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# An intrinsic approach to Lichnerowicz conjecture

AKHIL RANJAN

Department of Mathematics, Indian Institute of Technology, Mumbai 400 076, India
Email: aranjan@math.iitb.ernet.in

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**Abstract.** In this paper we give a proof of Lichnerowicz conjecture for compact simply connected manifolds which is intrinsic in the sense that it avoids the *nice embeddings* into eigenspaces of the Laplacian. Even if one wants to use these embeddings, this paper gives a more streamlined proof. As a byproduct, we get a simple criterion for a polynomial to be a Jacobi polynomial.

**Keywords.** Harmonic manifolds; Blaschke manifold; mean curvature; Jacobi differential equation; Ricci curvature; compact rank one symmetric spaces; nice embeddings.

## 1. Introduction

The object of this paper is to present an intrinsic proof of the Lichnerowicz's conjecture for the compact simply connected harmonic manifolds. For the definition of harmonic manifolds, see [1]. One of the characterizations is that the geodesic spheres around any point have constant mean curvature depending only on the radius of the sphere. Alternatively, there exist eigenfunctions of the Laplacian which depend only on the distance from the point, the so called radial eigenfunctions. It suffices to consider only small values of the radii. Lichnerowicz showed that for dimension less than or equal to 4, such a manifold must be either flat or a locally symmetric space of rank one (see [5, 1]). He quite naturally asked whether the same was true in higher dimensions. Great progress was made in the case of compact simply connected harmonic manifolds, a detailed account of which is given in Besse's book [1]. It is shown that these are all Blaschke manifolds and their Ricci tensor is proportional to the metric tensor or in other words they are Einstien manifolds. From topological point of view each such manifold has its (integral) cohomology ring isomorphic to precisely one of the compact rank one symmetric space to be referred to as its *model CROSS* henceforth. This result is due to Allemigeon. A particularly striking discovery about compact harmonic spaces is a family of isometric minimal immersions into the round spheres in eigenspaces of the Laplacian acting on the space of square-integrable functions. Moreover, any two geodesics were shown to be congruent to each other under some Euclidean isometry. These are now known as *Besse's nice embeddings* or helical immersions [12]. In 1990, Szabo [11] successfully used them along with other known facts about harmonic manifolds to answer Lichnerowicz's query affirmatively for compact simply connected harmonic manifolds. In contrast to compact case, Damek and Ricci in 1992 [4] produced a family of examples of homogeneous harmonic manifolds which are not locally symmetric. In Szabo's paper the key point was to show that the volume function of a compact simply connected harmonic manifold when expressed in terms of normal coordinates coincided with that of its *model CROSS*. To this end he goes through the following steps:

1. He establishes what he calls the *basic commutativity in harmonic spaces*. This implies in conjunction with Allamigeon's theorem that for any point $p$ on the manifold and any eigenvalue $\lambda$ of the Laplacian there exist eigenfunctions which depend only on radial distance from $p$. Moreover, starting with any eigenfunction and averageing over geodesic spheres around $p$ we get such a radial eigenfunction.
2. By moving the point $p$ along a geodesic and averaging in the said manner we get a parallely displaced family of functions in the eigenspace which is finite dimensional. This along with the fact that each geodesic is periodic of period *assumed* to be $2\pi$ enables one to conclude that the *radial* eigenfunctions alluded to above are polynomials in cosine of the radial distance.
3. At this stage the *nice embeddings* are used to pin down the volume function in geodesic normal coordinates.
4. Finally it follows that the first radial eigenfunction is linear of the form $A\cos r + B$ and studying the nice embedding in the first eigenspace one shows symmetry easily.

In this paper we show that the *nice embeddings* can be avoided in the step (3) above. Steps (1) and (2) do not require them anyway. Our proof of step (3) can be regarded as a streamlined version of that given by Szabo. As for the last step one can either use *nice embeddings* or the partial solution to a problem of Antonio Ros about the first eigenvalue of $P$-manifolds given in [8, 9]. Alternatively, one can use induction to complete the proof. In both the latter cases one has worked wholly within the manifold thereby proving Szabo's theorem intrinsically. It is also worth noting that the proof is over as soon as the step (3) is completed in those cases where the manifold is spherical by the Lichnerowicz–Obata's theorem or the Bonnet–Meyers' theorem.

