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Existence of solutions of neutral functional integrodifferential equation in Banach spaces

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Abstract. In this paper we prove the existence of mild solutions for neutral functional integrodifferential equation in a Banach space. The results are obtained by using the Schaefer fixed-point theorem.

Keywords. Neutral integrodifferential equation; Schaefer fixed point theorem.

1. Introduction

The theory of neutral delay differential equations has been extensively studied in the literature [1–3, 6, 8]. Recently Hernandez and Henriquez [4] obtained some existence results for neutral functional differential equations in Banach spaces. In [5] they have established the existence of periodic solutions for the same kind of equations. In both papers they have used the semigroup theory and the Sadovski fixed point principle.

The purpose of this paper is to prove the existence of mild solutions for neutral functional integrodifferential equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - g(t, x_t)] = Ax(t) + \int_0^t f(s, x_s)\mathrm{d}s, \quad t \in J = [0, b],$$
$$x_0 = \phi, \quad \text{on } [-r, 0], \tag{1}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in $X, f : J \times C \to X$ and $g : J \times C \to X$ are continuous functions. Here $C = C([-r, 0], X)$ is the Banach space of all continuous functions $\phi : [-r, 0] \to X$ endowed with the norm $\|\phi\| = \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}$. Also for $x \in C([-r, b], X)$ we have $x_t \in C$ for $t \in [0, b]$, $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$.

2. Preliminaries

In order to define the concept of mild solution for (1), by comparison with the abstract Cauchy problem

$$x'(t) = Ax(t) + f(t),$$

whose properties are well-known [9], we associate problem (1) to the integral equation

$$x(t) = \lambda T(t)[\phi(0) - g(0, \phi)] + \lambda g(t, x_t) + \lambda \int_0^t AT(t - s)g(s, x_s)\mathrm{d}s$$
$$+ \lambda \int_0^t T(t - s) \int_0^s f(\tau, x_\tau)\mathrm{d}\tau\, \mathrm{d}s, \quad t \in [0, b]. \tag{2}$$

DEFINITION

A function $x : (-r, b) \to X$, $b > 0$, is called a mild solution of the Cauchy problem (1) if $x_0 = \phi$; the restriction of $x(.)$ to the interval $[0, b)$ is continuous and for each $0 \leq t < b$ the function $AT(t-s)F(s, x_s)$, $s \in [0, t)$, is integrable and the integral equation (2) is satisfied.

We need the following fixed point theorem due to Schaefer [10].

Schaefer Theorem. *Let S be a convex subset of a normed linear space E and $0 \in S$. Let $F : S \to S$ be a completely continuous operator and let*

$$\zeta(F) = \{x \in S;\ x = \lambda Fx \quad \textit{for some } 0 < \lambda < 1\}.$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

Assume that:

(i) A is the infinitesimal generator of a compact semigroup of bounded linear operators $T(t)$ in X such that

$$|T(t)| \leq M_1, \quad \text{for some } M_1 \geq 1 \quad \text{and } |AT(t)| \leq M_2, M_2 \geq 0.$$

(ii) For each $t \in J$ the function $f(t, .) : C \to X$ is continuous and for each $x \in C$ the function $f(., x) : J \to X$ is strongly measurable.

(iii) For every positive integer k there exists $\alpha_k \in L^1(0, b)$ such that

$$\sup_{\|x\| \leq k} |f(t, x)| \leq \alpha_k(t), \quad \text{for } t \in J \text{ a.e.}$$

(iv) The function g is completely continuous and such that the operator

$$G : C([-r, 0], X) \to C([0, b], X)$$

defined by $(G\phi)(t) = g(t, \phi)$ is compact.

(v) There exists constants $c_1 < 1$ and $c_2 > 0$ such that

$$|g(t, \phi)| \leq c_1 \|\phi\| + c_2, \quad t \in J, \quad \phi \in C.$$

(vi) There exists an integrable function $m : [0, b] \to [0, \infty)$ such that

$$|f(t, \phi)| \leq m(t)\Omega(\|\phi\|), \quad 0 \leq t \leq b, \quad \phi \in C,$$

where $\Omega : [0, \infty) \to (0, \infty)$ is a continuous non-decreasing function.

