A result on the composition of distributions

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Abstract. Let $F$ be a distribution and let $f$ be a locally summable function. The distribution $F(f)$ is defined as the neutrix limit of the sequence $\{F_n(f)\}$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. The distribution $(x^r)^{-s}$ is evaluated for $r, s = 1, 2, \ldots$.

Keywords. Distribution; delta-function; composition of distributions; neutrix; neutrix limit.

In the following we let $N$ be the neutrix, see [1], having domain $N'$ the positive integers and range $N''$ the real numbers, with negligible functions which are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \ln^r n: \lambda > 0, r = 1, 2, \ldots$$

and all functions which converge to zero in the usual sense as $n$ tends to infinity.

Now let $p(x)$ be an infinitely differentiable function having the following properties:

(i) $p(x) = 0$ for $|x| \geq 1$,
(ii) $p(x) \geq 0$,
(iii) $p(x) = p(-x)$,
(iv) $\int_{-1}^{1} p(x) \, dx = 1$.

Putting $\delta_n(x) = np(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $D$ be the space of infinitely differentiable functions with compact support and let $D'$ be the space of distributions defined on $D$. Then if $f$ is an arbitrary distribution in $D'$, we define

$$f_n(x) = (f * \delta_n)(x) = (f(t), \delta_n(x - t))$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

The following definition was given in [3].

DEFINITION 1

Let $F$ be a distribution and let $f$ be a locally summable function. We say that the distribution $F(f(x))$ exists and is equal to $h$ on the open interval $(a, b)$ if
\[ N \lim_{n \to \infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)\,dx = \langle h(x), \varphi(x) \rangle \]

for all test functions \( \varphi \) with compact support contained in \((a, b)\).

The following theorems were proved in [3] and [4] respectively:

**Theorem 1.** The distributions \((x^\mu)_-^\lambda\) and \((x^\mu)_+^\lambda\) exists and

\[(x^\mu)_-^\lambda = (x^\mu)_+^\lambda = 0\]

for \( \mu > 0 \) and \( \lambda \mu \neq -1, -2, \ldots \) and

\[(x^\mu)_-^\lambda = (-1)^\lambda \mu (x^\mu)_-^\lambda = \frac{\pi \text{cosec}(\pi \lambda)}{2\mu(-\lambda \mu - 1)!} \delta^{(-\lambda \mu - 1)}(x)\]

for \( \mu > 0, \lambda \neq -1, -2, \ldots \) and \( \lambda \mu = -1, -2, \ldots \).

**Theorem 2.** The distribution \((x^\mu)_-^s\) exists and

\[(x^\mu)_-^s = \frac{(-1)^{rs+s} c(\rho)}{r(rs - 1)!} \delta^{(rs-1)}(x)\]

for \( r, s = 1, 2, \ldots, \) where

\[ c(\rho) = \int_0^1 \ln t \rho(t) \, dt. \]

In the previous theorem, the distribution \(x_-^s\) is defined by

\[ x_-^s = -\frac{(\ln x_-)^{(s)}}{(s - 1)!} \]

for \( s = 1, 2, \ldots \) and not as in Gel'fand and Shilov [5].

We need the following lemmas which can be easily proved by induction:

**Lemma 1.** If \( \varphi \) is an arbitrary function in \( D \) with support contained in the interval \([-1, 1]\), then

\[
\langle x^{-r}, \varphi(x) \rangle = \int_{-1}^{1} x^{-r} \left[ \varphi(x) - \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!} x^k \right] \, dx
\]

\[+ \sum_{k=0}^{r-1} \frac{(-1)^{r-k-1} - 1}{(r-k)k!} \varphi^{(k)}(0), \quad \text{(1)}\]

for \( r = 1, 2, \ldots \).

**Lemma 2.**

\[
\int_{-1}^{1} v^i \rho^{(r)}(v) \, dv = \begin{cases} 0, & 0 \leq i < r, \\ (-1)^r r!, & i = r \end{cases} \quad \text{(2)}
\]

for \( r = 0, 1, 2, \ldots \).
We now prove the following theorem.