## 2. Laplacian on radial functions

Let $\Theta(r)$ denote the volume function on a simply connected compact harmonic manifold $M$ in terms of normal coordinates and $\sigma(r)$ the mean curvature of any geodesic sphere of radius $r$. It is easily shown that

$$\frac{\Theta'(r)}{\Theta(r)} = \sigma(r).$$

Further for a point $p$ and an eigenvalue $\lambda$ of the Laplacian $\Delta$, let $u$ be an eigenfunction which depends only on radial distance $r$ from $p$. As shown by Szabo ([11], p. 8, eq. (2.1)) $u$ satisfies the following ODE

$$u'' + \sigma(r)u' + \lambda u = 0. \tag{2.1}$$

Here $'$ means derivative with respect to $r$. We would like to study how closely $\Theta$ and $\sigma$ agree with their anologues in its *model CROSS*. Let us first define the volume function on all of *real* line as follows. Consider a geodesic $\gamma$ through a point $p$. Let $J_2, \ldots, J_d$ be the Jacobi fields along $\gamma$ which vanish at $\gamma(0)$ and whose derivatives at $\gamma(0)$ form an orthonormal basis along with $\gamma'(0)$. Let $E_1, \ldots, E_d$ be parallel translation of the above orthonormal basis along $\gamma$, $E_1$ being $\gamma'(r)$. Now set

$$\Theta(r) = \langle J_2 \wedge \cdots \wedge J_d,\ E_2 \wedge \cdots \wedge E_d\rangle(r).$$

By virtue of it being a Blaschke manifold, $\Theta$ when considered as a function on whole of the real line has the following properties:

1. It is periodic of period $2\pi$.
2. It has zeroes of order $k-1$ at $r = n\pi$ for $n$ any odd integer and zeroes of order $d-1$ at $r = n\pi$ for $n$ any even integer. Here $d$ is the dimension of $M$ and $k$ is the degree of the generator of the cohomology ring of $M$.
3. $\Theta(r) = (-1)^{d-1}\Theta(-r)$.

This clearly allows us to write

$$\Theta(r) = \mathrm{e}^{\alpha(\cos r)} \sin^{d-1}(r/2) \cos^{k-1}(r/2)$$

or $\Theta(r) = \mathrm{e}^{\alpha(\cos r)}\Theta_0(r)$ where $\Theta_0(r)$ is the volume function of the *model CROSS* and $\alpha$ is a smooth (actually analytic) function on $[-1, 1]$ with $\alpha(1) = 0$. Hence $\sigma(r) = \sigma_0(r) - \sin(r)\alpha'(\cos r)$ since $\sigma = \Theta'/\Theta$.

*Caution*: The conventional volume function is the absolute value of the one we have defined. They both agree within the *injectivity radius* i.e. for $0 \le r \le \pi$. For $0 < r < \pi$, an easy calculation gives that

$$\sigma_0(r) = \frac{1}{2\sin r}[(d-1)(1+\cos r) - (k-1)(1-\cos r)].$$

By Lemma 4.2 of [11], $u$ in eq. (2.1) is of the form $u = f(\cos r)$ for some *polynomial* $f$. Inserting all this data in 2.1 and setting $\cos r = x$ we see that $f$ satisfies the following:

$$(1-x^2)f'' - \left[\frac{d}{2}(1+x) - \frac{k}{2}(1-x) + (1-x^2)\alpha'(x)\right] f' + \lambda f = 0, \quad -1 \le x \le 1. \tag{2.2}$$

In the above equation $'$ denotes derivative with respect to $x$.

