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Transformation semigroup compactifications and norm continuity of weakly almost periodic functions

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Abstract. We prove if there exists a separately continuous action of a topologically right simple semitopological semigroup S on a topological space X and if S acts topologically surjective on X then each weakly almost periodic function on X, with respect to S, is left norm continuous.

Keywords. Transformation semigroup; left norm continuous; weakly almost periodic.

Throughout the paper, (S, X) denotes a transformation semigroup, i.e. a semigroup S, a set X, and a map (called the action of S on X) $(s, x) \to sx$: $S \times X \to X$ such that $s(tx) = (st)x$ for all $s, t \in S$ and $x \in X$. If T is a sub-semigroup of S and Y is a T-invariant subset of X (i.e. $Y \supseteq TY = \{ty : t \in T, y \in Y\}$), then we say (T, Y) is a *sub-transformation semigroup* of (S, X). When S and X are topological spaces, we say (T, Y) is dense in (S, X) if T is dense in S and Y is dense in X.

For notation and terminology we shall follow Berglund *et al* [1], as far as possible. For a topological space $X, \mathcal{C}(X)$ is the C^*-algebra of bounded continuous complex-valued functions on X with supremum norm. For a C^*-subalgebra $\mathcal{F}$ of $\mathcal{C}(X), X^{\mathcal{F}}$ denotes the spectrum of $\mathcal{F}$(= the set of all multiplicative means on $\mathcal{F}$) which is weak *-compact in the topological dual $\mathcal{F}^*$ of $\mathcal{F}$. The evaluation map $\epsilon : X \to X^{\mathcal{F}}$, with a weak *-dense image, is defined by $\epsilon(x)f = f(x)$ $(x \in X, f \in \mathcal{F})$.

For $s, t \in S$ and $x \in X$, we define the translation maps $\lambda_s : S \to S, \rho_s : S \to S$, $\dot{\lambda}_s : X \to X$ and $\dot{\rho}_x : S \to X$ by $\lambda_s(t) = st = \rho_t(s)$ and $\dot{\lambda}_s(x) = sx = \dot{\rho}_x(s)$.

When S and X are topological spaces, we say (S, X) is *left topological* if λ_s and $\dot{\lambda}_{\mathbf{s}}$ are continuous for all $s \in S$, *right topological* if ρ_s and $\dot{\rho}_{\mathbf{x}}$ are continuous for all $s \in S$ and $x \in X$, *semitopological* if it is both left and right topological, *topological* if the multiplication in S and the action of S on X are continuous. If there is a separately (resp. jointly) continuous action of a semitopological (resp. topological) group S with identity e on a topological space X and $ex = x$ for all $x \in X$, (S, X) is called a semitopological (resp. topological) transformation group. (S, X) is said to be compact (resp. Hausdorff) if so are S and X.

For $s \in S$ and $x \in X$, we consider the translation operators $L_s = \lambda_s^*$ and $R_s = \rho_s^*$ on $\mathcal{C}(S)$; $\dot{\mathrm{L}}_s = (\dot{\lambda}_s)^*$ and $\dot{\mathrm{R}}_x = (\dot{\rho}_x)^*$ on $\mathcal{C}(X)$, which are bounded with norms ≤ 1. Trivially $\dot{\mathrm{L}}_{st} = \dot{\mathrm{L}}_t \dot{\mathrm{L}}_s, \dot{\mathrm{R}}_{sx} = R_s \dot{\mathrm{R}}_x, L_s \dot{\mathrm{R}}_x = \dot{\mathrm{R}}_x \dot{\mathrm{L}}_s$.

A subset $\mathcal{F}$ of $\mathcal{C}(S)$ (resp. $\mathcal{H}$ of $\mathcal{C}(X)$) is called *translation invariant* if $L_s\mathcal{F} \cup R_s\mathcal{F} \subseteq \mathcal{F}$ (resp. $\dot{\mathrm{L}}_{\mathbf{s}}\mathcal{H} \subseteq \mathcal{H}$) for all $s \in S$.

For a semitopological (S,X), we define

$$\mathcal{AP}(X) = \{f \in \mathcal{C}(X) : \{\dot{\mathrm{L}}_s f : s \in S\} \text{ is norm relatively compact in } \mathcal{C}(X)\},$$
$$\mathcal{WAP}(X) = \{f \in \mathcal{C}(X) : \{\dot{\mathrm{L}}_s f : s \in S\} \text{ is weak relatively compact in } \mathcal{C}(X)\},$$
$$\mathcal{LC}(X) = \{f \in \mathcal{C}(X) : \text{the map } s \to \dot{\mathrm{L}}_s f : S \to \mathcal{C}(X) \text{ is norm continuous}\},$$
$$\mathcal{RC}(X) = \{f \in \mathcal{C}(X) : \text{the map } x \to \dot{\mathrm{R}}_x f : X \to \mathcal{C}(S) \text{ is norm continuous}\}.$$

All of these function spaces are translation invariant C^*-subalgebras of $\mathcal{C}(X)$ containing the constant functions. Clearly $\mathcal{AP}(X) \subseteq \mathcal{WAP}(X)$.

