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Nonlinear boundary value problems

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Abstract. In this paper we consider two quasilinear boundary value problems. The first is vector valued and has periodic boundary conditions. The second is scalar valued with nonlinear boundary conditions determined by multivalued maximal monotone maps. Using the theory of maximal monotone operators for reflexive Banach spaces and the Leray–Schauder principle we establish the existence of solutions for both problems.

Keywords. Monotone operator; maximal monotone operator; demicontinuous operator; weakly coercive operator; surjective operator; periodic problem; Leray–Schauder principle; Sobolev space; compact embedding.

1. Introduction

In this paper we study the following two problems:

$$\begin{aligned} &-(\|x'(t)\|^{p-2}x'(t))' + f(t,x(t),x'(t)) = 0 \text{ a.e. on } T \\ &x(0) = x(b),\ x'(0) = x'(b),\ 2 \le p < \infty \end{aligned} \tag{1}$$

and

$$\begin{aligned} &-(|x'(t)|^{p-2}x'(t))' + f(t,x(t),x'(t)) = 0 \text{ a.e. on } T \\ &x'(0) \in \xi_1(x(0)),\ -x'(b) \in \xi_2(x(b)),\ 2 \le p < \infty. \end{aligned} \tag{2}$$

We shall study problem (1) in R^N and problem (2) in R. In problem (2) ξ_1, ξ_2 are two maximal monotone graphs in R^2. Recently quasilinear ordinary differential equations were studied by Boccardo–Drabek–Giachetti–Kucera [2], Drabek [4], DelPino–Elqueta–Manasevich [3], Guo [6] and Zhang [16]. With the exception of Zhang, these works deal with the scalar equation. Boccardo *et al*, Drabek, DelPino–Elqueta–Manasevich and Zhang study the Dirichlet problem and the first three assume that the function f is independent of the derivative x'. In this case the one dimensional p–Laplacian $A(x) = -(|x'|^{p-2}x')'$ is invertible and its inverse is compact from $C(T)$ into itself. So the Leray–Schauder degree theory can be applied. Guo considers the periodic and Neumann problems and in the Neumann problem f is independent of the derivative x'. For these two problems the quasilinear differential operator is not invertible and so some generalized degree theory, such as coincidence theory of Mawhin is needed. Note that these works (except Zhang [16]) assume that the vector field f is continuous in all variables (including the time variable). The semilinear case (i.e. $p = 2$) is covered in the well written books of Gaines–Mawhin [5] and Mawhin [11] where the interested reader can find a comprehensive account of the coincidence degree theory and in the papers of Knobloch [9] and Mawhin [10]. In the same vein we should also mention the relevant works of Mawhin–Ward [12], [13] which deal with the Lienard and Duffing equations using nonresonance hypotheses. Our approach here is different and relies heavily on the theory of maximal monotone operators.

2. Preliminaries

Since our approach is based on the theory of maximal monotone operators, in this section we recall some basic definitions and facts from this theory. Details can be found in Zeidler [15].

Let X be a reflexive Banach space and X^* its topological dual. A possibly multivalued map $A : D \subseteq X \to 2^{X^*}$ is said to be "monotone", if for any $x, y \in D$ $(x^* - y^*, x - y) \geq 0$ holds for all $x^* \in A(x)$ and $y^* \in A(y)$. Here by $(\cdot, \cdot)$ we denote the duality brackets for the pair (X^*, X). When $(x^* - y^*, x - y) = 0$ implies that $x = y$, then we say that A is "strictly monotone". A monotone map for which the inequalities $(x^* - y^*, x - y) \geq 0$ for all $[y, y^*] \in \mathrm{Gr} A$, imply $[x, x^*] \in \mathrm{Gr} A$ is said to be "maximal monotone". Here by $\mathrm{Gr} A$ we denote the graph of A, i.e. $\mathrm{Gr} A = \{[y, y^*] \in X \times X^* : y^* \in A(y)\}$. From this definition it follows that A is maximal monotone if its graph is maximal with respect to inclusion among the graphs of monotone maps (i.e. $\mathrm{Gr} A$ is not properly included in the graph of another monotone map). When $A(\cdot)$ is maximal monotone, then for every $x \in D$, $A(x)$ is closed and convex in X^* and the set $\mathrm{Gr} A$ is demiclosed, i.e. if $x_n \to x$ in X, $x_n^* \xrightarrow{w} x^*$ in X^* as $n \to \infty$ (or $x_n \xrightarrow{w} x$ in X, $x_n^* \to x^*$ in X^* as $n \to \infty$) and $x_n^* \in A(x_n)$, $n \geq 1$, then $x^* \in A(x)$. Now let $D = X$ and $A : X \to X^*$ be single valued. We say that $A(\cdot)$ is "demicontinuous" if it is sequentially continuous from X into X_w^*, where X_w^* denotes the dual space X^* furnished with the weak topology (i.e. if $x_n \to x$ in X, then $A(x_n) \xrightarrow{w} A(x)$ in X^* as $n \to \infty$). A monotone, demicontinuous map is maximal monotone. A map $A : D \subseteq X \to 2^{X^*}$ is said to be "weakly coercive", if D is bounded in X or D is unbounded and $\inf[\|x^*\|_* : x^* \in A(x)] \to \infty$ as $\|x\| \to \infty$, $x \in D$ (here by $\|\cdot\|$ and $\|\cdot\|_*$ we denote the norms of X and X^* respectively). A maximal monotone, weakly coercive $A : D \subseteq X \to 2^{X^*}$ is surjective, i.e. $R(A) = X^*$. In particular a monotone, demicontinuous and weakly coercive map $A : X \to X^*$ is surjective.