## 3. Jacobi differential equation

The differential equation

$$(1-x^2)u'' - [(1+b)(1+x) - (1+a)(1-x)]u' + \lambda u = 0 \tag{3.3}$$

has been studied classically as a (singular) Sturm–Liouville equation on $[-1, 1]$ and it is known that for $a$ and $b$ in $(-1, \infty)$ and under natural boundary conditions ($u$ bounded as $|x| \to 1$) solutions exist for $\lambda = n(n+a+b+1), n \in \mathbb{N}$ and for each such value of $\lambda$, $u$ is a polynomial of degree $n$. Moreover, $u$ is unique upto a scalar multiple. In fact these polynomials form a complete orthogonal system in $L^2([-1, 1], \rho \mathrm{d}x)$ where $\rho(x) = (1+x)^a(1-x)^b$ is the weight function. These are known as Jacobi polynomials (see [2], p. 289). This differential equation is known as Jacobi differential equation with parameters $a$ and $b$. We assume that $a, b > -1$.

In this section we consider a *perturbed* Jacobi equation where we have an extra term of the form $(1-x^2)\delta(x)$ as a coefficient of $u'$, $\delta$ being a continuous function on $[-1, 1]$. Comparing with the corresponding equation satisfied by the polynomial $f$ in the previous section we easily see that $1+b = d/2$, $1+a = k/2$ and $\delta = \alpha'$. We also know that $k$ cannot exceed $d$ and can only take values in $2, 4, 8, d$, hence $a, b > -1$ is clearly true. Now we are ready to state our main theorem.

**Theorem 3.1.** *If the perturbed Jacobi differential equation*

$$(1-x^2)u'' - [(1+b)(1+x) - (1+a)(1-x) + (1-x^2)\delta(x)]u' + \lambda u = 0 \tag{3.4}$$

*admits a nonconstant polynomial as a solution for some value of $\lambda$, then the perturbation term $\delta$ must vanish identically.*

COROLLARY 3.1

*The perturbation term $\alpha'$ in* 2.2 *vanishes. Consequently $\alpha$ is identically zero and hence $\sigma = \sigma_0$ as well as $\Theta = \Theta_0$.*

The proof of the above will be broken into two lemmas. Let $P$ be a polynomial which we assume to be nonconstant and monic which satisfies 3.4 for a suitable $\lambda$.

*Lemma* 3.1. *$\delta$ must be a rational function with the degree of the numerator being strictly less than that of the denominator.*

*Proof.*

$$\delta = \frac{LP + \lambda P}{(1-x^2)P'},$$

where

$$L = (1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - [(1+b)(1+x) - (1+a)(1-x)]\frac{\mathrm{d}}{\mathrm{d}x}$$

is the Jacobi differential operator (with parameters $a$ and $b$). Clearly both numerator and denominator of $\delta$ are polynomials with denominator nonvanishing and of degree strictly more than that of the numerator. Hence the claim. ■

*Lemma* 3.2. *Let $\delta = p/q$ as a quotient of relatively prime polynomials with $q$ being monic. Then*

1. *All the roots of $q$ are simple and in $\mathbb{C} \setminus [-1, 1]$.*
2. *$q|P'$ and $q|P$.*
3. *Let $q = \prod(x-\beta_i)$ and $m_i$ be the multiplicity of $\beta_i \in P$, then $m_i \geq 2$.*
4. *If we put $q_1 = \prod(x-\beta_i)^{m_i-1}$, then $\delta = q_1'/q_1$.*
5. *$\pm 1$ cannot be roots of $P$.*
6. *Roots of $P$ which are not common with those of $q$, are all simple.*

*Proof.* Let $P = \prod(x-\beta_i)^{m_i}$ where $\beta_i$ are distinct complex numbers and $m_i$ are natural numbers which are nonzero. Let

$$v = \frac{P'}{P} = \sum \frac{m_i}{x-\beta_i}. \tag{3.5}$$

Then $v$ satisfies the Riccati equation (see [2], p. 124)

$$v' + v^2 = \left[\frac{1+b}{1-x} - \frac{1+a}{1+x} + \delta(x)\right] v - \frac{\lambda}{1-x^2}. \tag{3.6}$$

From 3.5 we get

$$v' + v^2 = \sum_i \frac{m_i^2 - m_i}{(x-\beta_i)^2} + \sum_{i\neq j} \frac{2m_i m_j}{(\beta_i-\beta_j)(x-\beta_i)}. \tag{3.7}$$

