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$\xi - \zeta$ relation

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Abstract. In this note we prove a relation between the Riemann Zeta function, ζ and the ξ function (Krein spectral shift) associated with the harmonic oscillator in one dimension. This gives a new integral representation of the zeta function and also a reformulation of the Riemann hypothesis as a question in $L^1(\mathbb{R})$.

Keywords. Krein spectral shift; Riemann zeta.

1. Introduction

Inverse spectral theory in one dimension involves recovering a Schrödinger operator from the knowledge of spectrum and a spectral function as done by Gelfand–Levitan in the fifties. In recent years there is a great deal of progress achieved in parametrizing iso-spectral classes of potentials (see the reviews of Simon [12], Gesztesy [4], and the papers of Levitan [11], Kotani-Krishna [7], Craig [2] and Sodin-Yuditskii [14] for Schrödinger operators). One of the consequences of a general formulation obtained using the Krein spectral shift function by Gesztesy-Simon [5] is given in this paper.

The Riemann zeta function is a well studied object, for example, Titch-march [15] gives a detailed exposition of this function. There are several expressions for ζ, and in this note we present an integral representation for ζ, that comes from the Krein spectral shift formula of Krein [8, 9]. Recently Gesztesy-Simon [5] generalized the trace formulae for Schrödinger operators using the Krein spectral shift function, which they named the ξ function, as it is central to inverse spectral theories in one dimension and had several important applications in spectral theories of operators in one dimension. This work used the proof of the Krein formula, given in Simon [12], theorem I.10 and its generalizations. A proof of the formula for a slightly larger class is shown in Mohapatra-Sinha [13]. We refer to these papers for the history and other work on the Krein spectral shift function.

Finally we note that the reformulation we obtain for the Riemann hypothesis as a closure problem in the space $L^1(\mathbb{R})$. Though, via the powerful Wiener's theorem, the verification only requires exhibiting a single function g_σ (for each $\sigma \in (1/2, 1)$) to lie in an explicit subspace X_σ.

Beurling provided an equivalent condition earlier in his paper [1], (also see Donoghue [3]) which reformulates the Riemann hypothesis as a completeness problem in $L^2(0, 1)$. Lee Jungseob [10] gave another such reformulation.

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2. ξ function of the harmonic oscillator

In this section we recall the ξ function of the harmonic oscillator (from Gesztesy-Simon [5]).

We consider the harmonic oscillator, $H = 1/2(-(\mathrm{d}^2/\mathrm{d}x^2) + x^2 + 1)$, acting on $L^2(\mathbb{R})$ with the set $C_0^\infty(\mathbb{R})$ its domain of essential self-adjointness and normalized so that its spectrum is the positive integers $\mathbb{Z}^+$. We consider the operator $H_{\infty,x}$ defined on $L^2((-\infty,x)) + L^2((x,\infty))$ as H together with the Dirichlet condition $f(x) = 0$ at x, using the notation of [5, 6, 12]. We denote by H_∞ the operator $H_{\infty,0}$ in the following. Then the spectrum of H_∞ is even integers $2\mathbb{Z}^+$, with uniform multiplicity 2. The Krein spectral shift function $\xi(\lambda)$ for the pair of operators (H, H_∞) is given by

$$\xi(\lambda) = \sum_{m=1}^{\infty} \chi_{[2m-1,2m)}(\lambda),$$

where χ_X denotes the indicator function of X. In terms of the ξ function, Gesztesy-Simon [5], Simon [12] theorem I.10 (case $\alpha = \infty$), and Mohapatra-Sinha theorem 4.2, proved the trace formula,

$$\operatorname{Tr}(f(H) - f(H_\infty)) = -\int (f(\lambda))' \xi(\lambda)\,\mathrm{d}\lambda \tag{1}$$

with different smoothness and decay conditions on f. Fix $s = \sigma + it$ and consider the smooth function $f(\lambda) = \lambda^{-s}$, on $[1,\infty)$, and zero in $(-\infty, 0)$, then by functional calculus, it follows that,

$$f(H) = \int f(\lambda)\mathrm{d}E_H(\lambda), \quad \text{and} \quad f(H_\infty) = \int f(\lambda)\mathrm{d}E_{H_\infty}(\lambda)$$

are both trace class for $\sigma > 2$, since

$$\sum_{m=1}^{\infty} m^{-\sigma} \quad \text{and} \quad \sum_{m=1}^{\infty} (2m)^{-\sigma}$$

converge.

