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A result on the composition of distributions

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Abstract. Let F be a distribution and let f be a locally summable function. The distribution $F(f)$ is defined as the neutrix limit of the sequence $\{F_n(f)\}$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. The distribution $(x^r)^{-s}$ is evaluated for $r, s = 1, 2, \ldots$.

Keywords. Distribution; delta-function; composition of distributions; neutrix; neutrix limit.

In the following we let N be the neutrix, see [1], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n: \quad \lambda > 0,\ r = 1, 2, \ldots$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Now let $\rho(x)$ be an infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x - t)\rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

The following definition was given in [3].

DEFINITION 1

Let F be a distribution and let f be a locally summable function. We say that the distribution $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$N\text{-}\lim_{n\to\infty}\int_{-\infty}^{\infty} F_n(f(x))\varphi(x)\mathrm{d}x = \langle h(x),\varphi(x)\rangle$$

for all test functions φ with compact support contained in (a,b).

The following theorems were proved in [3] and [4] respectively:

Theorem 1. *The distributions* $(x_-^\mu)_-^\lambda$ *and* $(x_+^\mu)_-^\lambda$ *exists and*

$$(x_-^\mu)_-^\lambda = (x_+^\mu)_-^\lambda = 0$$

for $\mu > 0$ *and* $\lambda\mu \neq -1,-2,\dots$ *and*

$$(x_-^\mu)_-^\lambda = (-1)^{\lambda\mu}(x_+^\mu)_-^\lambda = \frac{\pi\operatorname{cosec}(\pi\lambda)}{2\mu(-\lambda\mu-1)!}\delta^{(-\lambda\mu-1)}(x)$$

for $\mu > 0$, $\lambda \neq -1,-2,\dots$ *and* $\lambda\mu = -1,-2,\dots$.

Theorem 2. *The distribution* $(x_+^r)_-^{-s}$ *exists and*

$$(x_+^r)_-^{-s} = \frac{(-1)^{rs+s}c(\rho)}{r(rs-1)!}\delta^{(rs-1)}(x)$$

for $r,s = 1,2,\dots,$ *where*

$$c(\rho) = \int_0^1 \ln t\,\rho(t)\,\mathrm{d}t.$$

In the previous theorem, the distribution x_-^{-s} is defined by

$$x_-^{-s} = -\frac{(\ln x_-)^{(s)}}{(s-1)!}$$

for $s = 1,2,\dots$ and not as in Gel'fand and Shilov [5].

We need the following lemmas which can be easily proved by induction:

Lemma 1. *If* φ *is an arbitrary function in* $\mathcal{D}$ *with support contained in the interval* $[-1,1]$, *then*

$$\langle x^{-r},\varphi(x)\rangle = \int_{-1}^{1} x^{-r}\left[\varphi(x) - \sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!}x^k\right]\mathrm{d}x + \sum_{k=0}^{r-1}\frac{(-1)^{r-k-1}-1}{(r-k)k!}\varphi^{(k)}(0), \tag{1}$$

for $r = 1,2,\dots$.

Lemma 2.

$$\int_{-1}^{1} v^i\rho^{(r)}(v)\mathrm{d}v = \begin{cases}0, & 0\le i<r,\\ (-1)^r r!, & i = r\end{cases} \tag{2}$$

for $r = 0,1,2,\dots$.

We now prove the following theorem.

Theorem 3. *The distribution* $(x^r)^{-s}$ *exists and*

$$(x^r)^{-s} = x^{-rs}, \tag{3}$$

for $r, s = 1, 2, \ldots$.

Proof. We first put

$$[(x^r)^{-s}]_n = (x^r)^{-s} * \delta_n(x) = \frac{(-1)^{s-1}}{(s-1)!}\int_{-1/n}^{1/n} \ln|x^r - t|\delta_n^{(s)}(t)\mathrm{d}t,$$

and note that

$$\int_{-1}^{1} x^k[(x^r)^{-s}]_n\,\mathrm{d}x = \begin{cases} 0, & rs-k \text{ odd}, \\ 2\int_0^1 x^k[(x^r)^{-s}]_n\mathrm{d}x, & rs-k \text{ even}. \end{cases} \tag{4}$$

