Proc. Indian Acad. Sci. (Math. Sci.), Vol. 109, No. 4, November 1999, pp. 441–452.

Stokes drag on axially symmetric bodies: a new approach

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MS received 9 October 1998; revised 6 May 1999

Abstract. In this paper a new approach to evaluate the drag force in a simple way on a restricted axially symmetric body placed in a uniform stream (i) parallel to its axis, (ii) transverse to its axis, is advanced when the flow is governed by the Stokes equations. The method exploits the well-known integral for evaluating the drag on a sphere. The method not only provides the value of the drag on prolate and oblate spheroids and a deformed sphere in axial flow which already exists in literature but also new results for a cycloidal body, an egg shaped body and a deformed sphere in transverse flow. The salient results are exhibited graphically. The limitations imposed on the analysis because of the lack of fore and aft symmetry in the case of an egg-shaped body is also indicated. It is also seen that the analysis can be extended to calculate the couple on a body rotating about its axis of symmetry.

Keywords. Stokes drag; spheroid; cyclical body moment.

1. Introduction

The drag force on a sphere placed in a slow uniform stream of viscous fluid was first determined by Stokes [5] in the year 1851 and is commonly known as Stokes drag. Because of their growing importance, particularly in the fields of biomechanics and chemical engineering slow flow problems have started engaging considerable attention in recent years. These problems are governed by the so called Stokes equations and the reader is referred to the book by Happel and Brennel [4] for some classical solutions. The basic singularity of this equation is the Stokeslet, a name proposed by Hancock [3] in 1953. It was utilized by Chawng and Wu [1] to generate solutions for a prolate spheroid in a variety of flows. It is worth noting that the most sought after quantity in such flow problems is the drag force rather than a detailed description of the flow field. In this study, we have proposed a new method for estimating the drag force on an axially symmetric body under some limitations. The method exploits the well-known drag formula for a sphere and the steps used in its derivation [p. 122, 4]. Using this method, first a simple formula is obtained for evaluating the drag force on an axially symmetric body, with continuously turning tangent, placed in an uniform stream along the axis of symmetry, and then the method is extended to the transverse flow situation. The formula, thus obtained, is used to estimate the values of the drag force on prolate and oblate spheroids and are seen to agree with known results [2, 4, 6] existing in the literature.

Having thus tested the formula, we use it to obtain the drag force for

(i) a deformed sphere,
(ii) an axis symmetric body obtained by rotation of cycloid and
(iii) an egg shaped body.

Further, this analysis is extended to calculate the couple on a body rotating about its axis of symmetry. The authors have not come across such results in the existing literature.

2. The method

Axial Flow: Let us consider the axially symmetric body of characteristic length L placed along its axis (x-axis, say) in a uniform stream U of viscous fluid of density ρ and kinematic viscosity ν. When the Reynolds number UL/ν is small, the motion is governed by Stokes equation [4],

$$0 = -(1/\rho)\,\mathrm{grad}\,p + \nu\nabla^2\mathbf{u},\ \mathrm{div}\,\mathbf{u} = 0, \tag{2.1}$$

subject to the no slip boundary condition. For the case of a sphere of radius R, the solution is easily obtained and on evaluating the stress, the drag force F comes out as [4]

$$F = (9/2)\pi\mu U \int_0^{\pi} R\sin^3\alpha\,\mathrm{d}\alpha = \lambda R, \tag{2.2}$$

where

$$\lambda = 6\pi\mu U. \tag{2.3}$$

This shows that the drag force increases linearly with the radius of the sphere. In other words, the difference between drag force on two spheres of radii y and $y + \mathrm{d}y$ is given by

$$\mathrm{d}F = \lambda \mathrm{d}y. \tag{2.4}$$

A sphere of radius b is obtained by rotating the curve $x = b\cos t, y = b\sin t\,(0 \le t \le \pi)$ about the x-axis and the force $F = \lambda b$ is obtained from (2.4) as $\int_0^b \lambda\,\mathrm{d}y$ exhibiting that the force system $\mathrm{d}F$ may be considered as lying in the xy plane.

