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A short note on weighted mean matrices

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Abstract. In the present paper we have established a relation between $(\overline{N}, p_n)$ and $(\overline{N}, q_n)$ weighted mean matrices, when considered as bounded operators on l^p, $1 < p < \infty$.

Keywords. Weighted mean matrices.

1. Introduction

Let $\sum a_n$ be an infinite series with partial sum s_n.

If $p_n \geq 0$, $p_0 > 0$, $\sum p_n = \infty$ (so that $P_n = p_0 + p_1 + \cdots + p_n \to \infty$ as $n \to \infty$), and

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \to s$$

when $n \to \infty$, then we say that

$$s_n \to s(\overline{N}, p_n).$$

A result concerning the relation between $(\overline{N}, p_n)$ and $(\overline{N}, q_n)$ weighted mean matrices has been given in Hardy [2, Theorem 14].

Theorem A [2]. *If $p_n > 0$, $q_n > 0$, $\sum p_n = \infty$, $\sum q_n = \infty$ and either*

(a) $$\frac{q_{n+1}}{q_n} \leq \frac{p_{n+1}}{p_n} \quad \textit{or} \tag{1}$$

(b) $$\frac{p_{n+1}}{p_n} \leq \frac{q_{n+1}}{q_n} \tag{2}$$

and also

$$\frac{P_n}{p_n} \leq H \frac{Q_n}{q_n}, \tag{3}$$

then $\sum a_n = s(\overline{N}, p_n)$ implies $\sum a_n = s(\overline{N}, q_n)$.

In the present paper we have established a relation between the $(\overline{N}, p_n)$ and $(\overline{N}, q_n)$ weighted mean matrices, when considered as bounded operators on l^p, $1 < p < \infty$.

2. Main theorem

Theorem. *Let $\{p_n\}$ and $\{q_n\}$ be positive sequences satisfying the conditions*

$$P_n \left(\frac{q_{n+1}}{p_{n+1}} - \frac{q_n}{p_n} \right) \approx (n+1)^{\alpha}, \quad \textit{for some } \alpha \geq 0, \tag{4}$$

$$\frac{P_n}{p_n} \cdot \frac{q_n}{Q_n} \leq H, \tag{5}$$

$$\{p_n\} \quad \textit{is a non decreasing sequence.} \tag{6}$$

Then $(\overline{N}, q_n)$ *matrix is a bounded linear operator on* l^p *whenever* $(\overline{N}, p_n)$ *matrix is a bounded linear operator on* l^p.

Remark. In view of our theorem, Theorem A [2, Theorem 14] of Hardy is partially true when $q_{n+1}/p_{n+1} \geq q_n/p_n$ for l^p spaces, $1 < p < \infty$.

In 1994, Rhoades [4] has obtained sufficeint conditions for certain weighted mean matrices to be equivalent to C, the Cesaro matrix of order 1, considered as bounded operators on l^p, $1 < p < \infty$.

Theorem B [4]. *Let* $\{a_n\}$ *be a positive sequence satisfying the condition* $(n+1)(a_{n+1} - a_n) \approx (n+1)^\alpha$ *for some* $\alpha \geq 0$. *Then* $(\overline{N}, a)$ *and* C *are equivalent over* l^p *for* $1 < p < \infty$.

When we take $q_n = a_n$ and $p_n = 1$ in our theorem, then the condition of Theorem B is satisfied.

Thus one part of theorem B of Rhoades that $(\overline{N}, a)$ matrix is a bounded linear operator on l^p whenever $(C, 1)$ matrix is a bounded linear operator on l^p is implied by our theorem.

A triangular matrix $A = (a_{nk})$ is said to be factorable if $a_{nk} = c_n d_k$, $0 \leq k \leq n$, and $a_{nk} = 0$, $k > n$.

3. Lemmas

For the proof of our theorem, we need the following lemmas.

Lemma 1 [1]. *Let* $1 < p < \infty$, q *the conjugate index of* p. *If* c_n, $d_k \geq 0$ *and*

$$c_n \sum_{k=0}^{n} d_k^q \leq K d_n^{1/p-1}$$

for $n = 0, 1, \ldots$ *and some constant* K, *then* A *is a bounded operator on* l^p.

Lemma 2. *If*

(i) $a_1 \leq a_2 \leq \cdots \leq a_n$ *and* $b_1 \leq b_2 \leq \cdots \leq b_n$, *then Chebychev's inequality states:*

(ii) $\left(\frac{1}{n}\sum_{v=1}^{n} a_v\right)\left(\frac{1}{n}\sum_{v=1}^{n} b_v\right) \leq \frac{1}{n}\sum_{v=1}^{n} a_v b_v$.

(iii) $1^p + 2^p + 3^p + \cdots + n^p > \frac{n^{p+1}}{p+1}$, $\quad p > 0$.

The proof may be found in [3], pages 16 and 97.

Lemma 3. *Let* $\{p_n\}$ *and* $\{q_n\}$ *be two positive sequences satisfying the conditions* (4), (5) *and* (6) *of the theorem. Then*

(iv) $Q_n \geq \frac{(n+1)^{\alpha+1}}{(\alpha+1)(\alpha+2)}$, $\quad$ for $\alpha \geq 0$.

Remark. It follows that $Q_n \to \infty$ as $n \to \infty$.

