

On (N, p, q) summability factors of infinite series

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Abstract. In this paper a necessary and sufficient condition has been obtained for $\Sigma a_n \epsilon_n$ to be summable $|\bar{N}, q|$ whenever Σa_n is bounded (N, p, q) .

Keywords. Summability factors; conservative matrix.

1. Introduction

Let Σa_n be a given infinite series with s_n for its n th partial sum. Let $\{t_n\}$ denote the sequence of (N, p, q) mean of the sequence $\{s_n\}$. The (N, p, q) transform of $s_n = \Sigma_{\nu=0}^n a_\nu$ is defined as follows:

$$t_n = \frac{1}{r_n} \sum_{\nu=0}^n p_{n-\nu} q_\nu s_\nu, \quad (1)$$

where

$$r_n = p_0 q_n + \cdots + p_n q_0 (\neq 0) \\ p_{-1} = q_{-1} = r_{-1} = 0.$$

Necessary and sufficient conditions for the (N, p, q) method to be regular, that is for $s_n \rightarrow s$ to imply $s_n \rightarrow s(N, p, q)$ are

- (i) $p_{n-\nu} q_\nu / r_n \rightarrow 0$ for each integer $\nu \geq 0$ as $n \rightarrow \infty$ and
- (ii) $\sum_{\nu=0}^n |p_{n-\nu} q_\nu| < H |r_n|$, where H is a positive number independent of n .

Let $\{T_n\}$ denote the sequence of (\bar{N}, q) mean of the sequence $\{s_n\}$ defined by

$$T_n = \frac{1}{Q_n} \sum_{\nu=0}^n q_\nu s_\nu; \quad (Q_n \neq 0) \quad (2)$$

where $Q_n = \sum_{\nu=0}^n q_\nu \rightarrow \infty$, as $n \rightarrow \infty$ ($Q_{-i} = q_{-i} = 0, i \geq 1$).

We define the sequence of constants $\{c_n\}$ formally by means of the identity

$$\left(\sum_{n=0}^{\infty} p_n x^n \right)^{-1} = \sum_{n=0}^{\infty} c_n x^n, \quad c_{-i} = 0, \quad i \geq 1. \quad (3)$$

We also write $c_n^{(1)} = c_0 + c_1 + \cdots + c_n$.

We denote by \mathcal{M} , the class of sequences $\{p_n\}$ for which the following holds:

$$p_n > 0, \quad \frac{p_{n+1}}{p_n} \leq \frac{p_{n+2}}{p_{n+1}} \leq 1 \quad (n = 0, 1, \dots). \quad (4)$$

Let $\{p_n\}$ and $\{q_n\}$ be positive sequences. A series Σa_n is said to be bounded (N, p, q) or $\Sigma a_n = O(1)(N, p, q)$ if

$$\sum_{\nu=1}^n p_{n-\nu} q_{\nu} s_{\nu} = O(r_n), \quad \text{as } n \rightarrow \infty. \quad (5)$$

If X and Y are any two methods of summability, we say (ϵ_n) belongs to the class $[X, Y]$, if $\Sigma a_n \epsilon_n$ is summable $- Y$ whenever Σa_n is summable X .

Recently Mishra [7], Sarigol and Bor [8] and Sulaiman [9] have obtained summability factor theorems of the type $[[\bar{N}, p_n|_k, |\bar{N}, q_n|_k], [|\bar{N}, p_n|, |\bar{N}, q_n|_k], [|\bar{N}, p_n|_k, |\bar{N}, q_n|]]$.

In 1966 Das [2] has proved the following theorem:

Theorem A. *Let $\{p_n\} \in \mathcal{M}$, $q_n \geq 0$. Then if Σa_n is summable $|N, p, q|$, it is summable $|\bar{N}, q|$.*

It is therefore, natural to find a summability factor ϵ_n so that $\Sigma a_n \epsilon_n$ is summable $|\bar{N}, q|$ whenever Σa_n is bounded (N, p, q) .