The element force $\mathrm{d}F$ may be decomposed into two parts $(1/2)\,\mathrm{d}F$, each acting over the upper half and lower half; $(1/2)\,\mathrm{d}F$ on the upper half acts at a height y (say) above the x-axis. The total force $F/2$ on the upper half, may be considered as made up of these differential forces $\mathrm{d}F/2$ acting over elements corresponding to a system of half spheres of radii increasing from 0 to b and spread over from A to A' (figure 1). The moment of this force system (taken to be in the xy plane) about O, now provides

$$h.(F/2) = M = (1/2)\int_0^b y\,\mathrm{d}F = (1/2)\lambda\int_0^b y\,\mathrm{d}y = (1/4)\lambda b^2,$$

or

$$F = (1/2)(\lambda b^2)/h, \tag{2.5}$$

where h is the height of centroid of the force system. In the case of a sphere of radius b we have $F = \lambda b$, and so we get from (2.5), $h = b/2$ as it should be.

Next, we can express (2.2) also as

$$F = \int_{\alpha=0}^{\pi} \mathrm{d}f, \tag{2.6}$$

where

$$\mathrm{d}f = (3/4)\lambda R\sin^3\alpha\,\mathrm{d}\alpha \tag{2.7}$$

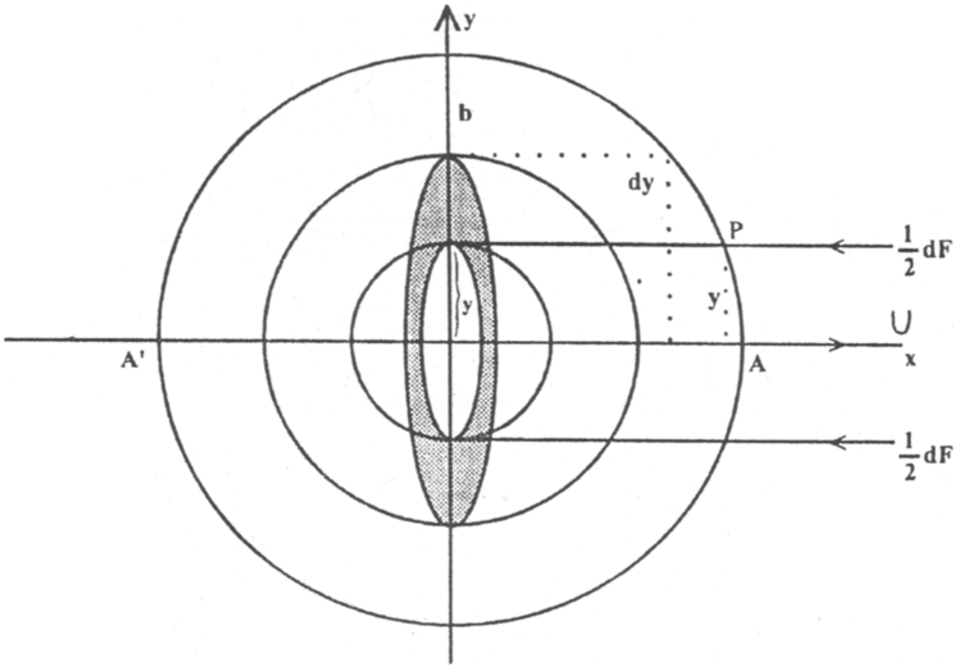

Figure 1. Elemental force system on the sphere.

is the elemental force on a circular ring element P (figure 2) [4, p. 122]. For the purpose of calculating $F/2$, the force on upper half, $(1/2)\mathrm{d}f$ may be taken to be acting at height η (say), above x-axis, given by

$$\begin{aligned} h &= \int \eta((1/2)\mathrm{d}f) \Big/ \int (1/2)\mathrm{d}f \\ &= \int_0^{\pi} \eta((3/4)\lambda R \sin^3\alpha\, \mathrm{d}\alpha) \Big/ \int_0^{\pi} (3/4)\lambda R \sin^3\alpha\, \mathrm{d}\alpha \\ &= (3/4) \int_0^{\pi} \eta \sin^3\alpha\, \mathrm{d}\alpha. \end{aligned}$$

Taking $\eta = R/2$, the result is seen to correspond to the value $h = b/2$ confirmed earlier. Thus we have

$$h = (3/8) \int_0^{\pi} R \sin^3\alpha\, \mathrm{d}\alpha. \tag{2.8}$$

It is proposed that the formula (2.8) holds good for an axially symmetric body also, when R is interpreted as the normal distance PM between the point P on the body and the point of intersection M of the normal at P with axis of symmetry and α as its slope (figure 2). On inserting the value of h from (2.8) in (2.5), we finally obtained

$$F_x = (1/2)(\lambda b^2)/h = (4/3)(\lambda b^2) \Big/ \int_0^{\pi} R \sin^3\alpha\, \mathrm{d}\alpha, \tag{2.9}$$

where the suffix x has been introduced to assert that the force is in the axial direction.