**Theorem 3.** The distribution \((x^r)^{-s}\) exists and

\[(x^r)^{-s} = x^{-rs}, \tag{3}\]

for \(r, s = 1, 2, \ldots\)

**Proof.** We first put

\[[(x^r)^{-s}]_n = (x^r)^{-s} \ast \delta_n(x) = \frac{(-1)^{s-1}}{(s-1)!} \int_{-1/n}^{1/n} \ln |x^r - t| \delta_n(t) \, dt, \]

and note that

\[
\int_{-1}^{1} x^k[(x^r)^{-s}]_n \, dx = \begin{cases} 
0 & \text{for } rs - k \text{ odd,} \\
2 \int_{0}^{1} x^k[(x^r)^{-s}]_n \, dx & \text{for } rs - k \text{ even.} 
\end{cases} \tag{4}
\]

Then

\[
(-1)^{s-1}(s-1)! \int_{0}^{1} x^k[(x^r)^{-s}]_n \, dx = \int_{0}^{1} x^k \int_{-1/n}^{1/n} \ln |x^r - t| \delta_n(t) \, dt \, dx
\]

\[
= \int_{-1/n}^{1/n} \delta_n(t) \int_{0}^{1/n} x^k \ln |x^r - t| \, dx \, dt + \int_{-1/n}^{1/n} \delta_n(t) \int_{n-1/n}^{1} x^k \ln |x^r - t| \, dx \, dt
\]

\[
= \frac{1}{r} n^{(rs-k-1)/r} \int_{-1}^{1} \rho^{(s)}(v) \int_{0}^{1} u^{-(r-k-1)/r} \ln |(u - v)/n| \, du \, dv
\]

\[
+ \frac{1}{r} n^{(rs-k-1)/r} \int_{-1}^{1} \rho^{(s)}(v) \int_{1}^{n} u^{-(r-k-1)/r} \ln |(u - v)/n| \, du \, dv
\]

\[
= I_1 + I_2, \tag{5}
\]

on using the substitutions \(u = nx^r\) and \(v = nt\).

It is easily seen that

\[
N - \lim_{n \to \infty} I_1 = 0, \tag{6}
\]

for \(k = 0, 1, \ldots, rs - 2\).

Now,

\[
\int_{-1}^{1} \rho^{(s)}(v) \int_{1}^{n} u^{-(r-k-1)/r} \ln |(u - v)/n| \, du \, dv
\]

\[
= \int_{-1}^{1} \rho^{(s)}(v) \int_{1}^{n} u^{-(r-k-1)/r} [\ln |1 - v/u| + \ln u - \ln n] \, du \, dv
\]

\[
= \int_{-1}^{1} \rho^{(s)}(v) \int_{1}^{n} u^{-(r-k-1)/r} \ln |1 - v/u| \, du \, dv,
\]

since \(\int_{-1}^{1} \rho^{(s)}(v) \, dv = 0\) for \(s = 1, 2, \ldots\), by Lemma 2. Further

\[
\int_{-1}^{1} \rho^{(s)}(v) \int_{1}^{n} u^{-(r-k-1)/r} \ln |1 - v/u| \, du \, dv
\]
\[ = - \sum_{i=1}^{\infty} \frac{1}{i} \int_{-1}^{1} v^{i} \rho^{(s)}(v) \int_{1}^{n} u^{(k+1)/r-i-1} \, du \, dv \]
\[ = - \sum_{i=1}^{\infty} \frac{r^{k+1}/r-i-1}{i(k+r+1)} \int_{-1}^{1} v^{i} \rho^{(s)}(v) \, dv, \]

and it follows that

\[ \lim_{n \to \infty} I_2 = \frac{1}{s(rs - k - 1)} \int_{-1}^{1} v^{s} \rho^{(s)}(v) \, dv \]
\[ = \frac{(-1)^{s}(s - 1)!}{rs - k - 1}, \quad (7) \]

on using Lemma 2, for \( k = 0, 1, \ldots, rs - 2 \).

Hence

\[ \lim_{n \to \infty} \int_{0}^{1} x^{k}[(x^{r})^{-s}]_{n} \, dx = - \frac{1}{rs - k - 1} \quad (8) \]

for \( k = 0, 1, \ldots, rs - 2 \), on using (5), (6) and (7). Then using (4) and (8), we see that

\[ \lim_{n \to \infty} \int_{-1}^{1} x^{k}[(x^{r})^{-s}] \, dx = \frac{(-1)^{rs-k-1}-1}{rs - k - 1}, \quad (9) \]

for \( k = 0, 1, \ldots, rs - 1 \).