Then

$$\begin{aligned}
(-1)^{s-1}(s-1)!\int_0^1 x^k[(x^r)^{-s}]_n\,\mathrm{d}x &= \int_0^1 x^k \int_{-1/n}^{1/n} \ln|x^r - t|\delta_n^{(s)}(t)\mathrm{d}t\,\mathrm{d}x \\
&= \int_{-1/n}^{1/n} \delta_n^{(s)}(t)\int_0^{n^{-1/r}} x^k \ln|x^r - t|\,\mathrm{d}x\,\mathrm{d}t + \int_{-1/n}^{1/n} \delta_n^{(s)}(t)\int_{n^{-1/r}}^{1} x^k \ln|x^r - t|\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{1}{r}n^{(rs-k-1)/r}\int_{-1}^{1}\rho^{(s)}(v)\int_0^1 u^{-(r-k-1)/r}\ln|(u-v)/n|\,\mathrm{d}u\,\mathrm{d}v \\
&\quad + \frac{1}{r}n^{(rs-k-1)/r}\int_{-1}^{1}\rho^{(s)}(v)\int_1^n u^{-(r-k-1)/r}\ln|(u-v)/n|\,\mathrm{d}u\,\mathrm{d}v \\
&= I_1 + I_2,
\end{aligned} \tag{5}$$

on using the substitutions $u = nx^r$ and $v = nt$.

It is easily seen that

$$\operatorname*{N-lim}_{n\to\infty} I_1 = 0, \tag{6}$$

for $k = 0, 1, \ldots, rs-2$.

Now,

$$\begin{aligned}
&\int_{-1}^{1}\rho^{(s)}(v)\int_1^n u^{-(r-k-1)/r}\ln|(u-v)/n|\,\mathrm{d}u\,\mathrm{d}v \\
&\quad = \int_{-1}^{1}\rho^{(s)}(v)\int_1^n u^{-(r-k-1)/r}[\ln|1-v/u| + \ln u - \ln n]\,\mathrm{d}u\,\mathrm{d}v \\
&\quad = \int_{-1}^{1}\rho^{(s)}(v)\int_1^n u^{-(r-k-1)/r}\ln|1-v/u|\,\mathrm{d}u\,\mathrm{d}v,
\end{aligned}$$

since $\int_{-1}^{1}\rho^{(s)}(v)\,\mathrm{d}v = 0$ for $s = 1, 2, \ldots$, by Lemma 2. Further

$$\int_{-1}^{1}\rho^{(s)}(v)\int_1^n u^{-(r-k-1)/r}\ln|1-v/u|\,\mathrm{d}u\,\mathrm{d}v$$

$$= -\sum_{i=1}^{\infty} \frac{1}{i} \int_{-1}^{1} v^i \rho^{(s)}(v) \int_{1}^{n} u^{(k+1)/r-i-1}\,du\,dv$$

$$= -\sum_{i=1}^{\infty} \frac{r[n^{(k+1)/r-i} - 1]}{i(k-ri+1)} \int_{-1}^{1} v^i \rho^{(s)}(v)\,dv,$$

and it follows that

$$N\text{-}\lim_{n\to\infty} I_2 = \frac{1}{s(rs-k-1)} \int_{-1}^{1} v^s \rho^{(s)}(v)\,dv$$

$$= \frac{(-1)^s (s-1)!}{rs-k-1}, \tag{7}$$

on using Lemma 2, for $k = 0, 1, \ldots, rs-2$.

Hence

$$N\text{-}\lim_{n\to\infty} \int_0^1 x^k[(x^r)^{-s}]_n\,dx = -\frac{1}{rs-k-1} \tag{8}$$

for $k = 0, 1, \ldots, rs-2$, on using (5), (6) and (7). Then using (4) and (8), we see that

$$N\text{-}\lim_{n\to\infty} \int_{-1}^{1} x^k[(x^r)^{-s}]_n\,dx = \frac{(-1)^{rs-k-1} - 1}{rs-k-1}, \tag{9}$$

for $k = 0, 1, \ldots, rs-1$.