Our method of proof will produce the solution via a fixed point argument based on the Leray–Schauder theorem, which for the convenience of the reader, we recall here (see Mawhin [11]):

Theorem 1. *If Y is a Banach space, $K : Y \to Y$ is compact and there exists $r > 0$ such that if $y = \mu K(y)$ with $0 < \mu < 1$, we have $\|y\| \leq r$ (a priori bound), then K has a fixed point, i.e. there exists $y \in Y$ such that $y = K(y)$.*

Finally in the sequel we will use the following elementary inequality

$$2^{2-p}|a - c|^p \leq (|a|^{p-2}a - |c|^{p-2}c)(a - c)$$

for all $a, c \in R$ and all $2 \leq p < \infty$.

3. Periodic vector problem

In this section we prove an existence theorem for problem (1). For this purpose we will need the following hypotheses on $f(t, x, y)$.

$H(f)_1$: $f : T \times R^N \times R^N \to R^N$ is a function such that

(i) for every $x, y \in R^N$, $t \to f(t, x, y)$ is measurable;
(ii) for almost all $t \in T$, $(x, y) \to f(t, x, y)$ is continuous;
(iii) for almost all $t \in T$ and all $x, y \in R^N$, we have

$$(f(t,x,y),x)_{R^N} \geq -a\|x\|^p - \beta\|x\|^r\|y\|^{p-r} - c(t)\|x\|^s$$

with $a, \beta \geq 0$, $1 \leq r, s < p$ and $c \in L^1(T)_+$;
(iv) there exists $M > 0$ such that if $\|x_0\| > M$ and $(x_0, y_0)_{R^N} = 0$, then we can find $\delta > 0$ and $c > 0$ such that for almost all $t \in T$ we have

$$\inf\left[(f(t,x,y),x)_{R^N} + \|y\|^p : \|x - x_0\| + \|y - y_0\| < \delta\right] \geq c;$$

(v) for almost all $t \in T$ and all $x, y \in R^N$, we have

$$\|f(t,x,y)\| \leq \gamma_1(t, \|x\|) + \gamma_2(t, \|x\|)\|y\|^{p-1}$$

with $\sup_{0 \leq r \leq k} \gamma_1(t,r) \leq \eta_{1,k}(t)$ a.e. on $T, \eta_{1,k}(\cdot) \in L^q(T)$ and $\sup_{0 \leq r \leq k} \gamma_2(t,r) \leq \eta_{2,k}(t)$ a.e. on $T, \eta_{2,k} \in L^\infty(T)$.

Remark. Hypothesis $H(f)_1$ (iv) is a suitable extension of the classical Nagumo–Hartman condition for continuous vector fields (see Hartman [7], p. 433 and Knobloch [9], Mawhin [10]).

DEFINITION

By a solution of (1) we mean a function $x \in C^1(T, R^N)$ such that $(\|x'(\cdot)\|^{p-2}x'(\cdot))' \in W^{1,q}(T, R^N)$ and satisfies (1).

Given $g \in L^q(T, R^N)$ we consider the following auxiliary problem:

$$\begin{aligned} &-(\|x'(t)\|^{p-2}x'(t))' + \|x(t)\|^{p-2}x(t) = g(t) \quad \text{a.e. on } T \\ &x(0) = x(b), \ x'(0) = x'(b). \end{aligned} \tag{3}$$

PROPOSITION 2

For every $g \in L^q(T, R^N)$ problem (3) has a unique solution.

Proof. Let $a \in R^N$ and consider the following boundary value problem:

$$\begin{aligned} &-(\|x'(t)\|^{p-2}x'(t))' + \|x(t)\|^{p-2}x(t) = g(t) \text{ a.e. on } T \\ &x(0) = x(b) = a. \end{aligned} \tag{4}$$

Setting $y(\cdot) = x(\cdot) - a$, problem (4) becomes a homogeneous Dirichlet problem in y:

$$\begin{aligned} &-(\|y'(t)\|^{p-2}y'(t))' + \|y(t) + a\|^{p-2}(y(t) + a) = g(t) \quad \text{a.e. on } T \\ &y(0) = y(b) = 0. \end{aligned} \tag{5}$$

We solve problem (5). To this end consider the operator $A_1 : W_0^{1,p}(T, R^N) \to W^{-1,q}(T, R^N)$ defined by

$$\langle A_1(y), z\rangle = \int_0^b (\|y'(t)\|^{p-2}(y'(t), z'(t))_{R^N} + \|y(t) + a\|^{p-2}(y(t) + a, z(t))_{R^N})\, dt.$$

Here by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(W_0^{1,p}(T, R^N), W^{-1,q}(T, R^N))$. We claim that $A_1(\cdot)$ is strictly monotone and demicontinuous. We have:

$$\begin{aligned}
&\langle A_1(y) - A_1(z), y - z\rangle \\
&= \int_0^b (\|y'(t)\|^{p-2}(y'(t), y'(t) - z'(t))_{R^N} - \|z'(t)\|^{p-2}(z'(t), y'(t) - z'(t))_{R^N})\,dt \\
&\quad + \int_0^b (\|y(t) + a\|^{p-2}(y(t) + a, y(t) - z(t))_{R^N} \\
&\quad - \|z(t) + a\|^{p-2}(z(t) + a, y(t) - z(t))_{R^N})\,dt.
\end{aligned}$$