Mazur and Orlicz [5] stated that, if a conservative (i.e. convergence preserving) matrix sums a bounded nonconvergent sequence, then it must sum an unbounded sequence. Zeller [10] obtained a proof of this theorem as a consequence of his study of the summability of slowly oscillating sequences whereas the proof of Mazur and Orlicz [6] was functional analytic, based on rather deep topological properties of FK-spaces. A simple direct proof of this theorem was also given by Fridy [3] which used only the well known Silverman–Toeplitz conditions for regularity.

In view of this remark we state and prove the following summability factor theorem.

Theorem 1. *Let $\{p_n\} \in \mathcal{M}$, $q_0 > 0$, $q_n \geq 0$ and let $\{q_n\}$ be monotonic non-increasing sequence for $n \geq 0$. The necessary and sufficient condition that $\Sigma a_n \epsilon_n$ should be summable $|\bar{N}, q|$, whenever*

$$\sum a_n = O(1)(N, p, q), \quad (6)$$

$$\sum_{n=0}^{\infty} \frac{q_n}{Q_n} |\epsilon_n| < \infty, \quad (7)$$

$$\sum_{n=0}^{\infty} |\Delta \epsilon_n| < \infty, \quad (8)$$

$$\sum_{n=0}^{\infty} \frac{Q_{n+1}}{q_{n+1}} |\Delta^2 \epsilon_n| < \infty, \quad (9)$$

is that

$$\sum_{n=1}^{\infty} \frac{q_n}{Q_n} |s_n| |\epsilon_n| < \infty. \quad (10)$$

Our theorem generalizes and unifies several known results of Mazhar [4], Daniel [1] and others.

2. Lemmas

We need the following lemmas for the proof of our theorem.

Lemma 1 [1]. Let $\{q_n\}$ be positive and monotonic non-increasing sequence. If $\{\epsilon_n\}$ is such that

- (i) $\Delta\epsilon_n = o(1)$, as $n \rightarrow \infty$, and
- (ii) $\sum \frac{Q_n}{q_{n+1}} |\Delta^2\epsilon_n| < \infty$,

then

$$\sum \frac{Q_n \Delta q_n}{q_n q_{n+1}} |\Delta\epsilon_n| < \infty.$$

Remark. This lemma holds only if $\{q_n\}$ is monotonic non-increasing.

If $\{q_n\}$ is not monotonic non-increasing then conclusion of the lemma may not be true.

Lemma 2 [2]. Let $\{p_n\} \in \mathcal{M}$. Then

- (iii) $c_0 > 0$, $c_n \leq 0$ ($n = 1, 2, \dots$),
- (iv) $\sum_{n=0}^{\infty} c_n x^n$ is absolutely convergent for $|x| \leq 1$,
- (v) $\sum_{n=0}^{\infty} c_n > 0$,

except when $\sum_{n=0}^{\infty} p_n = \infty$, in which case

- (vi) $\sum_{n=0}^{\infty} c_n = 0$.

Lemma 3 [2]. If

$$t_n = \frac{1}{r_n} \sum_{\nu=0}^n p_{n-\nu} q_{\nu} s_{\nu}.$$

then

$$s_n = \frac{1}{q_n} \sum_{\nu=0}^n c_{n-\nu} r_{\nu} t_{\nu}.$$

Lemma 4 [2].

$$\sum_{\mu=0}^n c_{n-\mu}^{(1)} r_{\mu} = Q_n,$$

where c_n, r_n and Q_n are defined as above.

3. Proof of theorem 1

Let

$$t_n = \frac{1}{r_n} \sum_{\nu=0}^n p_{n-\nu} q_{\nu} s_{\nu},$$

and

$$\begin{aligned} T_n &= \frac{1}{Q_n} \sum_{\nu=0}^n q_\nu \sum_{r=0}^{\nu} a_r \epsilon_r \\ &= \frac{1}{Q_n} \sum_{\nu=0}^n (Q_n - Q_{\nu-1}) a_\nu \epsilon_\nu. \end{aligned}$$

Then for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{q_n}{Q_n Q_{n-1}} \sum_{\nu=0}^n Q_{\nu-1} a_\nu \epsilon_\nu.$$