When $k = rs$ (5) still holds but now we have

$$I_1 = \frac{n^{-1/r}}{r} \int_{-1}^{1} \rho^{(s)}(v) \int_0^1 u^{-(r-k-1)/r} \ln|(u-v)/n)|\,du\,dv,$$

and it follows that for any continuous function ψ

$$\lim_{n\to\infty} \int_0^{n^{-1/r}} x^{rs}[(x^r)^{-s}]\psi(x)dx = 0. \tag{10}$$

Similarly

$$\lim_{n\to\infty} \int_{-n^{-1/r}}^{0} x^{rs}[(x^r)^{-s}]\psi(x)dx = 0. \tag{11}$$

Next, when $x^r \geq 1/n$, we have

$$(-1)^{s-1}(s-1)![(x^r)^{-s}]_n = \int_{-1/n}^{1/n} \ln|x^r - t|\delta_n^{(s)}(t)dt$$

$$= n^s \int_{-1}^{1} \ln|x^r - v/n|\rho^{(s)}(v)dv$$

$$= n^s \int_{-1}^{1} \left[\ln x^r - \sum_{i=1}^{\infty} \frac{v^i}{in^i x^{ri}}\right] \rho^{(s)}(v)dv$$

$$= -\sum_{i=s}^{\infty} \int_{-1}^{1} \frac{v^i}{in^{i-s}x^{ri}}\,\rho^{(s)}(v)dv.$$

It follows that

$$|(s-1)![(x^r)^{-s}]_n| \le \sum_{i=s}^{\infty} \int_{-1}^{1} \frac{|v|^i}{in^{i-s}x^{ri}} |\rho^{(s)}(v)|\,dv \le \sum_{i=s}^{\infty} \frac{K_s}{in^{i-s}x^{ri}},$$

where

$$K_s = \int_{-1}^{1} |\rho^{(s)}(v)|\,dv,$$

for $s = 1, 2, \ldots$.

If now $n^{-1/r} < \eta < 1$, then

$$\begin{aligned}(s-1)!\int_{n^{-1/r}}^{\eta} |x^{rs}[(x^r)^{-s}]_n|\,dx &\le K_s \sum_{i=s}^{\infty} \frac{n^{s-i}}{i} \int_{n^{-1/r}}^{\eta} x^{r(s-i)}dx \\ &= K_s \sum_{i=s}^{\infty} \frac{n^{-1/r}}{ri} \int_{1}^{n\eta^r} u^{s-i+1/r-1}\,du \\ &= \begin{cases} K_s \displaystyle\sum_{i=s}^{\infty} \frac{n^{-1/r}}{ri(s-i+1/r)}[(n\eta^r)^{s-i+1/r} - 1], & r \ne 1, \\ K_s \displaystyle\sum_{i=s,\, i\ne s+1}^{\infty} \frac{n^{-1}}{i(s-i+1)}[(n\eta)^{s-i+1} - 1] + K_s \frac{n^{-1}\ln(n\eta)}{s+1}, & r = 1. \end{cases}\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \int_{n^{-1/r}} |[(x^r)^{-s}]_n|\,dx = O(\eta),$$

for $r, s = 1, 2, \ldots$.

Thus, if ψ is a continuous function

$$\lim_{n\to\infty} \left| \int_{n^{-1/r}}^{\eta} x^{rs}[(x^r)^{-s}]_n \psi(x)dx \right| = O(\eta), \tag{12}$$

for $r, s = 1, 2, \ldots$.

Similarly,

$$\lim_{n\to\infty} \left| \int_{-\eta}^{-n^{-1/r}} x^{rs}[(x^r)^{-s}]_n \psi(x)dx \right| = O(\eta), \tag{13}$$

for $r, s = 1, 2, \ldots$.

Now let $\varphi(x)$ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1, 1]$. By Taylor's theorem we have

$$\varphi(x) = \sum_{k=0}^{rs-1} \frac{x^k}{k!}\varphi^{(k)}(0) + \frac{x^{rs}}{(rs)!}\varphi^{(rs)}(\xi x)$$

where $0 < \xi < 1$. Then

$$\begin{aligned}\langle [(x^r)^{-s}]_n, \varphi(x) \rangle &= \int_{-1}^{1} [(x^r)^{-s}]_n \varphi(x)dx \\ &= \sum_{k=0}^{rs-1} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^{1} x^k [(x^r)^{-s}]_n\,dx + \int_{-n^{-1/r}}^{n^{-1/r}} \frac{x^{rs}}{(rs)!}[(x^r)^{-s}]_n \varphi^{(rs)}(\xi x)dx\end{aligned}$$