First we estimate the first integral of the rhs of the above equation. We have:

$$\begin{aligned}
&\int_0^b (\|y'(t)\|^{p-2}(y'(t), y'(t) - z'(t))_{R^N} - \|z'(t)\|^{p-2}(z'(t), y'(t) - z'(t))_{R^N})\,dt \\
&\geq \int_0^b (\|y'(t)\|^p - \|y'(t)\|^{p-1}\|z'(t)\| - \|z'(t)\|^{p-1}\|y'(t)\| + \|z'(t)\|^p)\,dt \\
&= \int_0^b (\|y'(t)\|^{p-1}(\|y'(t)\| - \|z'(t)\|) - \|z'(t)\|^{p-1}(\|y'(t)\| - \|z'(t)\|))\,dt \\
&= \int_0^b (\|y'(t)\|^{p-1} - \|z'(t)\|^{p-1})(\|y'(t)\| - \|z'(t)\|)\,dt \\
&\geq 2^{2-p} \int_0^b |\|y'(t)\| - \|z'(t)\||^p\,dt \quad \text{(see § 2)}.
\end{aligned} \tag{6}$$

In a similar way, we obtain

$$\begin{aligned}
&\int_0^b (\|y(t) + a\|^{p-2}(y(t) + a, y(t) - z(t))_{R^N} \\
&\quad - \|z(t) + a\|^{p-2}(z(t) + a, y(t) - z(t))_{R^N})\,dt \\
&\geq 2^{2-p} \int_0^b |\|y(t) + a\| - \|z(t) + a\||^p\,dt.
\end{aligned} \tag{7}$$

From (6) and (7) we have the monotonicity of $A_1(\cdot)$. Moreover, if $\langle A_1(y) - A_1(z), y - z\rangle = 0$, we have $\|y'(t)\| = \|z'(t)\|$ a.e. on T and $\|y(t) + a\| = \|z(t) + a\|$ for all $t \in T$. Then it follows that

$$\begin{aligned}
&\int_0^b \|y'(t)\|^{p-2}\|y'(t) - z'(t)\|^2\,dt + \int_0^b \|y(t) + a\|^{p-2}\|y(t) - z(t)\|^2\,dt = 0 \\
&\Rightarrow \int_0^b \|y'(t)\|^{p-2}\|y'(t) - z'(t)\|^2\,dt = 0 \text{ and } \\
&\int_0^b \|y(t) + a\|^{p-2}\|y(t) - z(t)\|^2\,dt = 0
\end{aligned}$$

from which we deduce easily that $y = z$. Therefore $A_1(\cdot)$ is strictly monotone.

Next we show that $A_1(\cdot)$ is demicontinuous. To this end let $y_n \to y$ in $W_0^{1,p}(T, R^N)$ as $n \to \infty$. For every $z \in W_0^{1,p}(T, R^N)$ we have

$$|\langle A_1(y_n) - A_1(y), z\rangle|$$
$$\leq \left|\int_0^b (\|y_n'(t)\|^{p-2}(y_n'(t), z'(t))_{R^N} - \|y'(t)\|^{p-2}(y'(t), z'(t))_{R^N})\,dt\right|$$
$$+\left|\int_0^b (\|y_n(t)+a\|^{p-2}(y_n(t)+a, z(t))_{R^N} - \|y(t)+a\|^{p-2}(y(t)+a, z(t))_{R^N})dt\right|.$$

By passing to a subsequence if necessary we may assume that $y_n'(t) \to y'(t)$ a.e. on T and $y_n(t) \to y(t)$ for all $t \in T$ as $n \to \infty$ (the latter convergence follows from the fact that $W_0^{1,p}(T, R^N)$ is embedded continuously in $C(T, R^N)$). So by the extended dominated convergence theorem (see Ash [1], theorem 7.5.2, p. 295), we have

$$\left|\int_0^b (\|y_n'(t)\|^{p-2}(y_n'(t), z'(t))_{R^N} - \|y'(t)\|^{p-2}(y'(t), z'(t))_{R^N})\,dt\right| \to 0 \text{ and}$$
$$\left|\int_0^b (\|y_n(t) + a\|^{p-2}(y_n(t) + a, z(t))_{R^N} - \|y(t) + a\|^{p-2}(y(t) + a, z(t))_{R^N})\,dt\right|$$
$$\to 0 \text{ as } n \to \infty.$$

Therefore $|\langle A_1(y_n) - A_1(y), z\rangle| \to 0$ as $n \to \infty$ and since $z \in W_0^{1,p}(T, R^N)$ was arbitrary we conclude that $A_1(y_n) \xrightarrow{w} A_1(y)$ in $W^{-1,q}(T, R^N)$ as $n \to \infty$, i.e. $A_1(\cdot)$ is demicontinuous. Since $A_1(\cdot)$ is monotone and demicontinuous, it is maximal monotone (see § 2).

Finally we will show that $A_1(\cdot)$ is weakly coercive. We have:

$$\begin{aligned}\langle A_1(y), y\rangle &= \|y'\|_p^p + \int_0^b \|y(t) + a\|^{p-2}(y(t) + a, y(t))_{R^N}\,dt \\ &= \|y'\|_p^p + \|y + a\|_p^p - \int_0^b \|y(t) + a\|^{p-2}(y(t) + a, a)_{R^N}\,dt \\ &\geq \|y'\|_p^p + \|y + a\|_p^p - \int_0^b \|y(t) + a\|^{p-1}\|a\|\,dt \\ &\geq \|y + a\|_{1,p}^p - c\|y + a\|_p^{p-1} \quad \text{for some } c > 0,\end{aligned}$$

where $\|\cdot\|_{1,p}$ denotes the norm of the Sobolev space $W_0^{1,p}(T, R^N)$. So we proved that $A_1(\cdot)$ is weakly coercive. Hence $A_1(\cdot)$ is surjective. This means that we can find $y \in W_0^{1,p}(T, R^N)$ such that $A_1(y) = g$. Then for all $\phi \in C_0^\infty(T, R^N)$ we have

$$\int_0^b \|y'(t)\|^{p-2}(y'(t), \phi'(t))_{R^N} dt = \int_0^b (g(t) - \|y(t)+a\|^{p-2}(y(t)+a(g(t)+\alpha, \phi(t))_{R^N} dt.$$