Proof. For $\alpha \geq 0$, by condition (4)

$$P_k\left(\frac{q_{k+1}}{p_{k+1}} - \frac{q_k}{p_k}\right) \approx (k+1)^\alpha \Rightarrow \frac{q_{k+1}}{p_{k+1}} - \frac{q_k}{p_k} \approx \frac{(k+1)^\alpha}{P_k}.$$

Now taking summation

$$\sum_{k=0}^{n-1}\left(\frac{q_{k+1}}{p_{k+1}}-\frac{q_k}{p_k}\right)\approx\sum_{k=0}^{n-1}\frac{(k+1)^\alpha}{P_k}\Rightarrow\frac{q_n}{p_n}\approx\sum_{k=0}^{n-1}\frac{(k+1)^\alpha}{P_k}.$$

This implies

$$\frac{q_k}{p_k}\approx\sum_{r=0}^{k}\frac{(r+1)^\alpha}{P_r}\Rightarrow q_k\approx p_k\sum_{r=0}^{k}\frac{(r+1)^\alpha}{P_r}\Rightarrow\sum_{k=0}^{n}q_k\approx\sum_{k=0}^{n}p_k\sum_{r=0}^{k}\frac{(r+1)^\alpha}{P_r}.$$

Using Chebychev's inequality,

$$\begin{aligned}
Q_n &\geq n\left(\frac{1}{n}\sum_{k=0}^{n}p_k\right)\left(\frac{1}{n}\sum_{k=0}^{n}\sum_{r=0}^{k}\frac{(r+1)^\alpha}{P_r}\right)\\
&=\frac{P_n}{n}\sum_{k=0}^{n}\sum_{r=0}^{k}\frac{(r+1)^\alpha}{P_r}\\
&\geq\frac{P_n}{n}\sum_{k=0}^{n}\frac{1}{P_k}\sum_{r=0}^{k}(r+1)^\alpha\\
&\geq\frac{P_n}{n}\sum_{k=0}^{n}\frac{1}{P_k}\frac{(k+1)^{\alpha+1}}{(\alpha+1)},\quad\text{by Lemma 2,}\\
&\geq\frac{P_n}{nP_n}\sum_{k=0}^{n}\frac{(k+1)^{\alpha+1}}{(\alpha+1)}\\
&=\frac{1}{n(\alpha+1)}\sum_{k=0}^{n}(k+1)^{\alpha+1}\\
&\geq\frac{1}{n(\alpha+1)}\frac{(n+1)^{\alpha+2}}{(\alpha+2)},\quad\text{by Lemma 2,}\\
\Rightarrow Q_n &\geq\frac{(n+1)^{\alpha+1}}{(\alpha+1)(\alpha+2)}.
\end{aligned}$$

This proves (iv).

Proof of the theorem. Define

$$t_{n,q}=\frac{1}{Q_n}\sum_{k=0}^{n}q_k s_k, \tag{7}$$

$$t_{n,p}=\frac{1}{P_n}\sum_{k=0}^{n}p_k s_k. \tag{8}$$

Solving (8) for s_n and then substituting into (7), we have

$$t_{n,q}=\frac{1}{Q_n}\sum_{k=0}^{n}[P_k t_{k,p}-P_{k-1}t_{k-1,p}]\frac{q_k}{p_k}.$$

By partial summation formula

$$t_{n,q} = \frac{1}{Q_n}\left[\frac{q_{n+1}}{p_{n+1}} P_n t_{n,p} + \sum_{k=0}^{n}\left(\frac{q_k}{p_k} - \frac{q_{k+1}}{p_{k+1}}\right) P_k t_{k,p}\right].$$

Then $t_{n,q} = A_n(t_{n,p})$, where A is the lower triangular matrix with entries

$$a_{nk} = \begin{cases} \frac{P_k}{Q_n}\left(\frac{q_k}{p_k} - \frac{q_{k+1}}{p_{k+1}}\right), & 0 \le k < n \\ \frac{q_n}{p_n}\frac{P_n}{Q_n}, & k = n \\ 0. & k > n \end{cases}$$

Obviously $A = C + D$, where

$$c_{nk} = \frac{P_k}{Q_n}\left(\frac{q_k}{p_k} - \frac{q_{k+1}}{p_{k+1}}\right), \quad 0 \le k \le n$$

and D is the diagonal matrix with diagonal entries

$$d_{nn} = \frac{q_{n+1}}{p_{n+1}}\frac{P_n}{Q_n}.$$

To show that $A \in B(l^p)$ it is sufficient to show that $C \in B(l^p)$ and that D is bounded. In view of condition (5), D is bounded. In order to show that C is bounded, we use Lemma 1. Choose

$$c_n := \frac{1}{Q_n}$$

and

$$d_k := P_k\left(\frac{q_{k+1}}{p_{k+1}} - \frac{q_k}{p_k}\right) \ge 0.$$

Let $\alpha \ge 0$. Then the condition (4) of the theorem implies that $d_k \approx (k+1)^\alpha$.

By Lemma 1, if we show that

$$c_n \sum_{k=0}^{n} d_k^q \le K d_n^{1/p-1},$$

then C is bounded.

Now

$$\begin{aligned}\frac{c_n}{d_n^{1/p-1}} \sum_{k=0}^{n} d_k^q &= \frac{1}{Q_n[(n+1)^\alpha]^{q-1}} \sum_{k=0}^{n} (k+1)^{\alpha q} \\ &\approx \frac{(n+1)^{\alpha q+1}}{Q_n(n+1)^{\alpha q-\alpha}} = \frac{(n+1)^{\alpha+1}}{Q_n} \\ &\le \frac{(n+1)^{\alpha+1}(\alpha+1)(\alpha+2)}{(n+1)^{\alpha+1}} \quad \text{by condition (iv) of Lemma 3} \\ &\le K, \quad \text{for some constant } K.\end{aligned}$$

Thus $t_{n,q} \in l^p$ whenever $t_{n,p} \in l^p$. This proves the theorem.

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