(vii)

$$\int_0^b \hat{m}(s)\mathrm{d}s < \int_c^\infty \frac{\mathrm{d}s}{s + \Omega(s)},$$

where

$$c = \frac{1}{1 - c_1}[M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_2 + M_2 c_2 b],$$

and

$$\hat{m}(t) = \max\{M_2 c_1/(1 - c_1), M_1 m(t)/M_2 c_1\}.$$

3. Main result

Theorem. *If the assumptions (i) to (vii) are satisfied, then the problem (1) has a mild solution on $[-r, b]$.*

Proof. To prove the existence of a mild solution of (1) we apply Schaefer theorem. First we obtain *a priori* bounds for the solutions of the problem (3) as in [7]

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)-\lambda g(t,x_t)]=\lambda Ax(t)+\lambda\int_0^t f(s,x_s)\mathrm{d}s,\quad \lambda\in(0,1)\, t\in J=[0,b],$$
$$x_0=\phi. \tag{3}$$

Let x be a mild solution of the problem [2]. From

$$x(t)=\lambda T(t)[\phi(0)-g(0,\phi)]+\lambda g(t,x_t)+\lambda\int_0^t AT(t-s)g(s,x_s)\mathrm{d}s$$
$$+\lambda\int_0^t T(t-s)\int_0^s f(\tau,x_\tau)\mathrm{d}\tau\,\mathrm{d}s,$$

we have

$$|x(t)|\leq M_1[\|\phi\|+c_1\|\phi\|+c_2]+c_1\|x_t\|+c_2+M_2\int_0^t(c_1\|x_s\|+c_2)\mathrm{d}s$$
$$+M_1\int_0^t\int_0^s m(\tau)\Omega(\|x_\tau\|)\mathrm{d}\tau\,\mathrm{d}s.$$

We consider the function μ given by

$$\mu(t)=\sup\{|x(s)|:-r\leq s\leq t\},\quad 0\leq t\leq b.$$

Let $t^*\in[-r,t]$ be such that $\mu(t)=|x(t^*)|$. If $t^*\in[0,b]$ by the previous inequality we have

$$\mu(t)\leq M_1[\|\phi\|+c_1\|\phi\|+c_2]+c_1\mu(t)+c_2+M_2c_1\int_0^{t^*}\mu(s)\mathrm{d}s$$
$$+M_2c_2b+M_1\int_0^{t^*}\int_0^s m(\tau)\Omega(\mu(\tau))\mathrm{d}\tau\mathrm{d}s$$
$$\leq M_1[\|\phi\|+c_1\|\phi\|+c_2]+c_1\mu(t)+c_2+M_2c_1\int_0^t\mu(s)\mathrm{d}s$$
$$+M_2c_2b+M_1\int_0^t\int_0^s m(\tau)\Omega(\mu(\tau))\mathrm{d}\tau\mathrm{d}s$$

or

$$\mu(t)\leq\frac{1}{1-c_1}\Big\{M_1[\|\phi\|+c_1\|\phi\|+c_2]+c_2+M_2c_2b$$
$$+M_2c_1\int_0^t\mu(s)\mathrm{d}s+M_1\int_0^t\int_0^s m(\tau)\Omega(\mu(\tau))\mathrm{d}\tau\mathrm{d}s\Big\}.$$

If $t^*\in[-r,0]$ then $\mu(t)=\|\phi\|$ and the previous inequality holds since $M_1\geq 1$.

Denoting by $v(t)$ the right-hand side of the above inequality we have

$$c=v(0)=\frac{1}{1-c_1}\{M_1[\|\phi\|+c_1\|\phi\|+c_2]+c_2+M_2c_2b\},$$

$\mu(t)\leq v(t), 0\leq t\leq b$ and

$$v'(t)=\frac{1}{1-c_1}M_2c_1\mu(t)+\frac{1}{1-c_1}M_1\int_0^t m(\tau)\Omega(\mu(\tau))\mathrm{d}\tau$$

$$\leq \frac{1}{1-c_1} M_2 c_1 v(t) + \frac{1}{1-c_1} M_1 \int_0^t m(\tau)\Omega(v(\tau))\mathrm{d}\tau$$
$$\leq \frac{1}{1-c_1} M_2 c_1 \left\{ v(t) + \frac{M_1}{M_2 c_1} \int_0^t m(\tau)\Omega(v(\tau))\mathrm{d}\tau \right\}.$$

Let

$$w(t) = v(t) + \frac{M_1}{M_2 c_1} \int_0^t m(\tau)\Omega(v(\tau))\mathrm{d}\tau.$$