From the definition of the distributional derivative, we infer that

$$-(\|y'(t)\|^{p-2}y'(t))' = g(t) - \|y(t) + a\|^{p-2}(y(t) + a) \quad \text{a.e. on } T$$
$$y(0) = y(b) = 0.$$

Hence $y(\cdot)$ is a solution of (5) and so $x(\cdot) = y(\cdot) + a$ is a solution of (4). Moreover, by virtue of the strict monotonicity of A_1, this solution of (4) is unique. So we can define a map $\theta : R^N \to C^1(T, R^N)$ by setting $\theta(a)$ to be the unique solution of (4). Then introduce the map $\rho : R^N \to R^N$ defined by

$$\rho(a) = \|\theta'(a)(b)\|^{p-2}\theta'(a)(b) - \|\theta'(a)(0)\|^{p-2}\theta'(a)(0)$$
$$= \int_0^b (\|\theta'(a)(t)\|^{p-2}\theta'(a)(t))' \, dt.$$

Claim 1. $\rho(\cdot)$ is strictly monotone.

We need to show that for all $a, \beta \in R^N$ $(\rho(a) - \rho(\beta), a - \beta)_{R^N} \geq 0$ and if equality holds, then $a = \beta$. We have

$$\begin{aligned}
&(\rho(a) - \rho(\beta), a - \beta)_{R^N} \\
&= \int_0^b ((\|\theta'(a)(t)\|^{p-2}\theta'(a)(t) - \|\theta'(\beta)(t)\|^{p-2}\theta'(\beta)(t))',\ a - \theta(a)(t) \\
&\quad - (\beta - \theta(\beta)(t)))_{R^N} \, dt + \int_0^b ((\|\theta'(a)(t)\|^{p-2}\theta'(a)(t) \\
&\quad - \|\theta'(\beta)(t)\|^{p-2}\theta'(\beta)(t))', \theta(a)(t) - \theta(\beta)(t))_{R^N} \, dt. \qquad (8)
\end{aligned}$$

Since by definition $\theta(a)(\cdot)$ and $\theta(\beta)(\cdot)$ are solutions of (4) with boundary values a and β respectively, we can write that

$$\begin{aligned}
&(\|\theta'(a)(t)\|^{p-2}\theta'(a)(t) - \|\theta'(\beta)(t)\|^{p-2}\theta'(\beta)(t))' \\
&= \|\theta(a)(t)\|^{p-2}\theta(a)(t) - \|\theta(\beta)(t)\|^{p-2}\theta(\beta)(t) \quad \text{a.e. on } T.
\end{aligned}$$

Using this fact in the second integral of the rhs in (8) and performing an integration by parts on the first integral, we obtain

$$\begin{aligned}
&(\rho(a) - \rho(\beta), a - \beta)_{R^N} \\
&= \int_0^b [\|\theta'(a)(t)\|^{p-2}(\theta'(a)(t), \theta'(a)(t) - \theta'(\beta)(t))_{R^N} \\
&\quad - \|\theta'(\beta)(t)\|^{p-2}(\theta'(\beta)(t), \theta'(a)(t) - \theta'(\beta)(t))_{R^N}] \, dt \\
&\quad + \int_0^b (\|\theta(a)(t)\|^{p-2}\theta(a)(t) - \|\theta(\beta)(t)\|^{p-2}\theta(\beta)(t), \theta(a)(t) - \theta(\beta)(t))_{R^N} \, dt \\
&\geq 2^{2-p}\left[\int_0^b |\|\theta'(a)(t)\| - \|\theta'(\beta)(t)\||^p \, dt + \int_0^b |\|\theta(a)(t)\| - \|\theta(\beta)(t)\||^p dt\right].
\end{aligned}$$

From this last inequality as before, we infer that $\rho(\cdot)$ is indeed strictly monotone.

Claim 2. $\rho : R^N \to R^N$ is continuous.

Let $a_n \to a$ in R^N as $n \to \infty$ and set $x_n = \theta(a_n)$, $y_n = x_n - a_n$, $n \geq 1$. Taking the inner product of (5) with $y_n(t)$, integrating over T and finally performing an integration by parts, we obtain

$$\begin{aligned}
&\|y_n'\|_p^p + \|y_n + a_n\|_p^p \leq \|g\|_q(\|y_n + a_n\|_p + b^{1/p}\|a_n\|) + \|y_n + a_n\|_p^{p-1} b^{1/p}\|a_n\| \\
&\quad \Rightarrow \|y_n + a_n\|_{1,p}^p \leq \|g\|_q(\|y_n + a_n\|_p + b^{1/p}\|a_n\|) + \|y_n + a_n\|_p^{p-1} b^{1/p}\|a_n\|.
\end{aligned}$$

From this inequality it follows that $\{x_n = y_n + a_n\}_{n\geq 1}$ is bounded in $W^{1,p}(T, R^N)$. Hence $\{\|x_n\|^{p-2}x_n\}_{n\geq 1}$ and $\{\|x_n'\|^{p-2}x_n'\}_{n\geq 1}$ are bounded in $L^q(T, R^N)$. From these facts and eq. (4) it follows that $\{\|x_n'\|^{p-2}x_n'\}_{n\geq 1}$ is bounded in $W^{1,q}(T, R^N)$. Thus by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{x} u$ in $W^{1,p}(T, R^N)$ and $\|x_n'\|^{p-2}\, x_n' \xrightarrow{w} v$

in $W^{1,q}(T,R^N)$ as $n \to \infty$. In particular we have $\|x_n'\|^{p-2}x_n' \xrightarrow{w} \nu'$ in $L^q(T,R^N)$ as $n \to \infty$. Recall that $W^{1,p}(T,R^N)$ is embedded compactly in $C(T,R^N)$. So we have $x_n \to u$ in $C(T,R^N)$ as $n \to \infty$. Note that

$$\begin{aligned}&\|\|x_n(t)\|^{p-2}x_n(t) - \|u(t)\|^{p-2}u(t)\|\\ &\quad\le \|x_n(t)\|^{p-2}\|x_n(t) - u(t)\| + \|u(t)\|\|\|x_n(t)\|^{p-2} - \|u(t)\|^{p-2}|.\end{aligned}$$