$$+\int_{n^{-1/r}}^{\eta}\frac{x^{rs}}{(rs)!}[(x^r)^{-s}]_n\varphi^{(rs)}(\xi x)\mathrm{d}x+\int_{\eta}^{1}\frac{x^{rs}}{(rs)!}[(x^r)^{-s}]_n\varphi^{(rs)}(\xi x)\mathrm{d}x$$
$$+\int_{-\eta}^{-n^{-1/r}}\frac{x^{rs}}{(rs)!}[(x^r)^{-s}]_n\varphi^{(rs)}(\xi x)\mathrm{d}x+\int_{-1}^{-\eta}\frac{x^{rs}}{(rs)!}[(x^r)^{-s}]_n\varphi^{(rs)}(\xi x)\mathrm{d}x.$$

Using (11) to (15) and noting that the sequence $[(x^r)^{-s}]_n$ converges uniformly to x^{-rs} on the intervals $[-1,-\eta]$ and $[\eta,1]$, it follows that

$$\begin{aligned}N\text{-}\lim_{n\to\infty}\langle[(x^r)^{-s}]_n,\varphi(x)\rangle &= \sum_{k=0}^{rs-1}\frac{(-1)^{rs-k-1}-1}{(rs-k-1)k!}\varphi^{(k)}(0)+O(\eta)\\ &\quad+\int_{\eta}^{1}\frac{\varphi^{(rs)}(\xi x)}{(rs)!}\,\mathrm{d}x+\int_{-1}^{-\eta}\frac{\varphi^{(rs)}(\xi x)}{(rs)!}\,\mathrm{d}x.\end{aligned}$$

Since η can be made arbitrarily small, it follows that

$$\begin{aligned}N\text{-}\lim_{n\to\infty}\langle[(x^r)^{-s}]_n,\varphi(x)\rangle &= \sum_{k=0}^{rs-1}\frac{(-1)^{rs-k-1}-1}{(rs-k-1)k!}\varphi^{(k)}(0)+\int_{-1}^{1}\frac{\varphi^{(rs)}(\xi x)}{(rs)!}\,\mathrm{d}x\\ &= \int_{-1}^{1}x^{-rs}\left[\varphi(x)-\sum_{k=0}^{rs-1}\frac{x^k}{k!}\varphi^{(k)}(0)\right]\mathrm{d}x\\ &\quad+\sum_{k=0}^{rs-1}\frac{(-1)^{rs-k-1}-1}{(rs-k-1)k!}\varphi^{(k)}(0)\\ &= \langle x^{-rs},\varphi(x)\rangle,\end{aligned}$$

on using (1). This proves (3) on the interval $[-1,1]$. However, (3) clearly holds on any interval not containing the origin, and the proof is complete.

In the corollary, the distribution $(x+i0)^{-s}$ is defined by

$$(x+i0)^{-s}=x^{-s}+\frac{i\pi(-1)^s}{(s-1)!}\delta^{(s-1)}(x), \tag{14}$$

for $s=1,2,\ldots$, see Gel'fand and Shilov [5].

COROLLARY 3.1

The distribution $(x^r+i0)^{-s}$ *exists and*

$$(x^r+i0)^{-s}=(x+i0)^{-rs} \tag{15}$$

for $r=1,3,5,\ldots$ *and* $s=1,2,\ldots$ *and*

$$(x^r+i0)^{-s}=x^{-rs}, \tag{16}$$

for $r=2,4,6,\ldots$ *and* $s=1,2,\ldots$.

Proof. It was proved in [2] that

$$\delta^{(s)}(x^r)=\frac{s!}{r(rs+r-1)!}\delta^{(rs+r-1)}(x), \tag{17}$$

for $r=1,3,5,\ldots$ and $s=0,1,2,$ and

$$\delta^{(s)}(x^r)=0, \tag{18}$$

for $r = 2, 4, 6, \ldots$ and $s = 0, 1, 2, \ldots$. Equation (15) follows immediately from (3), (14) and (17) and (16) follows immediately from (3), (14) and (18).

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