In the above equation we have expanded the *cross terms* occuring in $v^2$ into partial fractions. Now let $q = \prod (x-\alpha_j)^{r_j}$ where $\alpha_j$ are distinct complex numbers and $r_j \geq 1$. Since $p$ and $q$ are relatively prime, if we expand $\delta = p/q$ in partial fractions, $1/(x-\alpha_j)^{r_j}$ will survive for each $j$. They will continue to survive after multiplication by $v = \sum m_i/(x-\beta_i)$ and further expansion into partial fractions. Now if we compare the rhs of (3.6) and (3.7) after expanding into partial fractions we find that we must have $\{\alpha_j\} \subset \{\beta_i\}$ and $r_j = 1$ for each $j$. This proves that the roots of $q$ are simple. Since $\delta = p/q$ is continuous on $[-1,1]$ roots of $q$ must be away from $[-1,1]$. This poves the first assertion.

From the second assertion $q|P$ is clear from above. To show that $q|P'$, we note that $LP + \lambda P = (p(1-x^2)P')/q$ is a polynomial. Hence $q|P'$ since it is relatively prime to $p$, $1-x$, and $1+x$. This gives the second claim.

Put $S = \{\beta_i\}, S' = \{\alpha_j\}$, then $S' \subset S$. Also put $S'' = S \setminus S'$. We can then write $q = \prod_{S'}(x-\beta_j)$ and hence $\delta = \sum_{S'} c_i/(x-\beta_i)$ where $c_i$ are nonzero numbers. Coming back to the third statement, let us compare the coefficient of $1/(x-\beta_i)^2$ in the rhs of (3.6) and (3.7) (after partial fractions) for $\beta_i \neq \pm 1$. We see that

$$c_i\ m_i = m_i^2 - m_i \text{ for } i \text{ s.t } \beta_i \in S' \text{ and } m_i^2 - m_i = 0 \text{ if } \beta_i \in S'' \setminus \{\pm 1\}.$$

From this we can conclude that

$$c_i = m_i - 1 \text{ for } i \text{ s.t. } \beta_i \in S'$$

and $m_i = 1$ for $i$ s.t. $\beta_i \in S'' \setminus \{\pm 1\}$ (since $m_i \neq 0$ for each $i$).

$$\frac{p}{q} = \sum_{S'} \frac{m_i - 1}{x - \beta_i}.$$

The first of these shows that $m_i \geq 2$ for $\beta_i \in S'$ because $c_i \neq 0$ and the second one can be rewritten as $p/q = q_1'/q_1$, where $q_1 = \prod_{S'}(x-\beta_i)^{m_i-1}$. These are just the third and the fourth assertions.

For the fifth claim, write

$$P(x) = \prod_S (x-\beta_i)^{m_i} = (x-1)^A (x+1)^B \prod_{S'} (x-\beta_i)^{m_i} \prod_{S''\setminus\{\pm 1\}} (x-\beta_i).$$

Comparing coefficients of $(x-1)^{-2}$ and $(x+1)^{-2}$ we see that

$$A^2 - A = -(1+b)A \text{ and } B^2 - B = -(1+a)B.$$

Since $a$ and $b$ are more than $-1$ and $A$ and $B$ are natural numbers we get $A = B = 0$. Thus $S = S' \cup S''$ ; the set of roots of $P$, is disjoint from $\{\pm 1\}$.

Finally, $S'' \setminus \{\pm 1\} = S''$ so that for $\beta_i \in S''$ we have $m_i = 1$ which is the sixth assertion. ■

## 4. Proof of theorem 3.1

We have the following facts: $P(x) = \prod_{S'}(x-\beta_i)^{m_i} \prod_{S''}(x-\beta_i)$, $m_i \geq 2.q_1(x) = \prod_{S'}(x-\beta_i)^{m_i-1}$. Clearly, $q_1|P'$. Let $P'/q_1 = n\prod_j (x-\gamma_j)^{M_j} = R(x)$, say. (Here $n = \deg P$.)