From an elementary inequality (see Rudin [14], p. 78), we have

$$\begin{aligned}&\|\|x_n(t)\|^{p-2}x_n(t) - \|u(t)\|^{p-2}u(t)\|\\ &\quad\le\begin{cases}\|x_n\|_\infty^{p-2}\|x_n - u\|_\infty + \|u\|_\infty\|x_n - u\|_\infty^{p-2} & \text{if } 2\le p<3\\ \|x_n\|_\infty^{p-2}\|x_n-u\|_\infty+(p-2)\|x_n - u\|_\infty(\|x_n\|_\infty+\|u\|_\infty)^{p-3} & \text{if } 3\le p<\infty\end{cases}\\ &\quad\Rightarrow \|x_n\|^{p-2}x_n \to \|u\|^{p-2}u \text{ in } C(T,R^N) \text{ as } n\to\infty.\end{aligned}$$

Since for every $n \ge 1$ we have $-(\|x_n'(t)\|^{p-2}x_n'(t))' + \|x_n(t)\|^{p-2}x_n(t) = g(t)$ a.e. on T in the limit as $n\to\infty$ we obtain $-\nu'(t) + \|u(t)\|^{p-2}u(t)= g(t)$ a.e. on T. Also from the compact embedding of $W^{1,q}(T,R^N)$ in $C(T,R^N)$, we have that $\|x_n'\|^{p-2}x_n' \to \nu$ in $C(T,R^N)$ as $n \to \infty$. Consider the map $\sigma : R^N \to R^N$ defined by $\sigma(z)= \|z\|^{p-2}z$. Clearly this map is strictly monotone, continuous and weakly coercive, thus surjective. Hence $\sigma^{-1} : R^N \to R^N$ is well-defined and is easily seen to be continuous. Then we have $x_n'(t) \to \sigma^{-1}(\nu(t))$ for all $t \in T$ as $n \to \infty$. Hence $x_n' \to \sigma^{-1}(\nu)$ in $L^p(T,R^N)$ as $n \to \infty$ and so in the limit we have $-(\|u'(t)\|^{p-2}u'(t))' + \|u(t)\|^{p-2}u(t)= g(t)$ a.e. on T, $u(0)= u(b)= a$, therefore $u = \theta(a)$. Since $W^{1,q}(T,R^N)$ is compactly embedded in $C(T,R^N)$ we have $\|x_n'\|^{p-2}x_n' \to \|u'\|^{p-2}u' = \nu$ in $C(T,R^N)$ and so $\rho(a_n) \to \rho(a)$ as $n \to \infty$, which proves the continuity of $\rho(\cdot)$.

Claim 3. $\rho(\cdot)$ is weakly coercive.

To establish this claim we argue as follows:

$$\begin{aligned}\frac{(\rho(a),a)_{R^N}}{\|a\|} &= \frac{(\|\theta'(a)(b)\|^{p-2}\theta'(a)(b) - \|\theta'(a)(0)\|^{p-2}\theta'(a)(0),a)_{R^N}}{\|a\|}\\ &= \frac{\int_0^b((\|\theta'(a)(t)\|^{p-2}\theta'(a)(t))',\theta(a)(t))_{R^N}\,dt + \|\theta'(a)\|_p^p}{\|a\|}\\ &\ge \frac{\|\theta(a)\|_p^p + \|\theta'(a)\|_p^p - \|h\|_q\|\theta(a)\|_p}{\|a\|} \quad \text{(see eq. (4)).}\end{aligned}$$

Using the mean value theorem for integrals (see Hewitt–Stromberg [8], Theorem 21.69, p. 420), we can find $0 < t_0 < b$ such that

$$b\|\theta(a)(t_0)\| = \int_0^b \|\theta(a)(t)\|\,dt.$$

Hence for every $t \in T$, we have

$$\begin{aligned}&\theta(a)(t) = \theta(a)(t_0)+\int_{t_0}^t \theta'(a)(s)\,ds\\ &\quad\Rightarrow \|\theta(a)(t)\| \le \|\theta(a)(t_0)\| + \|\theta'(a)\|_1 \le \frac{1}{b}\|\theta(a)\|_1 + \|\theta'(a)\|_1\\ &\quad\Rightarrow \|a\| \le \|\theta(a)\|_\infty \le \gamma_1\|\theta(a)\|_{1,p} \quad \text{for some } \gamma_1 > 0.\end{aligned}$$

So we can write

$$\frac{(\rho(a),a)_{R^N}}{\|a\|} \geq \frac{\|\theta(a)\|_p^p + \|\theta'(a)\|_p^p - \|h\|_q\|\theta(a)\|_p}{\gamma_1\|\theta(a)\|_{1,p}}$$
$$\geq \frac{\|\theta(a)\|_{1,p}(\|\theta(a)\|_{1,p}^{p-1} - \|h\|_q)}{\gamma_1\|\theta(a)\|_{1,p}}.$$

If $\|a\| \to \infty$, we have $\|\theta(a)\|_{1,p} \to \infty$. Indeed $\|a\| \leq \|\theta(a)\|_\infty \leq \gamma_2\|\theta(a)\|_{1,p}$ for some $\gamma_2 > 0$ (the last inequality being a consequence of the fact that $W^{1,p}(T,R^N)$ is embedded compactly in $C(T,R^N)$). So $\rho(\cdot)$ is weakly coercive as claimed.