Here is the primary relation between the ξ and ζ functions.

Theorem 2.1. *Let $\xi(\lambda)$ denote the Krein spectral shift function for the pair of operators (H, H_∞), defined above. Then the Riemann zeta function $\zeta(s)$ is related to ξ through the relation,*

$$(1 - 2^{(1-s)})\zeta(s) = s\int_1^\infty \lambda^{-s-1}\xi(\lambda)\mathrm{d}\lambda \tag{2}$$

valid for any $s = \sigma + it$, with $\sigma > 0$.

Proof. We consider $s = \sigma + it$ with $\sigma > 2$, then we take $f(x) = x^{(-s)}$. Then by definition $\zeta(s) = \mathrm{Trace}(f(H))$. Now we rewrite this as

$$\zeta(s) = \mathrm{Trace}\,(f(H) - f(H_\infty)) + \mathrm{Trace}\,(f(H_\infty))$$

by the linearity of the trace. Now we notice that since the spectral multiplicity of H_∞ is 2 and the spectrum is the even integers we have

$$\text{Trace}\,(f(H_\infty)) = 2\sum_{n=1}^{\infty}(2n)^{(-s)} = 2^{(1-s)}\sum_{n=1}^{\infty}(n)^{(-s)} = 2^{(1-s)}\zeta(s).$$

Using the above relations, and the trace formula in terms of the Krein spectral shift, we immediately see that for $\sigma > 2$, the theorem is valid, since the function $f(\lambda) = \lambda^{-s}$ is C^2 and satisfies $(1 + |\lambda|^2)f^{(j)}(\lambda) \in L^2(\mathbb{R}^+), j = 1, 2$, while its extension to $\sigma > 0$ follows from the analyticity of the left and right hand sides of eq. (2).

A simple change of variables $\ln \lambda = x$ in the expression for ζ given in the above theorem gives the following corollary. In the following we take,

$$\phi(x) = \sum_{n=1}^{\infty}\chi_{[\ln(2n-1),\ln(2n))}(x).$$

COROLLARY 2.2

The zeta function is given, in the region $\sigma > 0, s = \sigma + it$, by

$$\zeta(s) = \frac{s}{(1-2^{1-s})}\int_{0.}^{\infty} \mathrm{e}^{-sx}\phi(x)\,\mathrm{d}x.$$

Remark 1. Once the above relation is written down it is trivial to prove directly from the definition of the ζ function, using integration by parts, but the connection with the spectral problem is essential for the next part of the remark.

2. We could have considered the Dirichlet operator associated with any point $x \in \mathbb{R}$, in which case we can define a new family of functions $\zeta(x, s)$, given by

$$\zeta(x,s) = \text{Tr}\,(f(H) - f(H_{\infty,x})) = s\int_{1}^{\infty}\frac{1}{\lambda^{s+1}}\xi(\lambda,x)\mathrm{d}\lambda,$$

where $\xi(\lambda, x) = \sum_{n=1}^{\infty}\chi_{[n,\mu_n(x)]}(\lambda)$ is the ξ function associated with the pair $H, H_{\infty,x}$ and as before we first define the sum on the right hand side via the integrals for $\sigma > 2$ and then extend them to $\sigma > 0$. The expression for the right hand side agrees with the sum. Then the non-constant points of discontinuity of $\zeta(x, s)$ satisfy a differential equation in x, called the Dubrovin equation (familiar in inverse spectral theory, see [2, 5, 9, 14]),

$$\frac{\mathrm{d}\mu_i(x)}{\mathrm{d}x} = \sigma_i(x)\left(\frac{\partial g_\lambda(x,x)}{\partial\lambda}\bigg|_{\lambda=\mu_i(x)}\right)^{-1}, \quad i = 1, 2, 3, \ldots$$

where $g_\lambda(x, x) = \lim_{\epsilon\to 0} G(\lambda + i\epsilon, x, x)$, G being the Green function associated with H. This differential equation gives a curve in the space of analytic functions on $\mathbb{C}$ with the zeta function (times the factor $(1 - 2^{1-s})/s$) as the initial value. An explicit analysis of the equation should provide a new tool to study the zeta function.