Using Abel's transformation, we get

$$\begin{aligned} T_n - T_{n-1} &= \frac{q_n}{Q_n Q_{n-1}} \left[\sum_{\nu=0}^{n-1} s_\nu \Delta(Q_{\nu-1} \epsilon_\nu) + s_n \epsilon_n Q_{n-1} \right] \\ &= -\frac{q_n}{Q_n Q_{n-1}} \sum_{\nu=0}^{n-1} q_\nu s_\nu \epsilon_\nu + \frac{q_n}{Q_n Q_{n-1}} \sum_{\nu=0}^{n-1} Q_\nu s_\nu \Delta \epsilon_\nu + \frac{q_n}{Q_n} s_n \epsilon_n. \end{aligned}$$

Let

$$T_n - T_{n-1} = \sum_1 + \sum_2 + \sum_3, \quad \text{say.} \quad (11)$$

The theorem is proved if we show that $\sum |\sum_1|$ and $\sum |\sum_2|$ are convergent.

Now

$$\begin{aligned} \sum_1 &= -\frac{q_n}{Q_n Q_{n-1}} \sum_{\nu=0}^{n-1} q_\nu \epsilon_\nu s_\nu \\ &= -\frac{q_n}{Q_n Q_{n-1}} \sum_{\nu=0}^{n-1} q_\nu \epsilon_\nu \frac{1}{q_\nu} \sum_{\mu=0}^{\nu} c_{\nu-\mu} r_\mu t_\mu \\ &= -\frac{q_n}{Q_n Q_{n-1}} \sum_{\nu=0}^{n-1} \epsilon_\nu \sum_{\mu=0}^{\nu} c_{\nu-\mu} r_\mu t_\mu \\ &= -\frac{q_n}{Q_n Q_{n-1}} \sum_{\mu=0}^{n-1} r_\mu t_\mu \sum_{\nu=\mu}^{n-1} c_{\nu-\mu} \epsilon_\nu \\ &= -\frac{q_n}{Q_n Q_{n-1}} \sum_{\mu=0}^{n-1} r_\mu t_\mu \left[\sum_{\nu=\mu}^{n-1} \left(\sum_{k=0}^{\nu} c_{k-\mu} \right) \Delta \epsilon_\nu + \epsilon_n \sum_{k=0}^{n-1} c_{k-\mu} - \sum_{k=0}^{\mu-1} c_{k-\mu} \epsilon_\mu \right] \\ &= -\frac{q_n}{Q_n Q_{n-1}} \sum_{\mu=0}^{n-1} r_\mu t_\mu \left[\sum_{\nu=\mu}^{n-1} c_{\nu-\mu}^{(1)} \Delta \epsilon_\nu + \epsilon_n c_{n-1-\mu}^{(1)} \right] \\ &= -\frac{q_n}{Q_n Q_{n-1}} \sum_{\mu=0}^{n-1} r_\mu t_\mu \sum_{\nu=\mu}^{n-1} c_{\nu-\mu}^{(1)} \Delta \epsilon_\nu - \frac{q_n}{Q_n Q_{n-1}} \sum_{\mu=0}^{n-1} \epsilon_n c_{n-1-\mu}^{(1)} r_\mu t_\mu \\ &= \sum_{11} + \sum_{12}, \quad \text{say.} \end{aligned}$$

Then as $t_n = O(1)$, using lemmas 2, 3 and 4,

$$\begin{aligned}
 \sum_{n=1}^{\infty} \left| \sum_{11} \right| &\leq \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{\mu=0}^{n-1} r_{\mu} |t_{\mu}| \sum_{\nu=\mu}^{n-1} c_{\nu-\mu}^{(1)} |\Delta \epsilon_{\nu}|, \text{ since } c_{\nu-\mu}^{(1)} > 0 \text{ by Lemma 2.} \\
 &\leq K \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{\nu=0}^{n-1} |\Delta \epsilon_{\nu}| \sum_{\mu=0}^{\nu} c_{\nu-\mu}^{(1)} r_{\mu} \\
 &= K \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{\nu=0}^{n-1} |\Delta \epsilon_{\nu}| Q_{\nu} \\
 &= K \sum_{\nu=0}^{\infty} |\Delta \epsilon_{\nu}| Q_{\nu} \sum_{n=\nu+1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \\
 &= K \sum_{\nu=0}^{\infty} |\Delta \epsilon_{\nu}| < \infty.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \sum_{n=1}^{\infty} \left| \sum_{12} \right| &\leq K \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} |\epsilon_n| \sum_{\mu=0}^{n-1} c_{n-1-\mu}^{(1)} r_{\mu} \\
 &= K \sum_{n=1}^{\infty} \frac{q_n}{Q_n} |\epsilon_n| < \infty.
 \end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} \left| \sum_1 \right| < \infty.$$