Therefore $\rho(\cdot)$ is monotone, continuous, weakly coercive, hence surjective. So we can find $a \in R^N$ such that $\rho(a) = 0$, which implies that $\|\theta'(a)(0)\|^{p-2}\theta'(a)(0) = \|\theta'(a)(b)\|^{p-2}\theta'(a)(b)$. Acting with σ^{-1} on both sides of this equality, we obtain $\theta'(a)(0) = \theta'(a)(b)$ (recall $\sigma(z) = \|z\|^{p-2}z$). Thus $x = \theta(a)$ is a solution of (3).

Finally to show the uniqueness of this solution, let x, y be two solutions of (3). Via an integration by parts and using the periodic boundary conditions, we obtain

$$0 = \int_0^b (\|x'(t)\|^{p-2}x'(t) - \|y'(t)\|^{p-2}y'(t), x'(t) - y'(t))_{R^N}\,dt$$
$$+ \int_0^b (\|x(t)\|^{p-2}x(t) - \|y(t)\|^{p-2}y(t), x(t) - y(t))_{R^N}\,dt$$
$$\geq 2^{2-p}\left[\int_0^b |\|x'(t)\| - \|y'(t)\||^p\,dt + |\|x(t)\| - \|y(t)\||^p\,dt\right].$$

From this inequality, as before, we infer that $x = y$. Hence the solution of (3) is unique.
Using proposition 2, we can prove an existence theorem for problem (1).

Theorem 3. *If hypotheses* $\mathrm{H(f)}_1$ *hold, then problem* (1) *has a solution.*

Proof. Let $D = \{x \in C^1(T,R^N) : \|x'(\cdot)\|^{p-2}x'(\cdot) \in W^{1,q}(T,R^N), x(0) = x(b), x'(0) = x'(b)\}$ and let $A : D \subseteq L^p(T,R^N) \to L^q(T,R^N)$ be defined by $A(x) = -(\|x'\|^{p-2}x')'$.

Claim 1. $A(\cdot)$ is a maximal monotone operator.

Let $(\cdot,\cdot)_{pq}$ denote the duality brackets for $(L^p(T,R^N), L^q(T,R^N))$. For every $x, y \in D$ we have

$$(A(x) - A(y), x - y)_{pq}$$
$$= \int_0^b (-(\|x'(t)\|^{p-2}x'(t))' + (\|y'(t)\|^{p-2}y'(t))', x(t) - y(t))_{R^N}\,dt$$
$$= \int_0^b (\|x'(t)\|^{p-2}x(t)x'(t) - \|y'(t)\|^{p-2}y'(t), x'(t) - y'(t))_{R^N}\,dt$$

(integration by parts)

$$\geq \int_0^b (\|x'(t)\|^{p-1} - \|y'(t)\|^{p-1})(\|x'(t)\| - \|y'(t)\|)\,dt \geq 0.$$

So $A(\cdot)$ is monotone (in fact strictly monotone). To prove the maximality of $A(\cdot)$ it suffices to show that if $J : L^p(T,R^N) \to L^q(T,R^N)$ is defined by $J(x)(\cdot) = \|x(\cdot)\|^{p-2}x(\cdot)$, then $R(A+J)=L^q(T,R^N)$. Indeed suppose that $(A+J)(\cdot)$ is surjective and let $y \in L^p(T,R^N)$, $\nu \in L^q(T,R^N)$ be such that

$$(A(x)-\nu, x-y)_{pq} \geq 0 \quad \text{for all } x \in D. \tag{9}$$

Since we have assumed that $R(A+J)=L^q(T,R^N)$, we can find $x_1 \in D$ such that

$$A(x_1)+J(x_1)=\nu+J(y).$$

In (9) we set $x=x_1 \in D$. We obtain

$$0 \leq (A(x_1)-A(x_1)-J(x_1)+J(y), x_1-y)_{pq}=(J(y)-J(x_1), x_1-y)_{pq}. \tag{10}$$

But $J(\cdot)$ being the Frechet derivative of the strictly convex map $x \to 1/p\|x\|_p^p$ on $L^p(T,R^N)$ is maximal monotone and strictly monotone. So from (5) it follows that $(J(y)-J(x_1), x_1-y)_{pq}=0$ and so $x_1=y$. Hence $y\in D$ and $\nu=A(x_1)$. This proves the maximality of $A(\cdot)$.

Therefore we have to show that $R(A+J)=L^q(T,R^N)$. This is equivalent to saying that for every $g \in L^q(T,R^N)$ problem (3) has a solution. But this is exactly proposition 2. Moreover, since $J(\cdot)$ is strictly monotone, we see at once that $(A+J)^{-1}: L^q(T,R^N) \to D \subseteq W^{1,p}(T,R^N)$ is well-defined. In what follows we set $L=(A+J)^{-1}$.

Claim 2. $L: L^q(T,R^N) \to D \subseteq W^{1,p}(T,R^N)$ is compact.