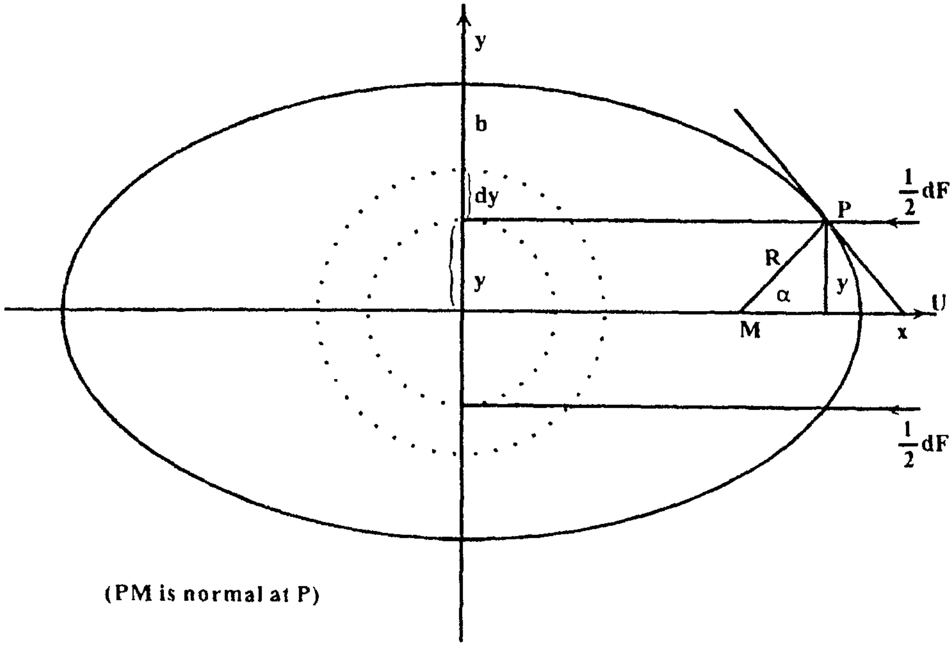

Figure 2. Force system on axially symmetric body.

While using (2.9) it should be kept in mind that b denotes intercept between the meridian curve and the axis of the normal perpendicular to the axis i.e., $b = R$ at $\alpha = \pi/2$.

Transverse flow: We set up a polar coordinate system (R, β, γ) with β as the polar angle with the y-axis and γ the azimuthal angle in $z - x$ plane. Since the y-axis is not the axis of symmetry for the body we cannot make use of circular ring elemental force $(3/4)\lambda R \sin^3\beta \, d\beta$ corresponding to (2.7). But we can easily write down the elemental force on the element $R^2 \sin\beta d\beta \, d\gamma$ as

$$\delta f = (3\lambda R/8\pi) \sin^3\beta d\beta \, d\gamma.$$

Transforming the above to the polar coordinates (R, θ, ϕ) with the x-axis as the polar axis, we have

$$\delta f = (3\lambda R/8\pi)(1 - \sin^3\alpha \cos^2\phi) \sin\alpha \, d\alpha \, d\phi,$$

as the force on the element $R^2 \sin\alpha \, d\alpha \, d\phi$. On integrating over ϕ from 0 to 2π, we get

$$df_y = (3\lambda R/8)(2 - \sin^2\alpha) \sin\alpha \, d\alpha, \tag{2.10}$$

where the suffix y has been placed to designate the force due to the external flow along the y-axis, the transverse direction.

Integrating df_y over the surface of the sphere, we get

$$F_y = (3\lambda/8) \int_0^{\pi} (2 \sin\alpha - \sin^3\alpha) d\alpha = \lambda R, \tag{2.11}$$

agreeing with the correct value. This suggests we can take the force df_y as given by (2.10) as the element force on the circular ring element at P. Although the force F_y is along the y

direction, we have reduced it to elemental forces on a system of spheres centered on the x-axis. Since F_y and $\mathrm{d}f_y$ themselves are scalar quantities, on comparing (2.10) with (2.7), we can use the analysis as in the axial flow case with h replaced by

$$h_y = (3/16)\int_0^{\pi} R(2\sin\alpha - \sin^3\alpha)\mathrm{d}\alpha. \tag{2.12}$$

Thus, we get from (2.5)

$$F_y = (\lambda b^2)/(2h_y). \tag{2.13}$$

In addition, it seems that formula (2.5) should be valid only for a curve with continuously varying tangent, but not for a curve with edges or other kind of nodes.