Then $\gamma_j$ are those roots are $P'$ which are not common with those of $P$. This is so because the roots of $P$ in $S''$ are all simple. Hence

$$\{\gamma_j\} \bigcap S = \emptyset = S \bigcap \{\pm 1\}. \tag{4.8}$$

Substituting the expressions obtained for $P, P'$ and $\delta$ in the equation (3.4), dividing by $(1 - x^2)P'$ and simplifying we get

$$\sum \frac{M_j}{x - \gamma_j} + \frac{a+1}{x+1} + \frac{b+1}{x+1} = -\frac{\lambda \prod_S (x - \beta_i)}{n \prod (x - \gamma_j)^{M_j}}. \tag{4.9}$$

Putting $x = \beta \in S$ and using (4.8) we find that

$$\sum_j \frac{M_j}{\beta - \gamma_j} + \frac{a+1}{\beta+1} + \frac{b+1}{\beta-1} = 0, \quad \text{for each such } \beta. \tag{4.10}$$

This in turn implies that $\beta \in \text{conv}\{\gamma_j, \pm 1\}$ (where conv denotes the *convex hull*). Or $S \subset \text{conv}\{\gamma_j, \pm 1\}$. On the other hand by Lucas' theorem $\text{conv}\{\gamma_j\} \subset \text{conv}\, S$. Now the argument of Lemma 4.6 ([11], p. 23) goes through and shows that $S \subset (-1, 1)$. Also as observed earlier $S'$ is disjoint from $[-1, 1]$. Therefore, $S'$ must be empty so that $q$ is constant 1 and hence $p = \delta$ vanishes identically as $\deg(p) < \deg(q)$. ■

An interesting corollary is the following charcterization of the Jacobi polynomials which should be of theoretical interest.

COROLLARY 4.2

*If $P$ is a nonconstant polynomial such that for some complex number $\lambda$ the rational function $(P'' + \lambda P)/(1 - x^2)P'$ has simple poles at $\pm 1$ with positive residues $1 + a, 1 + b$ respectively and no other pole in the real interval $(-1, +1)$, then $P$ is a Jacobi polynomial with parameters $a$ and $b$.* ■

## 5. Proof of the conjecture

Let us first recall a theorem of Antonio Ros ([9], Theorem 4.2, p. 402).

**Theorem 5.2.** *Let $M$ be an $n$-dimensional $P_{2\pi}$-manifold and suppose that the Ricci tensor, $S$, and the metric, $\langle , \rangle$, on $M$ verify the relation $S \geq k\langle , \rangle$, where $k$ is a real constant. Let $\lambda_1$ be the first eigenvalue of the Laplacian of $M$. Then we have*

$$\lambda_1 \geq \frac{1}{3}(2k + n + 2).$$

The proof is short and elegant and independent of the other computations done in his paper. It also follows that, in case the equality holds, any first eigenfunction $f$ has the property that for any geodesic $\gamma_u, f(\gamma_u(t)) = A_u \cos t + B_u \sin t + C_u$. This property will be referred to as Ros' property in what follows. Also here and in the sequel $\gamma_u$ will always denote the geodesic with initial condition $u$.

Ros also noted that for *CROSSES* the equality holds. He naturally asked as to what restrictions apply to $M$ if equality held.

As a partial answer to the above question the following theorem has been proved in [8] and [9]:

**Theorem 5.3.** *If equality holds in the Ros' estimate for $\lambda_1$ of a P-manifold and if M admits a corresponding eigenfunction without saddle points, then M is a CROSS.*

Also see [7] for another related result.

*Proof of the conjecture.* Now from corollary 3.1 to our main theorem $\Theta = \Theta_0$ and hence $\mathrm{Ric}_M = \mathrm{Ric}_0$ and $\lambda_1(M) = \lambda_1(M_0)$ where $M_0$ denotes the *model CROSS*. Moreover, from any point on $M$ the first radial eigenfunction is of the form $\cos r + C$ and hence without saddle points. The proof of the Lichnerowicz conjecture is now complete. ■

It is however, possible to avoid the use of this theorem and give a more elementary inductive proof of the conjecture. It should be noted that in the spherical case the proof is already over by the well known Lichnerowicz–Obata theorem [6] or more easily by the rigidity part of the Bonnet–Meyers' theorem [3]. So let $M$ be of projective type.