Because of the reflexivity of $L^q(T,R^N)$ to prove the claim it suffices to show that if $\nu_n \xrightarrow{w} \nu$ in $L^q(T,R^N)$, then $L(\nu_n) \to L(\nu)$ in $W^{1,p}(T,R^N)$ as $n \to \infty$ (complete continuity of $L(\cdot)$). To this end let $x_n=L(\nu_n)$, $n \geq 1$. Then $x_n \in D$ and we have

$$\begin{aligned} A(x_n)+J(x_n)=\nu_n,\ n\geq 1 &\Rightarrow (A(x_n),x_n)_{pq}+(J(x_n),x_n)_{pq}=(\nu_n,x_n)_{pq} \\ &\Rightarrow \|x_n'\|_p^p+\|x_n\|_p^p \leq \|\nu_n\|_q\|x_n\|_p \\ &\Rightarrow \|x_n\|_{1,p}^{p-1} \leq \sup_{n\geq 1}\|\nu_n\|_q < \infty. \end{aligned}$$

So we see that $\{x_n\}_{n\geq 1}$ is bounded in $W^{1,p}(T,R^N)$. By passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}(T,R^N)$. Since $W^{1,p}(T,R^N)$ is embedded compactly in $L^p(T,R^N)$, we also have that $x_n \to x$ in $L^p(T,R^N)$ as $n \to \infty$. So

$$\begin{aligned} &\lim(A(x_n)+J(x_n),x_n-x)_{pq}=\lim(\nu_n,x_n-x)_{pq}=0 \\ &\Rightarrow \limsup(A(x_n),x_n-x)_{pq} \leq 0 \\ &\quad (\text{since } J: L^p(T,R^N) \to L^q(T,R^N) \text{ is continuous}) \\ &\Rightarrow \limsup\|x_n'\|_p^p \leq \limsup(A(x_n),x)_{pq}. \end{aligned}$$

Because $W^{1,p}(T,R^N)$ is compactly embedded in $C(T,R^N)$, we have that $x_n \to x$ in $C(T,R^N)$ as $n \to \infty$. So $x(0)=x(b)$. Using this and Green's formula (integration by parts), we have

$$\begin{aligned} (A(x_n),x)_{pq} &= \int_0^b \|x_n'(t)\|^{p-2}(x_n'(t),x'(t))_{R^N}\,dt \leq \|x_n'\|_p^{p-1}\|x'\|_p \\ &\Rightarrow \limsup\|x_n'\|_p^p \leq \limsup\|x_n'\|_p^{p-1}\|x'\| \\ &\Rightarrow \limsup\|x_n'\|_p \leq \|x'\|_p. \end{aligned} \tag{11}$$

On the other hand from the weak lower semicontinuity of the norm functional, we have that

$$\|x'\|_p \leq \liminf\|x_n'\|_p. \tag{12}$$

From (11) and (12) it follows that $\|x_n'\|_p \to \|x'\|_p$ as $n \to \infty$. Since $x_n' \xrightarrow{w} x'$ in $L^p(T, R^N)$ as $n \to \infty$ and $L^p(T, R^N)$ is uniformly convex, we deduce that $x_n' \to x'$ in $L^p(T, R^N)$ as $n \to \infty$ (Kadec–Klee property). Hence $x_n \to x$ in $W^{1,p}(T, R^N)$ as $n \to \infty$. Because $A(x_n) + J(x_n) = \nu_n$, we have $[x_n, \nu_n - J(x_n)] \in \operatorname{Gr} A$, $n \geq 1$. From the maximal monotonicity of $A(\cdot)$ (see claim 1), $\operatorname{Gr} A$ is demiclosed in $L^p(T, R^N) \times L^q(T, R^N)$ (see § 2) and so $[x, \nu - J(x)] \in \operatorname{Gr} A$, hence $A(x) + J(x) = \nu$, i.e. $x = L(\nu)$. This proves that $L(\cdot)$ is compact as claimed.

Now let $N : W^{1,p}(T, R^N) \to L^q(T, R^N)$ be the Nemitsky (superposition) operator corresponding to the function $f(t, x, y)$; i.e. $N(x)(\cdot) = f(\cdot, x(\cdot), x'(\cdot))$. Using hypothesis $\mathrm{H(f)}_1(v)$ it is easy to see that $N(\cdot)$ is continuous and bounded (i.e. maps bounded sets to bounded sets). Let $N_1(x) = -N(x) + J(x)$. This is a continuous and bounded map. Evidently the resolution of problem (1), is equivalent to solving the following abstract fixed point problem

$$x = LN_1(x). \tag{13}$$

Note that the continuity and boundedness of N_1 combined with claim 2, imply that $LN_1 : W^{1,p}(T, R^N) \to W^{1,p}(T, R^N)$ is a compact map. Then by virtue of theorem 1, we will be able to solve (13) if we show the validity of the following claim:

Claim 3. $\Gamma = \{x \in D \subseteq W^{1,p}(T, R^N) : x = \lambda LN_1(x), 0 < \lambda < 1\}$ is bounded in $W^{1,p}(T, R^N)$.

So let $x \in \Gamma$. By definition we have

$$\begin{aligned} x &= \lambda LN_1(x) \quad \text{for some} \quad 0 < \lambda < 1 \\ &\Rightarrow (A + J)\left(\frac{1}{\lambda}x\right) = N_1(x) \\ &\Rightarrow -(\|x'(t)\|^{p-2}x'(t))' = -\lambda^{p-1}f(t, x(t)), x'(t)) \\ &\quad + (\lambda^{p-1} - 1)\|x(t)\|^{p-2}x(t) \quad \text{a.e. on } T \\ x(0) &= x(b), x'(0) = x'(b). \end{aligned}$$

Take the inner product with $x(t)$ and then integrate over T. We have

$$\begin{aligned} &\int_0^b (-(\|x'(t)\|^{p-2}x'(t))', x(t))_{R^N}\, dt \\ &= \lambda^{p-1} \int_0^b (-f(t, x(t), x'(t)), x(t))_{R^N}\, dt + (\lambda^{p-1} - 1)\|x\|_p^p. \end{aligned} \tag{14}$$