We are working to improve our proposed conjecture and results will be presented in a future problem so that above stated limitations could be removed and results would have been true for general axi-symmetric bodies.

3. Flow past a spheroid

Prolate spheroid: A prolate spheroid is generated by the rotation of an ellipse about the x-axis. The parametric equation of the ellipse may be taken as

$$x = a\cos t, \quad y = b\sin t. \tag{3.1}$$

This provides the following values:

$$\sin\alpha = (b\sin t)/R, \cos\alpha = ((b^2/a)\cos t)/R, \ R = (b/a)(b^2\cos^2 t + a^2\sin^2 t)^{1/2}. \tag{3.2}$$

Therefore, for the axial flow, using (2.8), we get

$$h = (3/8)(b^2/a)\int_0^{\pi} \sin^3 t/(1 - e^2\cos^2 t)\mathrm{d}t \tag{3.3}$$

$$= (3b^2/16ae^3)[(1 + e^2)L - 2e], \tag{3.4}$$

where $L = \log\{(1 + e)/(1 - e)\}$. Substituting the above value of h in (2.9) and setting $\lambda = 6\pi\mu U$, we get the well-known results [2]

$$F_x = 16\pi\mu U a e^3[(1 + e^2)L - 2e]^{-1}. \tag{3.5}$$

When $e \to 0$, it may be confirmed that we get the classical result $6\pi\mu Ua$ for a sphere placed in a uniform stream U. Next, using the values (3.2) in (2.12), we get

$$h_y = (3/32)(b^2/ae^3)[(3e^2 - 1)L + 2e] \tag{3.6}$$

and hence from (2.13), we have for the transverse flow

$$F_y = 32\pi\mu U a e^3[2e + (3e^2 - 1)L]^{-1}. \tag{3.7}$$

The above result is again seen to be in agreement with that obtained by Chwang and Wu [2]. When $e \to 0$, we get the classical result $6\pi\mu Ua$, for a sphere.

Oblate spheroid: The oblate spheroid is obtained by a rotation about the x-axis of the ellipse with parametric equations

$$x = b\cos t, \quad y = a\sin t. \tag{3.8}$$

Notice that the cross-section at O is now a circle of radius a and not b. This provides the following values:

$$\sin\alpha = (a\cos t)/R, \cos\alpha = (a^2/b)(\sin t)/R, R = (a/b)(a^2\sin^2 t + b^2\cos^2 t)^{1/2}. \tag{3.9}$$

Next, using the above values in (2.9), we get

$$h = (3/8)(a/e^3)[e(1-e^2)^{1/2} - (1-2e^2)\sin^{-1} e],$$

on substituting the above value of h in (2.9) with b replaced by a we get the axial drag force as

$$F_x = (1/2)(\lambda a^2/h) = (8\pi\mu U a e^3)[e(1-e^2)^{1/2} - (1-2e^2)\sin^{-1} e]^{-1}, \tag{3.10}$$

when $e \to 0$, it may be confirmed that we get the classical result $F_x = 6\pi\mu U a$ for a sphere. For the transverse flow we have from (3.6)

$$h_y = (3/16)(a/e^3)[(1+2e^2)\sin^{-1} e - e(1-e^2)^{1/2}],$$

and so from (2.12) with b replaced by a

$$F_y = (\lambda a^2/2h_y) = 16\pi\mu U a e^3/[(1+2e^2)\sin^{-1} e - e(1-e^2)^{1/2}]. \tag{3.11}$$

The values (3.10) and (3.11) are seen to agree with known results [4].

4. Flow past a deformed sphere

Consider the axially symmetric body defined by

$$r = a\left[1 + \epsilon\left\{d_0 + d_2P_2(\mu) + \sum_{k=0}^{\infty} d_{2k+1}P_{2k+1}(\mu)\right\}\right], \quad \mu = \cos\theta, \tag{4.1}$$

where (r,θ) are spherical polar coordinates and $P_n(\mu)$ are Legendre functions. For a small value of parameter ϵ, this equation (4.1) represents a spheroid of small eccentricity $e = \sqrt{(3\epsilon d_2)}$ when $d_n = 0\,(n \geq 3)$. This provides the following values (figure 2):

$$\sin\alpha = \sin\theta + \epsilon\left\{d_2P_2'(\mu) + \sum_{k=0}^{\infty} d_{2k+1}P_{2k+1}'(\mu)\right\}\sin\theta\cos\theta + 0(\epsilon^2), \tag{4.2}$$