Then $w(0) = v(0)$, $v(t) \leq w(t)$, and

$$w'(t) = v'(t) + \frac{M_1}{M_2 c_1} m(t)\Omega(v(t))$$
$$\leq \frac{1}{1-c_1} M_2 c_1 w(t) + \frac{M_1}{M_2 c_1} m(t)\Omega(w(t))$$
$$\leq \hat{m}(t)\{w(t) + \Omega(w(t))\}.$$

This implies

$$\int_{w(0)}^{w(t)} \frac{\mathrm{d}s}{s+\Omega(s)} \leq \int_0^b \hat{m}(s)\mathrm{d}s < \int_c^\infty \frac{\mathrm{d}s}{s+\Omega(s)}, \quad 0 \leq t \leq b.$$

This inequality implies that there is a constant K such that $v(t) \leq K$, $t \in [0,b]$ and hence $\mu(t) \leq K$, $t \in [0,b]$, $\|x_t\| \leq \mu(t)$, we have

$$\|x\|_1 = \sup\{|x(t)| : -r \leq t \leq b\} \leq K,$$

where K depends only on b and on the functions m and Ω.

In the second step we rewrite the problem (1) as follows. For $\phi \in C$ define $\hat{\phi} \in C_b$, $C_b = C([-r,b],X)$ by

$$\hat{\phi}(t) = \begin{cases} \phi(t), & -r \leq t \leq 0, \\ T(t)\phi(0), & 0 \leq t \leq b. \end{cases}$$

If $x(t) = y(t) + \hat{\phi}(t)$, $t \in [-r,b]$, it is easy to see that y satisfies

$$y_0 = 0$$
$$y(t) = -T(t)g(0,\phi) + g(t, y_t + \hat{\phi}_t) + \int_0^t AT(t-s)g(s, y_s + \hat{\phi}_s)\mathrm{d}s$$
$$+ \int_0^t T(t-s)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\mathrm{d}\tau\,\mathrm{d}s$$

if and only if x satisfies

$$x(t) = T(t)[\phi(0) - g(0,\phi)] + g(t,x_t) + \int_0^t AT(t-s)g(s,x_s)\mathrm{d}s$$
$$+ \int_0^t T(t-s)\int_0^s f(\tau, x_\tau)\mathrm{d}\tau\,\mathrm{d}s$$

and $x_0 = \phi$.

Define $C_b^0 = \{y \in C_b : y_0 = 0\}$ and $F : C_b^0 \to C_b^0$, by

$$(Fy)(t) = 0, \quad -r \le t \le 0$$

$$(Fy)(t) = -T(t)g(0,\phi) + g(t, y_t + \hat{\phi}_t) + \int_0^t AT(t-s)g(s, y_s + \hat{\phi}_s)\mathrm{d}s + \int_0^t T(t-s)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\mathrm{d}\tau\,\mathrm{d}s, \quad 0 \le t \le b.$$

It will now be shown that F is a completely continuous operator.

Let $B_k = \{y \in C_b^0 : \|y\|_1 \le k\}$ for some $k \ge 1$. We first show that F maps B_k into an equicontinuous family. Let $y \in B_k$ and $t_1, t_2 \in [0, b]$. Then if $0 < t_1 < t_2 < b$,

$$\begin{aligned}
|(Fy)(t_1) - (Fy)(t_2)| &\le |T(t_1) - T(t_2)||g(0,\phi)| + |g(t_1, y_{t_1} + \hat{\phi}_{t_1}) - g(t_2, y_{t_2} + \hat{\phi}_{t_2})| \\
&\quad + \left| \int_0^{t_1} A[T(t_1 - s) - T(t_2 - s)]g(s, y_s + \hat{\phi}_s)\mathrm{d}s \right| \\
&\quad + \left| \int_{t_1}^{t_2} AT(t_2 - s)g(s, y_s + \hat{\phi}_s)\mathrm{d}s \right| \\
&\quad + \left| \int_0^{t_1} [T(t_1 - s) - T(t_2 - s)] \int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\mathrm{d}\tau\,\mathrm{d}s \right| \\
&\quad + \left| \int_{t_1}^{t_2} T(t_2 - s) \int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\mathrm{d}\tau\,\mathrm{d}s \right| \\
&\le |T(t_1) - T(t_2)||g(0,\phi)| + |g(t_1, y_{t_1} + \hat{\phi}_{t_1}) - g(t_2, y_{t_2} + \hat{\phi}_{t_2})| \\
&\quad + \int_0^{t_1} |A[T(t_1 - s) - T(t_2 - s)]|(c_1\|y_s + \hat{\phi}_s\| + c_2)\mathrm{d}s \\
&\quad + \int_{t_1}^{t_2} |AT(t_2 - s)|(c_1\|y_s + \hat{\phi}_s\| + c_2)\mathrm{d}s| \\
&\quad + \int_0^{t_1} |T(t_1 - s) - T(t_2 - s)| \int_0^s \alpha_{k'}(\tau)\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \int_{t_1}^{t_2} |T(t_2 - s)| \int_0^s \alpha_{k'}(\tau)\mathrm{d}\tau\,\mathrm{d}s,
\end{aligned}$$