*Claim I.* For any $p \in M$, $C(p)$- the cut locus of $p$ is totally geodesic.

*Proof.* Let $f = \cos r + C$ be a radial first eigenfunction from $p$, where $r = d(p, .),$. Let $u \in T_q C(p)$ be a unit vector. Then $\langle \nabla f, u\rangle = 0$ and $\langle (\nabla^2 f)(u), u\rangle = 0$, where $\nabla^2$ denotes the Hessian. Hence by Ros' property $f(\gamma_u(t)) = f(\gamma_u(0)) = f(C(p))$ for every $t$. It follows that $\gamma_u$ lies in the level set $C(p)$. ■

*Claim II.* $C(p)$ is itself harmonic.

*Proof.* Let $q \in C(p)$ and $h$ be a radial first eigenfunction from $q$. Further, let $\sigma$ be a geodesic in $C(p)$ starting at $q$ and $q' = \sigma(s_0)$ be a point on it. Let $N_{q'}$ be the normal space to $T_{q'}C(p)$. For any unit vector $v \in N_{q'}$, $\gamma_v(\pi) = p$, and hence $h(\gamma_v(\pi))$ is independent of $v$. Now $\nabla h$ at $q'$ points in the direction of $\sigma'(s_0)$ and thus is perpendicular to $v$. It follows that $h(\gamma_v(t)) = A_v \cos t + C_v$, since $B_v = 0$. Moreover,

$$h(q') = h(\gamma_v(0)) = A_v + C_v = \cos s_0 + C$$
$$h(p) = h(\gamma_v(\pi)) = -A_v + C_v = -1 + C, \quad \text{as } d(p,q) = \pi$$

Hence, it follows that $A_v$ depends only on $s_0 = d(q, q')$ and is equal to the value it takes in the model CROSS in a parallel situation. (Observe that the constant $C$ is same as in the model.) Let $\Delta_c$ denote the Laplacian in the cut locus $C(p)$. At the point $q'$,

$$\Delta_c h = \Delta h - \mathrm{tr}_{N_{q'}}(\nabla^2 h).$$

This implies that

$$\Delta_c h = \lambda_1(M_0)h - \sum A_{v_i},$$

where $\{v_i\}$ form an orthonormal basis of $N_{q'}$. If $\cos r + C'$ gives the radial first eigenfunction in the cut locus of the model CROSS $M_0$ with eigenvalue $\lambda_{1,c}$, it follows that on setting $h' = h + (C' - C)$, it becomes a radial eigenfunction in $C(p)$ of eigenvalue $\lambda_{1,c}$. Thus $C(p)$ is a harmonic manifold as claimed. ■

*Claim III.* $M$ is a symmetric space.

*Proof.* By induction hypothesis, $C(p)$ is a CROSS of sectional curvatures lying in $[\frac{1}{4}, 1]$. Here we have assumed that $M$ is not spherical so that $C(p)$ is not a point. Now let $c$ be any geodesic in $C(p)$ and $\mathcal{R}$ be the curvature operator along $c$ given by $\mathcal{R}(u) = R_{u,c'}c'$. Obviously, it leaves $T_{c(t)}C(p)$ and $N_{c(t)}$ invariant. These bundles are also parallel along $c$. This forces the Jacobi fields along $c$ which vanish at $c(0)$ and whose initial derivative is in $N_{c(0)}$ to remain in $N$ throughout. Since the trace of $\mathcal{R}$ gives the Ricci curvature, it follows that $\mathrm{tr}\,\mathcal{R}|N$ equals $k/4$ where $k$ is the degree of the generator of $H^*(M,Z)$. It is also the rank of the normal bundle $N$ of $C(p)$. Applying the Bonnet–Meyers' method to these normal Jacobi fields and noting that these cannot vanish again before $2\pi$, (the whole complement of $(k-1)$ fields which vanish at $\pi$ is tangent to $C(p)$) we find that each of these must be of the form $\sin(t/2)E$ for some parallel field $E$. Hence all the Jacobi fields along $c$ are as in the model CROSS. Finally, since any geodesic can be made to lie in some cut locus, symmetry of $M$ follows. ■

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