The following collects some basic facts about these spaces.

PROPOSITION 1.1

Let (S,X) be semitopological. Then

(i) $\mathcal{AP}(X) \subseteq \mathcal{LC}(X) \cap \mathcal{RC}(X)$.
(ii) *If S (resp. X) is compact then $\mathcal{AP}(X) = \mathcal{LC}(X)$ (resp. $\mathcal{RC}(X)$).*
(iii) *If (S, X) is compact then, $\mathcal{AP}(X) = \mathcal{LC}(X) = \mathcal{RC}(X)$ and $\mathcal{WAP}(X) = \mathcal{C}(X)$.*
(iv) *If (S, X) is compact and topological then, $\mathcal{AP}(X) = \mathcal{LC}(X) = \mathcal{RC}(X) = \mathcal{WAP}(X) = \mathcal{C}(X)$.*
(v) *If S (resp. X) is compact and Hausdorff then $\mathcal{C}(X) = \mathcal{RC}(X)$ (resp. $\mathcal{LC}(X)$) if and only if the action of S on X is (jointly) continuous.*
(vi) *If (S, X) is compact and Hausdorff then $\mathcal{LC}(X) = \mathcal{RC}(X) = \mathcal{C}(X)$ if and only if the action of S on X is continuous.*

Proof. (i) follows from Lemma 1.1(d) of [3]. The proof of (ii) is easy. (iii) follows from (ii) and Lemma 1.1(a) of [3]. (iv) is a consequence of (iii) and Lemma 1.1(b) of [3]. To prove (v), note that the action of S on X is continuous if and only if for each $f \in \mathcal{C}(X)$ the map $(s,x) \to f(sx) : S \times X \to \mathbb{C}$ is continuous. By ([1], B.3), this is equivalent to norm continuity of $x \to \dot{\mathrm{R}}_x f : X \to \mathcal{C}(S)$ (resp. $s \to \dot{\mathrm{L}}_s f : S \to \mathcal{C}(X)$) when S (resp. X) is compact. (vi) follows from (v). □

There is no inclusion relation between $\mathcal{WAP}(X)$ and $\mathcal{LC}(X)$ or $\mathcal{RC}(X)$ that holds for every semitopological (S,X). But if S (resp. X) is compact then, by Proposition 1.1(ii), $\mathcal{LC}(X)$ (resp. $\mathcal{RC}(X)) \subseteq \mathcal{WAP}(X)$. Here we are going to obtain some conditions for establishing the reverse inclusion.

By a *homomorphism* from (S,X) into a transformation semigroup (T,Y) we mean a pair (ϕ,ψ), where $\phi : S \to T$ is a semigroup homomorphism and $\psi : X \to Y$ is a map with the property $\psi(sx) = \phi(s)\psi(x)$ for each $s \in S$ and $x \in X$. We say (ϕ,ψ) is one-to-one (resp. onto) if both ϕ and ψ are one-to-one (resp. onto). A transformation semigroup homomorphism that is one-to-one and onto is called an *isomorphism*. When S and X are topological spaces, we say (ϕ,ψ) is continuous if both ϕ and ψ are continuous.

By a right (resp. left) topological compactification of a semitopological (S,X) we mean a pair $((\phi,\psi),(T,Y))$, where (T,Y) is a compact Hausdorff right(resp. left) topological transformation semigroup, and $(\phi,\psi) : (S,X) \to (T,Y)$ is a continuous homomorphism such that $(\phi(S),\psi(X))$ is a dense semitopological sub-transformation semigroup of (T,Y). If (T,Y) is semitopological (resp. topological), $((\phi,\psi),(T,Y))$ is a semitopological (resp. topological) compactification of (S,X).

Let $((\phi,\psi),(T,Y))$ and $((\phi',\psi'),(T',Y'))$ be right(resp. left) topological compactifications of (S,X). We call a continuous homomorphism (π,γ) of (T,Y) onto (T',Y') such that $\pi\circ\phi=\phi',\gamma\circ\psi=\psi'$, a *homomorphism* of $((\phi,\psi),(T,Y))$ onto $((\phi',\psi'),\ (T',Y'))$. If such a homomorphism exists we say $((\phi,\psi),(T,Y))$ is an *extension* of $((\phi',\psi'),(T',Y'))$ and we write

$$(\pi,\gamma):((\phi,\psi),(T,Y))\to((\phi',\psi'),(T',Y')).$$

If (π,γ) is also one-to-one then it is an *isomorphism* of compactifications.