Again,

$$\begin{aligned}
 \sum_2 &= \frac{q_n}{Q_n Q_{n-1}} \sum_{\nu=0}^{n-1} Q_{\nu} \Delta \epsilon_{\nu} \frac{1}{q_{\nu}} \sum_{\mu=0}^{\nu} c_{\nu-\mu} r_{\mu} t_{\mu} \\
 &= \frac{q_n}{Q_n Q_{n-1}} \sum_{\mu=0}^{n-1} r_{\mu} t_{\mu} \sum_{\nu=\mu}^{n-1} \frac{Q_{\nu}}{q_{\nu}} \Delta \epsilon_{\nu} c_{\nu-\mu} = \frac{q_n}{Q_n Q_{n-1}} \sum_{\mu=0}^{n-1} r_{\mu} t_{\mu} L_1, \quad \text{say,}
 \end{aligned}$$

where

$$\begin{aligned}
 L_1 &= \sum_{\nu=\mu}^{n-1} \frac{Q_{\nu}}{q_{\nu}} \Delta \epsilon_{\nu} c_{\nu-\mu} \\
 &= \sum_{\nu=\mu}^{n-1} \left(\sum_{k=0}^{\nu} c_{k-\mu} \right) \Delta \left(\frac{Q_{\nu}}{q_{\nu}} \Delta \epsilon_{\nu} \right) + \frac{Q_n}{q_n} \Delta \epsilon_n \sum_{k=0}^{n-1} c_{k-\mu} \\
 &= \sum_{\nu=\mu}^{n-1} \left(\sum_{k=\mu}^{\nu} c_{k-\mu} \right) \left[\Delta \left(\frac{Q_{\nu}}{q_{\nu}} \right) \Delta \epsilon_{\nu} + \frac{Q_{\nu+1}}{q_{\nu+1}} \Delta^2 \epsilon_{\nu} \right] + \frac{Q_n}{q_n} \Delta \epsilon_n \sum_{k=\mu}^{n-1} c_{k-\mu} \\
 &= \sum_{\nu=\mu}^{n-1} \left(\sum_{k=\mu}^{\nu} c_{k-\mu} \right) \left[-\frac{q_{\nu+1}}{q_{\nu}} \Delta \epsilon_{\nu} + Q_{\nu+1} \Delta \left(\frac{1}{q_{\nu}} \right) \Delta \epsilon_{\nu} + \frac{Q_{\nu+1}}{q_{\nu+1}} \Delta^2 \epsilon_{\nu} \right] + \frac{Q_n}{q_n} \Delta \epsilon_n \sum_{k=\mu}^{n-1} c_{k-\mu}
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{\nu=\mu}^{n-1} \left[-c_{\nu-\mu}^{(1)} \frac{q_{\nu+1}}{q_\nu} \Delta \epsilon_\nu + c_{\nu-\mu}^{(1)} Q_{\nu+1} \Delta \left(\frac{1}{q_\nu} \right) \Delta \epsilon_\nu + c_{\nu-\mu}^{(1)} \right. \\
&\quad \left. \times \frac{Q_{\nu+1}}{q_{\nu+1}} \Delta^2 \epsilon_\nu \right] + \frac{Q_n}{q_n} \Delta \epsilon_n c_{n-1-\mu}^{(1)}.
\end{aligned}$$