Once we have the above expression for ζ, we can use the Wiener's L^1 Tauberian theorem (see Wiener [16], § 14, Theorem 9) to obtain the following reformulation of the Riemann hypothesis.

PROPOSITION 2.3

Consider the function $f_\sigma(\lambda) = \mathrm{e}^{-\sigma\lambda}\phi(\lambda)$. Fix, $1 > \sigma > 1/2$. Let X_{f_σ} denote the subspace generated by finite linear combinations of the translates of f_σ. Then, $\hat{f}_\sigma$ is zero free if and only if $X_{f_\sigma} = L^1(\mathbb{R})$.

Proof. The expression for ζ is given by the above corollary as

$$\frac{(1-2^{1-\sigma-it})}{\sigma+it}\zeta(\sigma+it)=\int_0^{\infty} e^{-it\lambda} f_\sigma(\lambda)d\lambda.$$

Since $1/2<\sigma<1$, it is clear by inspection that the right hand side integral vanishes for any t if and only if $\sigma+it$ is a zero of the function ζ. On the other hand for a fixed σ, the right hand side of the above equation is just $\hat{f}_\sigma(t)$. Wiener's theorem gives a precise condition for the Fourier transform of a function to be zero free, which when applied yields the result.

Theorem 2.4. *Let f_σ and X_{f_σ} be as in the above Proposition. Then the Riemann hypothesis is valid if and only if for each $\sigma \in (1/2,1)$, there is a function g_σ with zero free Fourier transform such that $g_\sigma \in X_{f_\sigma}$.*

Proof. Suppose σ is such that $\hat{f}_\sigma(t_0)=0$, but there is a $g_\sigma \in L^1(\mathbb{R})$ such that $\hat{g}_\sigma(t)\neq 0$ for any $t\in\mathbb{R}$ but $g_\sigma \in X_{f_\sigma}$. Then since $g_\sigma\in X_\sigma$, we can find for any $\epsilon>0$, complex numbers $c_1,\ldots,c_{n(\epsilon)}$ and real numbers $x_1,\ldots,x_{n(\epsilon)}$ such that

$$\left\|\sum_{i=1}^{n} c_n f_\sigma(\cdot - x_i) - g_\sigma(\cdot)\right\|_1 < \epsilon.$$

But

$$|\hat{g}_\sigma(t_0)| = |\hat{g}_\sigma(t_0) - \sum_{i=1}^{n} c_n e^{it_0x_i}\hat{f}_\sigma(t_0)| \tag{3}$$

$$\leq \left\|\sum_{i=1}^{n} c_n f_\sigma(\cdot - x_i) - g_\sigma(\cdot)\right\|_1 < \epsilon. \tag{4}$$

The epsilon being arbitrary, it follows that $g_\sigma(t_0)=0$, gives a contradiction. Therefore $\hat{f}_\sigma$ is zero free under the assumption of the theorem. Now the equivalence follows from Wiener's Tauberian theorem.

Since the Fourier transform of the convolution of two L^1 functions is the product of their Fourier transforms it is obvious that in the above theorem we could replace f_σ by its convolution with any integrable function of zero free Fourier transform. This we state as a corollary.

COROLLARY 2.5

Let f_σ and X_{f_σ} be defined as in the above theorem. Then the Riemann hypothesis is valid if and only if for each $\sigma\in(1/2,1)$, there is a pair of functions h_σ, g_σ (not necessarily distinct), with zero free Fourier transforms such that $g_\sigma \in X_{f_\sigma * h_\sigma}$.

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