We call a compactification of (S,X) having a property *P*, a *P-compactification* of (S,X). By a *universal P-compactification* of (S,X) we mean a *P-compactification* of (S,X) that is an extension of every other *P-compactification* of *(S, X)*, which is unique up to isomorphism.

We say a compactification $((\phi,\psi),(T,Y))$ of (S,X) has *the joint continuity property* if the actions of S on T and on Y, i.e. the maps

$$(s,t)\to\phi(s)t:S\times T\to T,(s,y)\to\phi(s)y:S\times Y\to Y,$$

are (jointly) continuous.

For a semitopological (S,X), by an application of Theorem 1.3 of [3], $((\epsilon,\delta),\ (S^{\mathcal{WAP}},\ X^{\mathcal{WAP}})$ (resp. $(S^{\mathcal{AP}},\ X^{\mathcal{AP}})))$ is a universal semitopological (resp. topological) compactification of (S,X).

Furthermore, it is shown that $((\epsilon,\delta),(S^{\mathcal{LC}},X^{\mathcal{LC}})$ (resp. $(S^{\mathcal{RC}},X^{\mathcal{RC}})))$ is a right(resp. left) topological compactification of (S,X) that is universal with respect to the joint continuity property (see [2]).

Now, we aim to describe our main results. In [4] and [5], some conditions which entail the left norm continuity of weakly almost periodic functions on a transformation semigroup are discussed. Here we are going to give more precise conditions to the same effect. For this we first need a lemma.

Lemma 2.1. *Let (S,X) be a compact Hausdorff semitopological transformation semigroup.*

(i) *If the subsemigroup $T:=\{t\in S:tS=S,tX=X\}$ is dense in S, then for each $f\in\mathcal{C}(X)$, the map $s\to\dot{L}_sf:S\to\mathcal{C}(X)$ is norm continuous at each point of T. Hence the action of S on X is continuous at each point of $T\times X$, and so (T,X) is a topological sub-transformation semigroup of (S,X).*
(ii) *If the subsemigroup $T':=\{t\in S:St=S\}$ is dense in S, then for each $f\in\mathcal{C}(X)$, the map $x\to\dot{R}_xf:X\to\mathcal{C}(S)$ is norm continuous at each point of $Y:=\{y\in X:Sy=X\}$. Hence the action of S on X is continuous at each point of $S\times Y$, and so (T',Y) is a topological sub-transformation semigroup of (S,X).*

Proof. To prove (i), suppose that $f\in\mathcal{C}(X),t_0\in T$ and $\epsilon>0$. By ([1]; B.1), the function $(s,x)\to f(sx):S\times X\to\mathbb{C}$ is (jointly) continuous at each point of $\{s_0\}\times X$ for some $s_0\in S$. Hence, by ([1]; B.3), the set $N:=\{s\in S:\|\dot{L}_sf-\dot{L}_{s_0}f\|<\epsilon/2\}$ is a neighborhood of s_0.

By definition of $T,s_0=t_0u_0$ for some $u_0\in S$. Since T is dense in S and $t_0S=S,t_0T$ must be dense in S. Choose $t\in T$ such that $t_0t\in N$. Then $\rho_t^{-1}(N)$ is a neighborhood of t_0, and if $s\in\rho_t^{-1}(N)$ we have

$$\begin{aligned}\|\dot{\mathrm{L}}_s f - \dot{\mathrm{L}}_{t_0} f\| &= \sup_{x\in X} |f(stx) - f(t_0tx)| \\ &\le \|\dot{\mathrm{L}}_{st} f - \dot{\mathrm{L}}_{t_0u_0} f\| + \|\dot{\mathrm{L}}_{t_0u_0} f - \dot{\mathrm{L}}_{t_0t} f\| \\ &< \epsilon/2 + \epsilon/2 = \epsilon,\end{aligned}$$

since $s_0 = t_0u_0$ and $st, t_0t \in N$.

Therefore $s \to \dot{\mathrm{L}}_s f : S \to \mathcal{C}(X)$ is norm continuous at t_0. By ([1]; B.3), the action of S on X is continuous at each point of $T \times X$.