So

$$\begin{aligned}
\sum_2 &= -\frac{q_n}{Q_n Q_{n-1}} \sum_{\mu=0}^{n-1} r_\mu t_\mu \sum_{\nu=\mu}^{n-1} c_{\nu-\mu}^{(1)} \frac{q_{\nu+1}}{q_\nu} \Delta \epsilon_\nu \\
&\quad + \frac{q_n}{Q_n Q_{n-1}} \sum_{\mu=0}^{n-1} r_\mu t_\mu \sum_{\nu=\mu}^{n-1} c_{\nu-\mu}^{(1)} Q_{\nu+1} \Delta \left(\frac{1}{q_\nu} \right) \Delta \epsilon_\nu \\
&\quad + \frac{q_n}{Q_n Q_{n-1}} \sum_{\mu=0}^{n-1} r_\mu t_\mu \sum_{\nu=\mu}^{n-1} c_{\nu-\mu}^{(1)} \frac{Q_{\nu+1}}{q_{\nu+1}} \Delta^2 \epsilon_\nu + \frac{q_n}{Q_n Q_{n-1}} \sum_{\mu=0}^{n-1} r_\mu t_\mu c_{n-1-\mu}^{(1)} \frac{Q_n}{q_n} \Delta \epsilon_n \\
&= \sum_{21} + \sum_{22} + \sum_{23} + \sum_{24}, \quad \text{say.}
\end{aligned}$$

Therefore to show that

$$\sum_{n=1}^{\infty} \left| \sum_2 \right| < \infty$$

it is enough to show that

$$\sum_{n=1}^{\infty} \left| \sum_{2i} \right| < \infty, \quad i = 1, 2, 3, 4.$$

Now as $c_k^{(1)} > 0$ for $k > 0$ and as $t_n = O(1)$

$$\begin{aligned}
\sum_{n=1}^{\infty} \left| \sum_{21} \right| &\leq \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{\mu=0}^{n-1} r_\mu |t_\mu| \sum_{\nu=\mu}^{n-1} c_{\nu-\mu}^{(1)} \frac{q_{\nu+1}}{q_\nu} |\Delta \epsilon_\nu| \\
&\leq K \left[\sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{\mu=0}^{n-1} r_\mu \sum_{\nu=\mu}^{n-1} c_{\nu-\mu}^{(1)} \frac{q_{\nu+1}}{q_\nu} |\Delta \epsilon_\nu| \right] \\
&= K \left[\sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{\nu=0}^{n-1} \frac{q_{\nu+1}}{q_\nu} |\Delta \epsilon_\nu| \sum_{\mu=0}^{\nu} r_\mu c_{\nu-\mu}^{(1)} \right] \\
&= K \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{\nu=0}^{n-1} \frac{q_{\nu+1}}{q_\nu} |\Delta \epsilon_\nu| Q_\nu \\
&= K \sum_{\nu=0}^{\infty} \frac{q_{\nu+1}}{q_\nu} Q_\nu |\Delta \epsilon_\nu| \sum_{n=\nu+1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \\
&= K \sum_{\nu=0}^{\infty} \frac{q_{\nu+1}}{q_\nu} Q_\nu |\Delta \epsilon_\nu| \frac{1}{Q_\nu} = K \sum_{\nu=0}^{\infty} \frac{q_{\nu+1}}{q_\nu} |\Delta \epsilon_\nu| \\
&\leq K \sum_{\nu=0}^{\infty} |\Delta \epsilon_\nu| < \infty, \quad \text{as } \{q_n\} \text{ is non-increasing.}
\end{aligned}$$

Note that K is a positive constant not necessarily same at each occurrence.