$$\cos\alpha = \cos\theta - \epsilon\left\{d_2P'2(\mu) + \sum_{k=0}^{\infty} d_{2k+1}P_{2k+1}'(\mu)\right\}\sin^2\theta + 0(\epsilon^2), \tag{4.3}$$

$$R = (r\cdot\sin\theta)/\sin\alpha. \tag{4.4}$$

Now, we have

$$b = R_{\alpha=\pi/2} = a[1 + \epsilon(d_0 - d_2/2) + \epsilon\{d_3.P_3(0) + d_4.P_4(0) + d_5.P_5(0) + \ldots\} + 0(\epsilon^2)],$$

if it is to be same as that for a spheroid, we must have

$$d_3P_3(0) + d_4P_4(0) + d_5P_5(0) + \cdots = 0,$$

and then

$$b = a[1 + \epsilon(d_0 - d_2/2)]. \tag{4.5}$$

Therefore, for the axial flow, using (2.8), we get

$$h = (a/2)[1 + \epsilon(d_0 - (4/5)d_2)].$$

Substituting the above value of h in (2.7), we get the result derived by Usha and Nigam mentioned in (4), viz.,

$$F_x = 6\pi\mu Ua\{1 + \epsilon(d_0 - d_2/5) + 0(\epsilon^2)\}. \tag{4.6}$$

As $\epsilon \to 0$, we get the classical result for a sphere with radius a placed in a uniform stream U. Next, for the transverse flow, using the values (4.2), (4.3), (4.4) in (2.12), we get

$$h_y = (a/2)[1 + \epsilon\{d_0 - (11/10)d_2\}], \tag{4.7}$$

providing

$$F_y = (\lambda b^2)/(2h_y) = 6\pi\mu Ua[1 + \epsilon\{d_0 - (1/10)d_2\}]. \tag{4.8}$$

For $\epsilon \to 0$, we again get the classical result for a sphere. It is interesting to note that so long the transverse length b remains unaltered, the drag values F_x and F_y do not differ, from those of a spheroid.

5. Flow past a cycloidal body of revolution

Case I: Let us take the inverted cycloid

$$x = a(t + \sin t), y = a(1 + \cos t), -\pi \le t \le \pi, \tag{5.1}$$

with vertex at $(0, 2a)$, and revolve it about x-axis, the base, to generate the cycloidal body of revolution. In this case we have

$$\alpha = (\pi/2) - (t/2), R = 2a.\cos(t/2), \tag{5.2}$$

and so we get from (2.9) by replacing b by $2a$,

$$h = (9/32)a\pi. \tag{5.3}$$

The formula (2.9) provides, for the axial flow, the drag force,

$$Fc_x = (128/3)\mu Ua, \tag{5.4}$$

where the label c stands for cycloid. For the transverse flow, we have from (2.12)

$$h_y = (15/64)a\pi, \tag{5.5}$$

and then from (2.13) and (5.4), we have

$$Fc_y = (256/5)\mu Ua = (6/5)Fc_x = 1.20Fc_x. \tag{5.6}$$

The transverse drag force on the cycloid is 1.2 times the axial drag force. This may be used experimentally to confirm the conjectures proposed in this paper.

Case II: Next, we consider the body generated by the rotation about x-axis of the curve composed of arcs of two cycloidal parts represented parametrically by

$$x = a(1+\cos t), \quad y = a(t+\sin t), \quad 0 \le t \le \pi; \tag{5.7a}$$

$$x = -a(1+\cos t), \quad y = a(t+\sin t), \quad 0 \le t \le \pi. \tag{5.7b}$$

Thus, we have, for the first part

$$\alpha = t/2, \sin\alpha = \sin(t/2), R = a(t+\sin t)/\sin(t/2), \tag{5.8}$$

$$\alpha = \pi - t/2, \quad \sin\alpha = \sin t/2, \quad R = a(t+\sin t)/\sin t/2, \tag{5.9}$$

using the above values in (2.8), we have

$$h = (3/8)\left[\int_{\alpha=0}^{\pi/2} R\sin^3\alpha\,\mathrm{d}\alpha + \int_{\alpha=\pi/2}^{\pi} R\sin^3\alpha\,\mathrm{d}\alpha\right]$$

$$= (a/32)[3\pi^2+16], \tag{5.10}$$

and hence from (2.9), we have the axial drag

$$Fc_x = (96\pi^3\mu Ua)/(3\pi^2+16). \tag{5.11}$$

Similarly, for the transverse flow, we have from (2.12) with R given by (5.9)