where $k' = k + \|\hat{\phi}\|$. The right hand side is independent of $y \in B_k$ and tends to zero as $t_2 - t_1 \to 0$, since g is completely continuous and the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology. Thus F maps B_k into an equicontinuous family of functions.

Notice that we considered here only the case $0 < t_1 < t_2$, since the other cases $t_1 < t_2 < 0$ or $t_1 < 0 < t_2$ are very simple.

It is easy to see that the family FB_k is uniformly bounded. Next, we show $\overline{FB_k}$ is compact. Since we have shown FB_k is an equicontinuous collection, it suffices by the Arzela–Ascoli theorem to show that F maps B_k into a precompact set in X.

Let $0 < t \le b$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $y \in B_k$ we define

$$(F_\epsilon y)(t) = -T(t)g(0,\phi) + g(t, y_t + \hat{\phi}_t) + \int_0^{t-\epsilon} AT(t-s)g(s, y_s + \hat{\phi}_s)\mathrm{d}s + \int_0^{t-\epsilon} T(t-s)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\mathrm{d}\tau\,\mathrm{d}s$$

$$= -T(t)g(0,\phi) + g(t, y_t + \hat{\phi}_t) + T(\epsilon)\int_0^{t-\epsilon} AT(t-s-\epsilon)g(s, y_s + \hat{\phi}_s)\mathrm{d}s$$

$$+ T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\mathrm{d}\tau\,\mathrm{d}s.$$

Since $T(t)$ is a compact operator, the set $Y_\epsilon(t) = \{(F_\epsilon y)(t) : y \in B_k\}$ is precompact in X for every ϵ, $0 < \epsilon < t$. Moreover for every $y \in B_k$ we have

$$|(Fy)(t) - (F_\epsilon y)(t)| \leq \int_{t-\epsilon}^t |AT(t-s)g(s, y_s + \hat{\phi}_s)|\mathrm{d}s$$

$$+ \int_{t-\epsilon}^t |T(t-s)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\mathrm{d}\tau|\mathrm{d}s$$

$$\leq \int_{t-\epsilon}^t |AT(t-s)g(s, y_s + \hat{\phi}_s)|\mathrm{d}s$$

$$+ \int_{t-\epsilon}^t |T(t-s)| \int_0^s \alpha_{k'}(\tau)\mathrm{d}\tau\,\mathrm{d}s.$$

Therefore there are precompact sets arbitrarily close to the set $\{(Fy)(t) : x \in B_k\}$. Hence the set $\{(Fy)(t) : y \in B_k\}$ is precompact in X.

It remains to show that $F : C_b^0 \to C_b^0$ is continuous. Let $\{y_n\}_0^\infty \subseteq C_b^0$ with $y_n \to y$ in C_b^0. Then there is an integer r such that $\|y_n(t)\| \leq r$ for all n and $t \in J$, so $y_n \in B_r$ and $y \in B_r$. By (iii), $f(t, y_{n_t} + \hat{\phi}_t) \to f(t, y_t + \hat{\phi}_t)$ for each $t \in J$ and since $|f(t, y_{n_t} + \hat{\phi}_t) - f(t, y_t + \hat{\phi}_t|$ $\leq 2\alpha_{r'}(t)$, $r' = r + \|\hat{\phi}\|$ and also g is completely continuous, we have by dominated convergence theorem

$$\|Fy_n - Fy\| = \sup_{t\in J} |[g(t, y_{n_t} + \hat{\phi}_t) - g(t, y_t + \hat{\phi}_t)]$$