To prove (ii), let $f \in \mathcal{C}(X), y_0 \in Y$ and $\epsilon > 0$. By ([1]; B.1), the function $(x, s) \to f(sx) : X \times S \to \mathbb{C}$ is (jointly) continuous at each point of $\{x_0\} \times S$ for some $x_0 \in X$. Hence, by ([1]; B.3), the set $B := \{x \in X : \|\dot{\mathrm{R}}_x f - \dot{\mathrm{R}}_{x_0} f\| < \epsilon/2\}$ is a neighborhood of x_0.

The assumptions imply that for any $y_0 \in Y$ there exists an $s_0 \in S$ such that $x_0 = s_0y_0$, and $T'y_0$ is dense in X. Hence $ty_0 \in B$ for some $t \in T'$, and so $\lambda_t^{-1}(B)$ is a neighborhood of y_0. Now if $x \in \lambda_t^{-1}(B)$, then

$$\begin{aligned}\|\dot{\mathrm{R}}_x f - \dot{\mathrm{R}}_{y_0} f\| &= \sup_{s\in S} |f(stx) - f(sty_0)| \\ &\le \|\dot{\mathrm{R}}_{tx} f - \dot{\mathrm{R}}_{s_0y_0} f\| + \|\dot{\mathrm{R}}_{s_0y_0} f - \dot{\mathrm{R}}_{ty_0} f\| \\ &< \epsilon/2 + \epsilon/2 = \epsilon,\end{aligned}$$

since $x_0 = s_0y_0$ and $tx, ty_0 \in N$. Thus $x \to \dot{\mathrm{R}}_x f : X \to \mathcal{C}(S)$ is norm continuous at y_0. By ([1]; B.3), the action of S on X is continuous at each point of $S \times Y$. □

For a semitopological (S, X), we say S acts *transitively* (resp. *point transitively*) on X if $Sx = X$ for all (resp. for some) $x \in X$. S acts *topologically transitively* on X, if Sx is dense in X for all $x \in X$. Also, we say S acts *surjectively* (resp. *topologically surjectively*) on X if $sX = X$ (resp. sX is dense in X) for all $s \in S$.

The following extends Theorem 4.10 of [1] to transformation semigroup setting.

Theorem 2.2. *Let (S, X) be semitopological.*

(i) *If S is topologically right simple and if S acts topologically surjective on X, then $\mathcal{WAP}(X) \subseteq \mathcal{LC}(X)$.*
(ii) *If S is topologically left simple and if S acts topologically transitive on X, then $\mathcal{WAP}(X) \subseteq \mathcal{RC}(X)$.*

Proof. To prove (i), let S be topologically right simple and let S act topologically surjective on X. Let $((\epsilon, \delta), (T, Y))$ denote the universal semitopological compactification of (S, X). Then the subsemigroup $T_1 = \{t \in T : tT = T, tY = Y\}$ of T contains $\epsilon(S)$, and hence by Lemma 2.1(i), for each $f \in \mathcal{C}(Y)$ the map $t \to \dot{\mathrm{L}}_t f : T \to \mathcal{C}(Y)$ is norm continuous at each point of $\epsilon(S)$. Since

$$\delta^*(\dot{\mathrm{L}}_{\epsilon(s)} f) = \dot{\mathrm{L}}_s \delta^*(f)\ (s \in S, f \in \mathcal{C}(Y)),$$

where $\delta^* : \mathcal{C}(Y) \to \mathcal{WAP}(X)$ denotes the dual of the evaluation map $\delta : X \to X^{\mathcal{WAP}}$, it follows that

$$\mathcal{WAP}(X) = \delta^*\mathcal{C}(Y) \subseteq \mathcal{LC}(X).$$

The proof of (ii) is similar. □

COROLLARY 2.3

Let (S,X) be a semitopological transformation group. Then $\mathcal{WAP}(X) \subseteq \mathcal{LC}(X)$. In particular if S acts point transitively on X then $\mathcal{WAP}(X) \subseteq \mathcal{LC}(X) \cap \mathcal{RC}(X)$.

Proof. Suppose that (S,X) is a semitopological transformation group. By Theorem 1.1.17 of [1], S is left simple and right simple. Let e be the identity of S and $s \in S, x \in X$, then $x = ex = ss^{-1}x \in sX$ and so $sX = X$. This means that S acts surjectively on X and so, by Theorem 2.2(i), the first assertion holds. To prove the second assertion, let $Sx_0 = X$ for some $x_0 \in X$. Then for each $y \in X$ we have $y = s_0x_0$ for some $s_0 \in S$, and so $Sy = Ss_0x_0 = Sx_0 = X$, i.e. S acts transitively on X. Now the conclusion follows from Theorem 2.2(ii). □

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