When \( k = rs \) (5) still holds but now we have

\[ I_1 = \frac{n^{-1/r}}{r} \int_{-1}^{1} \rho^{(s)}(v) \int_{0}^{1} u^{-(r-s-k+1)/r} \ln |(u-v)/n| \, du \, dv, \]

and it follows that for any continuous function \( \psi \)

\[ \lim_{n \to \infty} \int_{0}^{\frac{n-1}{r}} x^{rs}[(x^{r})^{-s}] \psi(x) \, dx = 0. \quad (10) \]

Similarly

\[ \lim_{n \to \infty} \int_{-\frac{n-1}{r}}^{0} x^{rs}[(x^{r})^{-s}] \psi(x) \, dx = 0. \quad (11) \]

Next, when \( x^{r} \geq 1/n \), we have

\[ (-1)^{t-1}(s - 1)![x^{r}]^{-s} = \int_{-1/n}^{1/n} \ln |x^{r} - t| \delta^{(s)}(t) \, dt \]
\[ = n^{t} \int_{-1}^{1} \ln |x^{r} - v|/n \rho^{(s)}(v) \, dv \]
\[ = n^{t} \int_{-1}^{1} \left[ \ln x^{r} - \sum_{i=1}^{\infty} \frac{v^{i}}{i^{n}} \right] \rho^{(s)}(v) \, dv \]
\[ = - \sum_{i=1}^{\infty} \int_{-1}^{1} \frac{v^{i}}{i^{n-1}x^{r}} \rho^{(s)}(v) \, dv. \]
It follows that

\[(s-1)![(x')^{-s}]_n| \leq \sum_{i=s}^{\infty} \int_{-1}^{1} \frac{|v|^i}{i! (n-1)^i} |\rho^{(s)}(v)| dv \leq \sum_{i=s}^{\infty} \frac{K_s}{i! (n-1)^i},\]

where

\[K_s = \int_{-1}^{1} |\rho^{(s)}(v)| dv,\]

for \(s = 1, 2, \ldots,\)

If now \(n^{-1/r} < \eta < 1,\) then

\[(s-1)! \int_{n^{-1/r}}^{\eta} |x^s[(x')^{-s}]_n| dx \leq K_s \sum_{i=s}^{\infty} \frac{n^{1-i}}{i} \int_{n^{-1/r}}^{\eta} x^{(s-1)i} dx\]

\[= \left\{ \begin{array}{ll}
K_s \sum_{i=s}^{\infty} \frac{n^{1-i}}{ri(s-i+1/r)} [(n\eta)^{s-i+1/r} - 1], & r \neq 1, \\
K_s \sum_{i=s}^{\infty} \frac{n^{1-i}}{i(s-i+1/r)} [(n\eta)^{s-i+1} - 1] + K_s \frac{n^{-1} \ln(n\eta)}{s+1}, & r = 1.
\end{array} \right.\]

It follows that

\[\lim_{n \to \infty} \int_{n^{-1/r}}^{\eta} |[(x')^{-s}]_n| dx = O(\eta),\]

for \(r, s = 1, 2, \ldots.\)

Thus, if \(\psi\) is a continuous function

\[\lim_{n \to \infty} \left| \int_{n^{-1/r}}^{\eta} x^r[(x')^{-s}]_n \psi(x) dx \right| = O(\eta),\]

(12)

for \(r, s = 1, 2, \ldots.\)

Similarly,

\[\lim_{n \to \infty} \left| \int_{-\eta}^{-n^{-1/r}} x^r[(x')^{-s}]_n \psi(x) dx \right| = O(\eta),\]

(13)

for \(r, s = 1, 2, \ldots.\)