$$+ \int_0^t AT(t-s)[g(s, y_{n_s} + \hat{\phi}_s) - g(s, y_s + \hat{\phi}_s)]\mathrm{d}s$$

$$+ \int_0^t T(t-s)\int_0^s [f(\tau, y_{n_\tau} + \hat{\phi}_\tau) - f(\tau, y_\tau + \hat{\phi}_\tau)]\mathrm{d}\tau\,\mathrm{d}s|$$

$$\leq |g(t, y_{n_t} + \hat{\phi}_t) - g(t, y_t + \hat{\phi}_t)|$$

$$+ \int_0^b |AT(t-s)||g(s, y_{n_s} + \hat{\phi}_s) - g(s, y_s + \hat{\phi}_s)|\mathrm{d}s$$

$$+ \int_0^b |T(t-s)| \int_0^s |f(\tau, y_{n_\tau} + \hat{\phi}_\tau) - f(\tau, y_\tau + \hat{\phi}_\tau)|\mathrm{d}\tau\,\mathrm{d}s \to 0.$$

Thus F is continuous. This completes the proof that F is completely continuous.

Finally the set $\zeta(F) = \{y \in C_b^0 : y = \lambda Fy, \lambda \in (0,1)\}$ is bounded, since for every solution y in $\zeta(F)$ the function $x = y + \hat{\phi}$ is a mild solution of (3), for which we have proved that $\|x\|_1 \leq K$ and hence

$$\|y\|_1 \leq K + \|\hat{\phi}\|.$$

Consequently by Schaefer's theorem the operator F has a fixed point in C_b^0. This means that the problem (1) has a mild solution.

4. Example

Consider the following partial integrodifferential equation of the form

$$\frac{\partial}{\partial t}[z(y,t) - p(s, z(y, t-r))] = \frac{\partial^2}{\partial y^2} z(y,t) + \int_0^t q(s, z(y, s-r))\mathrm{d}s,$$

$$0 \le y \le \pi, \quad t \in J \qquad (4)$$

$$z(0,t) = z(\pi, t) = 0, \quad t \ge 0$$
$$z(t,y) = \phi(y,t), \quad -r \le t \le 0$$

where ϕ is continuous. Let

$$f(t, w_t)(y) = q(t, w(t-y)), \quad 0 \le y \le \pi$$

and

$$g(t, w_t)(y) = p(t, w(t-y)).$$

Take $X = L^2[0,\pi]$ and define $A : X \to X$ by $Aw = w''$ with domain $D(A) = \{w \in X, w, w'$ are absolutely continuous, $w'' \in X$, $w(0) = w(\pi) = 0\}$. Then

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \quad w \in D(A)$$

where $w_n(s) = \sqrt{2/\pi}\sin ns$, $n = 1, 2, 3, \ldots$ is the orthogonal set of eigenvectors of A. It is well-known that A is the infinitesimal generator of an analytic semigroup $T(t)$, $t \ge 0$ in X and is given by

$$T(t)w = \sum_{n=1}^{\infty} \exp(-n^2 t)(w, w_n) w_n, \quad w \in X.$$

Since the analytic semigroup $T(t)$ is compact there exist constants $N \ge 1$ and $N_1 > 0$ such that $|T(t)| \le N$ and $|AT(t)| \le N_1$ for each $t \ge 0$.

Further the function $p : J \times [0,\pi] \to [0,\pi]$ is completely continuous and there exist constants $n_1 < 1$ and $n_2 > 0$ such that

$$\|p(t, w(t-y))\| \le n_1(\|w\|) + n_2,$$

and also there exists an integrable function $l : J \to [0, \infty)$ such that

$$\|q(t, w(t-y))\| \le l(t)\Omega_1(\|w\|),$$

where $\Omega_1 : [0,\infty) \to (0,\infty)$ is continuous and nondecreasing and

$$\int_0^b \hat{n}(s)\mathrm{d}s < \int_c^{\infty} \frac{\mathrm{d}s}{s + \Omega_1(s)},$$

where

$$c = \frac{1}{1-n_1}[N(\|\phi\| + n_1\|\phi\| + n_2) + n_2 + N_1 n_2 b]$$

and

$$\hat{n}(t) = \max\{N_1 n_1/(1-n_1), Nl(t)/N_1 n_1\}.$$

Since all the conditions of the above theorem are satisfied, the system (4) has a mild solution on $[-r, b]$.

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