$$h_y = (3/16)\left[\int_{\alpha=0}^{\pi/2}\{2R\sin\alpha - R\sin^3\alpha\}\mathrm{d}\alpha + \int_{\alpha=\pi/2}^{\pi}\{2R\sin\alpha - R\sin^3\alpha\}\mathrm{d}\alpha\right]$$

$$= (a/64)[9\pi^2+32], \tag{5.12}$$

and hence from (2.13), we have transverse drag

$$Fc_y = \lambda(a\pi)^2/2h_y = (92\pi^3\mu Ua)/(9\pi^2+32). \tag{5.13}$$

In Case II, we have

$$Fc_x/Fc_y = (9\pi^2+32)/(6\pi^2+32) \text{ or } Fc_x \approx 1.32\, Fc_y. \tag{5.14}$$

Keeping in view that in Case II, the axis have been interchanged, it is seen that result (5.14) compares with the result (5.6).

It will be interesting to compare that results for cycloid with those of a spheroid. For this purpose we consider a spheroid with major and minor axes of lengths $2\pi a$ and $4a$ respectively, corresponding to the maximum and minimum diameters of the cycloidal body. The eccentricity of such spheroid is $e = \sqrt{(1-(4/\pi^2))} = 0.77$. Therefore, we have from (5.4) and (3.5) with a replaced by $a\pi$ and evaluated at $e = 0.77$ and the label p stands for the prolate spheroid.

$$Fc_x/Fp_x = (8/3\pi)[\{(1+e^2)L - 2e\}/e^3] \approx 1.00. \tag{5.15}$$

From (5.6) and (3.7) with a replaced by $a\pi$,

$$Fc_y/Fp_y = (8/5\pi^2)[\{2e + (3e^2-1)L\}/e^3] \approx 1.1. \tag{5.16}$$

Next, from (5.11) and (3.10) with a replaced by $a\pi$, we have

$$Fc_x/Fo_x = (12\pi^2/(3\pi^2+16))[\{e(1-e^2)^{1/2} - (1-2e^2)\sin^{-1}e\}/e^3] \approx 3.7, \tag{5.17}$$

and from (5.13) and (3.11) with a replaced by $a\pi$, we have

$$Fc_y/Fo_y = (12\pi^2/(9\pi^2 + 32))[\{-e(1-e^2)^{1/2} + (1+2e^2)\sin^{-1} e\}/e^3] \approx 3.06, \tag{5.18}$$

where the subcript o stands for oblate spheroid. It is interesting to note that while the ratios for the prolate case given by (5.15) and (5.16) also close to unity, the forces on the cycloidal body as given by (5.17) and (5.18) are much larger than those on the oblate spheroid.

6. Flow past an egg-shaped body

Continuing in the same manner, we have calculated the drag for an egg-shaped body in which the right portion is in the shape of a half prolate spheroid given parametrically by

$$x = a\cos t, \quad y = a\sin t, \quad 0 \le t \le \pi/2, \tag{6.1}$$

and left portion is a hemisphere given by

$$x = b\cos t, \quad y = b\sin t, \quad \pi/2 \le t \le \pi. \tag{6.2}$$

Now for the spheroidal portion, we have

$$\begin{aligned} &\sin\alpha = (b\sin t)/R, \cos\alpha = (b\cos t)/(b^2\cos^2 t + a^2\sin^2 t)^{1/2}, \\ &\quad R = (b/a)(a^2\sin^2 t + b^2\cos^2 t)^{1/2}, \end{aligned} \tag{6.3}$$

and for the spherical part, we have

$$R = b. \tag{6.4}$$

Next, for (2.8), we can obtain

$$\begin{aligned} h &= (3/8)\int_{\alpha=0}^{\pi} R\sin^3\alpha\, d\alpha = (3/8)\left[\int_{\alpha=0}^{\pi/2} R\sin^3\alpha\, d\alpha + \int_{\alpha=\pi/2}^{\pi} R\sin^3\alpha\, d\alpha\right] \\ &= (3/8)[(b^2/4ae^3)\{-2e + (1+e^2)\log(1+e/1-e)\} + (2/3)b], \end{aligned} \tag{6.5}$$

and then on substituting the above value of h in (2.9), we have axial drag

$$\begin{aligned} F_x &= 8\pi\mu Ua(1-e^2)^{1/2}[(2/3) + (1-e^2)^{1/2}/4e^3 \\ &\quad \times\{-2 + (1+e^2)\log(1+e/1-e)\}]^{-1}. \end{aligned} \tag{6.6}$$