Now let \(\varphi(x)\) be an arbitrary function in \(D\) with support contained in the interval \([-1, 1]\). By Taylor's theorem we have

\[\varphi(x) = \sum_{k=0}^{r-1} \frac{x^k}{k!} \varphi^{(k)}(0) + \frac{x^r}{(rs)!} \varphi^{(rs)}(\xi x)\]

where \(0 < \xi < 1.\) Then

\[\langle [(x')^{-s}]_n, \varphi(x) \rangle = \int_{-1}^{1} [(x')^{-s}]_n \varphi(x) dx\]

\[= \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^{1} x^k [(x')^{-s}]_n dx + \int_{-n^{-1/r}}^{n^{-1/r}} \frac{x^r}{(rs)!} [(x')^{-s}]_n \varphi^{(rs)}(\xi x) dx\]
\[
+ \int_{-\eta}^{\eta} \frac{X^s}{(rs)!} [(x^s)^{-n}] \varphi^{(rs)}(\xi x) \, dx + \int_{-\eta}^{\eta} \frac{X^s}{(rs)!} [(x^s)^{-n}] \varphi^{(rs)}(\xi x) \, dx \\
+ \int_{-\eta}^{-n+1/rs} \frac{X^s}{(rs)!} [(x^s)^{-n}] \varphi^{(rs)}(\xi x) \, dx + \int_{-1}^{-\eta} \frac{X^s}{(rs)!} [(x^s)^{-n}] \varphi^{(rs)}(\xi x) \, dx.
\]

Using (11) to (15) and noting that the sequence \([(x^s)^{-n}]\) converges uniformly to \(x^{-rs}\) on the intervals \([-1, -\eta]\) and \([\eta, 1]\), it follows that

\[
\lim_{n \to \infty} \langle [(x^s)^{-n}], \varphi(x) \rangle = \sum_{k=0}^{rs-1} \frac{(-1)^{rs-k-1} - 1}{(rs - k - 1) k!} \varphi^{(k)}(0) + O(\eta) \\
+ \int_{\eta}^{1} \frac{\varphi^{(rs)}(\xi x)}{(rs)!} \, dx + \int_{-1}^{-\eta} \frac{\varphi^{(rs)}(\xi x)}{(rs)!} \, dx.
\]

Since \(\eta\) can be made arbitrarily small, it follows that

\[
\lim_{n \to \infty} \langle [(x^s)^{-n}], \varphi(x) \rangle = \sum_{k=0}^{rs-1} \frac{(-1)^{rs-k-1} - 1}{(rs - k - 1) k!} \varphi^{(k)}(0) + \int_{-1}^{1} \frac{\varphi^{(rs)}(\xi x)}{(rs)!} \, dx \\
= \int_{-1}^{1} x^{-rs} \left[ \varphi(x) - \sum_{k=0}^{rs-1} \frac{x^k}{k!} \varphi^{(k)}(0) \right] \, dx \\
+ \sum_{k=0}^{rs-1} \frac{(-1)^{rs-k-1} - 1}{(rs - k - 1) k!} \varphi^{(k)}(0) \\
= \langle x^{-rs}, \varphi(x) \rangle,
\]

on using (1). This proves (3) on the interval \([-1, 1]\). However, (3) clearly holds on any interval not containing the origin, and the proof is complete.

In the corollary, the distribution \((x + i0)^{-s}\) is defined by

\[
(x + i0)^{-s} = x^{-s} + \frac{\pi(-1)^s}{(s-1)!} \delta^{(s-1)}(x),
\]

for \(s = 1, 2, \ldots\), see Gel’fand and Shilov [5].

**COROLLARY 3.1**

The distribution \((x^r + i0)^{-s}\) exists and

\[
(x^r + i0)^{-s} = (x + i0)^{-rs}
\]

for \(r = 1, 3, 5, \ldots\) and \(s = 1, 2, \ldots\) and

\[
(x^r + i0)^{-s} = x^{-rs},
\]

for \(r = 2, 4, 6, \ldots\) and \(s = 1, 2, \ldots\).

**Proof.** It was proved in [2] that

\[
\delta^{(s)}(x^r) = \frac{s!}{r(rs + r - 1)!} \delta^{(rs+r-1)}(x),
\]

for \(r = 1, 3, 5, \ldots\) and \(s = 0, 1, 2, \ldots\), and

\[
\delta^{(s)}(x^r) = 0,
\]

for \(r = 1, 3, 5, \ldots\) and \(s = 0, 1, 2, \ldots\).
for $r = 2, 4, 6, \ldots$ and $s = 0, 1, 2, \ldots$. Equation (15) follows immediately from (3), (14) and (17) and (16) follows immediately from (3), (14) and (18).

References