\alpha_{11}\alpha_{12}}{l^2}, \quad W_{11}(a_{11}) = \frac{H_{11}l_1^*l_2l_3l_4\alpha_{12}}{l^2},\\
W_{12}(a_{12}) &= \frac{H_{12}l_1^*l_2l_3l_4\alpha_{11}}{l^2}, \quad W_2(a_2) = \frac{H_2l_3l_4\alpha_{21}\alpha_{22}}{l^2},\\
W_{21}(a_{21}) &= \frac{H_{21}l_1l_2^*l_3l_4\alpha_{22}}{l^2}, \quad W_{22}(a_{22}) = \frac{H_{22}l_1l_2^*l_3l_4\alpha_{21}}{l^2},\\
W_3(a_3) &= \frac{H_3l_1l_4\alpha_{31}\alpha_{32}}{l^2}, \quad W_{31}(a_{31}) = \frac{H_{31}l_1l_2l_3^*l_4\alpha_{32}}{l^2},\\
W_{32}(a_{32}) &= \frac{H_{32}l_1l_2l_3^*l_4\alpha_{31}}{l^2}, \quad W_4(a_4) = \frac{H_4l_1l_2\alpha_{41}\alpha_{42}}{l^2},\\
W_{41}(a_{41}) &= \frac{H_{41}l_1l_2l_3l_4^*\alpha_{42}}{l^2}, \quad W_{42}(a_{42}) = \frac{H_{42}l_1l_2l_3l_4^*\alpha_{41}}{l^2}.
\end{aligned}$$

For convenience, let us denote by $B_4 = l_1l_2l_3l_4$, then we have

$$\begin{aligned}
W_1^*(b_1) &= \frac{H_1^*B_4\beta_1}{l^2}, \quad W_2^*(b_2) = \frac{H_2^*B_4\beta_2}{l^2}, \quad W_3^*(b_3) = \frac{H_3^*B_4\beta_3}{l^2},\\
W_4^*(b_4) &= \frac{H_4^*B_4\beta_4}{l^2}, \quad W_5^*(b_5) = \frac{H_5^*B_4\beta_5}{l^2}, \quad W_6^*(b_6) = \frac{H_6^*B_4\beta_6}{l^2},
\end{aligned}$$

where H_r^* $(r = 1, 2, \ldots, 6)$ are normalizing constants. It may be verified easily that the properties for wedges specified in article 1.5 of [8] hold for the above construction of the wedges.

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