When $e \to 0$, it may be confirmed that we get the classical result $F_x = 6\pi\mu Ua$ for a sphere. Similarly, for transverse flow, from (2.12), we have

$$\begin{aligned} h_y &= (3/16)\int_0^{\pi} R(2\sin\alpha - \sin^3\alpha)d\alpha \\ &= (3/16)\left[\int_{\alpha=0}^{\pi/2} R(2\sin\alpha - \sin^3\alpha)d\alpha + \int_{\alpha=\pi/2}^{\pi} R(2\sin\alpha - \sin^3\alpha)d\alpha\right] \\ &= (3/16)[(4/3) + (1-e^2)^{1/2}/4e^3\{2e + (3e^2-1)\log(1+e/1-e)\}]. \end{aligned} \tag{6.7}$$

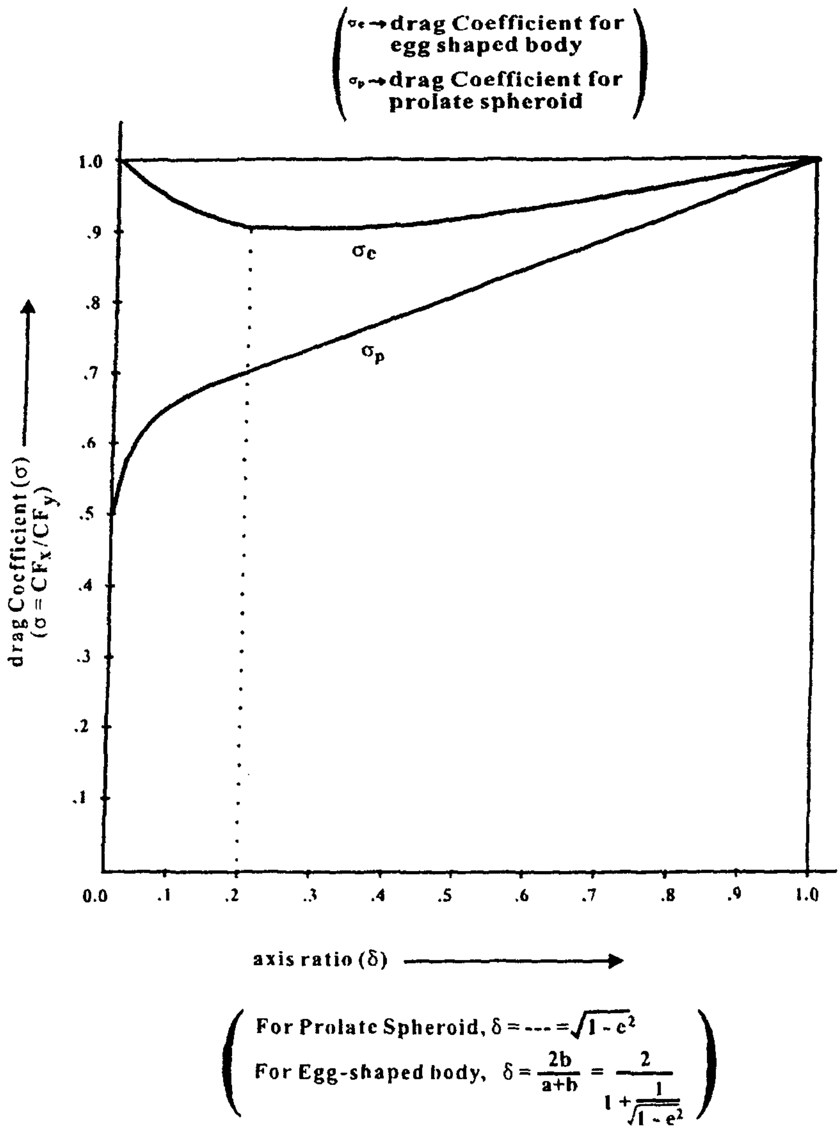

Figure 3. Effect of axis ratio δ on drag coefficient ratio σ.

On substituting the above h_y in (2.13), we get the transverse drag force

$$F_y = (16\pi\mu Ua)(1-e^2)^{1/2}[(4/3)+(1-e^2)^{1/2}/4e^3 \times\{2e+(3e^2-1)\log(1+e/1-e)\}]^{-1}. \tag{6.8}$$

When $e \to 0$, it may be confirmed that we get the classical result for a sphere. Since Stokes flow does not distinguish between fore and aft flow patterns, the same results will be obtained when the direction of flow is reversed.