Similarly

$$\begin{aligned}
 \sum_{n=1}^{\infty} \left| \sum_{22} \right| &\leq \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{\mu=0}^{n-1} r_{\mu} |t_{\mu}| \sum_{\nu=\mu}^{n-1} c_{\nu-\mu}^{(1)} \frac{Q_{\nu+1} \Delta q_{\nu}}{q_{\nu} q_{\nu+1}} |\Delta \epsilon_{\nu}| \\
 &\leq K \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{\mu=0}^{n-1} r_{\mu} \sum_{\nu=\mu}^{n-1} c_{\nu-\mu}^{(1)} \frac{Q_{\nu+1} \Delta q_{\nu}}{q_{\nu} q_{\nu+1}} |\Delta \epsilon_{\nu}| \\
 &= K \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{\nu=0}^{n-1} \frac{Q_{\nu+1} \Delta q_{\nu}}{q_{\nu} q_{\nu+1}} |\Delta \epsilon_{\nu}| \sum_{\mu=0}^{\nu} r_{\mu} c_{\nu-\mu}^{(1)} \\
 &= K \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{\nu=0}^{n-1} \frac{Q_{\nu+1} Q_{\nu} \Delta q_{\nu}}{q_{\nu} q_{\nu+1}} |\Delta \epsilon_{\nu}| \\
 &= K \sum_{\nu=0}^{\infty} \frac{Q_{\nu+1} Q_{\nu} \Delta q_{\nu}}{q_{\nu} q_{\nu+1}} |\Delta \epsilon_{\nu}| \sum_{n=\nu+1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \\
 &= K \sum_{\nu=0}^{\infty} \frac{Q_{\nu+1} \Delta q_{\nu}}{q_{\nu} q_{\nu+1}} |\Delta \epsilon_{\nu}| < \infty, \text{ by Lemma 1,} \\
 \sum_{n=1}^{\infty} \left| \sum_{23} \right| &\leq K \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{\mu=0}^{n-1} r_{\mu} \sum_{\nu=\mu}^{n-1} c_{\nu-\mu}^{(1)} \frac{Q_{\nu+1}}{q_{\nu+1}} |\Delta^2 \epsilon_{\nu}| \\
 &= K \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{\nu=0}^{n-1} \frac{Q_{\nu+1}}{q_{\nu+1}} |\Delta^2 \epsilon_{\nu}| \sum_{\mu=0}^{\nu} r_{\mu} c_{\nu-\mu}^{(1)} \\
 &= K \sum_{\nu=0}^{\infty} \frac{Q_{\nu+1} Q_{\nu}}{q_{\nu+1}} |\Delta^2 \epsilon_{\nu}| \times \frac{1}{Q_{\nu}} \\
 &= K \sum_{\nu=0}^{\infty} \frac{Q_{\nu+1}}{q_{\nu+1}} |\Delta^2 \epsilon_{\nu}| < \infty
 \end{aligned}$$

and

$$\begin{aligned}
 \sum_{n=1}^{\infty} \left| \sum_{24} \right| &\leq K \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} |\Delta \epsilon_n| \frac{Q_n}{q_n} \sum_{\mu=0}^{n-1} r_{\mu} c_{n-1-\mu}^{(1)} \\
 &= K \sum_{n=1}^{\infty} \frac{1}{Q_{n-1}} |\Delta \epsilon_n| Q_{n-1} \\
 &= K \sum_{n=1}^{\infty} |\Delta \epsilon_n| < \infty.
 \end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} \left| \sum_2 \right| < \infty$$

and the proof of the theorem is completed.

Theorem 2. Let $\{p_n\} \in \mathcal{M}$, $q_0 > 0$, $q_n \geq 0$ and let $\{q_n\}$ be monotonic non-increasing sequence for $n \geq 0$. The necessary and sufficient condition that $\Sigma a_n \epsilon_n$ should be

summable $|\bar{N}, q|$ whenever

$$\sum a_n = O(\mu_n)(N, p, q), \quad (12)$$

where $\{\mu_n\}$ is positive and monotonic non-decreasing and $\{\epsilon_n\}$ is such that

$$\sum_{n=0}^{\infty} \frac{q_n}{Q_n} |\epsilon_n| \mu_n < \infty, \quad (13)$$

$$\sum_{n=0}^{\infty} |\Delta \epsilon_n| \mu_n < \infty, \quad (14)$$

$$\sum_{n=0}^{\infty} \frac{Q_{n+1}}{q_{n+1}} |\Delta^2 \epsilon_n| \mu_n < \infty, \quad (15)$$

is that

$$\sum_{n=1}^{\infty} \frac{q_n}{Q_n} |s_n| |\epsilon_n| < \infty. \quad (16)$$

The proof of theorem 2 is similar to that of theorem 1.

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