It is instructive to analyse figure 3 depicting the ratio $\sigma = F_x/F_y$ for a prolate spheroid and an egg-shaped body. It is seen that while σ steadily increases with the axis ratio δ (for prolate spheroid, $\delta = \sqrt{(1-e^2)}$ and for egg-shaped body $\delta = 2/\{1 + 1/\sqrt{(1-e^2)}\}$ for the former from the value 0.5 at $\delta = 0$, corresponding to a needle, to its maximum value 1 for a sphere now for the latter it begins with the maximum value 1, for a needle decreasing to its lowest value 0.90 at $\delta = 0.2$ and then rises again to the maximum value 1 for a sphere. This discrepancy indicates that the results for the egg-shaped body are not tenable for small values of δ (at least up to $\delta = 0.2$). The cause of this discrepancy may be due to the tremendous lack of fore and aft symmetry when δ is small. It is worth noting that not only the ratio b/a of right and left axial lengths is very large, the ratio of corresponding curvatures is also large.

7. Moment on a rotating body

The moment on a sphere of radius b with angular velocity Ω is given by the formula [4]

$$M = 6\pi\mu b^3 \int_{\alpha=0}^{\pi} \sin^3\alpha\, d\alpha = \int_0^{\pi} dm(\text{say}), \tag{7.1}$$

where

$$dm = 6\pi\mu b^3 \sin^3\alpha\, d\alpha. \tag{7.2}$$

Comparing it with the elemental force df (with R replaced by b) as given by (2.7) and keeping in mind that two forces constitute a couple, we have

$$dm = (1/2)(3/4)(b^2\Omega/U)df = (2/3)a^2(1-e^2)(\Omega/U)df. \tag{7.3}$$

Prolate spheroid

Since the moment of a force is linearly related to the force, the relation (7.3) enables us to write down at once, on making use of the result (3.5), the moment on a prolate spheroid rotating about x-axis as

$$M_x = (32/3)\pi\mu a^3 e^3 \Omega(1-e^2)[-2e + (1+e^2)L]^{-1}. \tag{7.4}$$

Oblate spheroid

In a similar fashion, the moment on an oblate spheriod rotating with angular velocity Ω about x-axis is found to be given by

$$M_x = (16/3)\pi\mu a^3 e^3 \Omega[e(1-e^2)^{1/2} - (1-2e^2)\sin^{-1}e]^{-1}. \tag{7.5}$$

The above results (7.4), (7.5) are in agreement with classical results in the literature [4].

Cycloidal body

In a similar fashion, the moment on the cycloidal body (5.1) rotating with angular velocity Ω about x-axis is found to be given by

$$M_x = (1024/9)\mu\Omega a^3. \tag{7.6}$$

Again for the cycloidal body (5.7), the moment is found to be given by

$$M_x = (64\mu\Omega\pi^5 a^3)/(3\pi^2 + 16). \tag{7.7}$$

Egg-shaped body

Next, the moment on an egg-shaped body rotating with angular velocity Ω about x-axis is found to be given by

$$M_x = (16/3)\mu\Omega\pi a^3(1-e^2)^{3/2})[(2/3)+\sqrt{(1-e^2)}/4e^3(-2e+(1+e^2)L]^{-1}, \tag{7.8}$$

where L is given by (3.4).

Deformed sphere

In the end, the moment on a deformed sphere rotating with angular velocity Ω about x-axis is found by using (7.3) and (4.6) as

$$M_x = 4\pi\mu\Omega a^3\{1+\varepsilon(3d_0-(6/5)d_2)+o(\varepsilon^2)\}. \tag{7.9}$$

For $\varepsilon \to 0$, we get the classical result, $4\pi\mu\Omega a^3$, for a sphere with radius a and rotating with angular velocity Ω.

8. Conclusion

The simple drag formula (2.9) for axial free stream providing exact results for a spheroid may be fortuitous but the formula (2.13) derived by the help of (2.9), giving exact result for the non-axisymmetric situation cannot be dismissed as due to chance, Thus, it may be conjectured that the formulas do provide approximations to Stokes drag for axisymmetric bodies. These are, of course, subject to restrictions on the geometry of the meridional body profile $y(x)$ of continuously turning tangent implying that $y'(x)$ is continuous $y''(x) \neq 0$, thereby avoiding corners and straight line portions.

Since both axial and transverse flows have been considered in a free stream results of the force at an oblique angle of attack may be resolved into its components to get the required result. Flow with paraboloidal free stream has been represented through average velocity [2]. This may be exploited to generate a drag formula for axial symmetric bodies for more complex flows.

Acknowledgement

The authors are thankful to the referee for his suggestions.

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