Using Green's identity on the lhs, we have

$$\int_0^b (-(\|x'(t)\|^{p-2}x'(t))', x(t))_{R^N}\, dt = \|x'\|_p^p. \tag{15}$$

Also from hypothesis $\mathrm{H(f)}_1$ (iii) we have

$$\begin{aligned} &\lambda^{p-1} \int_0^b (-f(t, x(t), x'(t)), x(t))_{R^N}\, dt \\ &\leq \lambda^{p-1} a\|x\|_p^p + \lambda^{p-1}\beta \int_0^b \|x(t)\|^r \|x'(t)\|^{p-r}\, dt + \lambda^{p-1}\|c\|_1\|x\|_\infty^s. \end{aligned}$$

Let $\tau = p - r$ and set $\eta = p/r$, $\eta' = p/\tau$. Using Hölder's inequality, we have

$$\int_0^b \|x(t)\|^r \|x'(t)\|^\tau \,\mathrm{d}t \le \left(\int_0^b \|x(t)\|^{r\eta} \,\mathrm{d}t\right)^{1/\eta} \left(\int_0^b \|x'(t)\|^{\tau\eta'}\right)^{1/\eta'} = \|x\|_p^r \|x'\|_p^\tau.$$

So we have

$$\begin{aligned}\lambda^{p-1} &\int_0^b (-f(t,x(t),x'(t)),x(t))_{R^N} \,\mathrm{d}t \\ &\le \lambda^{p-1} a\|x\|_p^p + \lambda^{p-1}\beta\|x\|_p^r\|x'\|_p^\tau + \lambda^{p-1}\|c\|_1\|x\|_\infty^s. \end{aligned} \tag{16}$$

We will also show that for all $x \in \Gamma$, we have $\|x\|_\infty \le M$ where $M > 0$ is as in hypothesis $\mathrm{H(f)}_1$ (iv). To this end let $r(t) = \|x(t)\|^p$ and let $t_0 \in T$ be the point where $r(\cdot)$ attains its maximum and suppose that $r(t_0) > M^p$. First assume that $0 < t_0 < b$. We have $r'(t_0) = p\|x(t_0)\|^{p-2}(x'(t_0),x(t_0))_{R^N} = 0 \Rightarrow (x'(t_0),x(t_0))_{R^N} = 0$. So by virtue of hypothesis $\mathrm{H(f)}_1$ (iv), we can find $c, \delta > 0$ such that for almost all $t \in T$,

$$\inf[(f(t,x,y),x) + \|y\|^p : \|x(t_0) - x\| + \|x'(t_0) - y\| < \delta] \ge c.$$

Note that $x \in \Gamma \subseteq D$ and so $\|x'(\cdot)\|^{p-2}x'(\cdot) \in W^{1,q}(T,R^N)$. Since $W^{1,q}(T,R^N)$ is embedded continuously (in fact compactly) in $C(T,R^N)$, we have that $t \to \|x'(t)\|^{p-2} x'(t) \in C(T,R^N)$ and so $t \to \sigma^{-1}(\|x'(t)\|^{p-2}x'(t)) = x'(t) \in C(T,R^N)$ (recall that $\sigma(z) = \|z\|^{p-2}z$; see the proof of proposition 2). Hence $\Gamma \subseteq C^1(T,R^N)$ and so we can find $\delta_1 > 0$ such that if $t_0 < t \le t_0 + \delta_1$ we have

$$\|x(t_0) - x(t)\| + \|x'(t_0) - x'(t)\| < \delta.$$

So for almost all $t \in (t_0, t_0 + \delta_1]$, we have

$$\begin{aligned} &\lambda^{p-1}(f(t,x(t),x'(t)),x(t))_{R^N} + \lambda^{p-1}\|x'(t)\|^p \ge \lambda^{p-1}c \\ &\Rightarrow ((\|x'(t)\|^{p-2}x'(t))',x(t))_{R^N} + (\lambda^{p-1}-1)\|x(t)\|^p + \lambda^{p-1}\|x'(t)\|^p \ge \lambda^{p-1}c, \\ &\Rightarrow \int_{t_0}^t ((\|x'(s)\|^{p-2}x'(s))',x(s))_{R^N} \,\mathrm{d}s + (\lambda^{p-1} - 1)\int_{t_0}^t \|x(s)\|^p \,\mathrm{d}s \\ &\quad + \lambda^{p-1}\int_{t_0}^t \|x'(s)\|^p \,\mathrm{d}s \ge \lambda^{p-1}c(t - t_0). \end{aligned} \tag{17}$$

Using Green's identity (integration by parts) on the first integral of the lhs, of the last inequality, we obtain

$$\begin{aligned} \int_{t_0}^t ((\|x'(s)\|^{p-2}x'(s))',x(s))_{R^N} \,\mathrm{d}s &= \|x'(t)\|^{p-2}(x'(t),x(t))_{R^N} \\ &- \|x'(t_0)\|^{p-2}(x'(t_0),x(t_0))_{R^N} - \int_{t_0}^t \|x'(s)\|^p \,\mathrm{d}s. \end{aligned} \tag{18}$$

Using (18) in (17) and since $0 < \lambda < 1$ and $(x'(t_0),x(t_0))_{R^N} = 0$, we have

$$\begin{aligned} &\|x'(t)\|^{p-2}(x'(t),x(t))_{R^N} \ge \lambda^{p-1}c(t - t_0) > 0 \quad \text{for all} \quad t_0 < t \le t_0 + \delta_1 \\ &\Rightarrow r'(t) > 0, \quad \text{i.e.} \quad r(t) > r(t_0). \end{aligned}$$

This contradicts the choice of $t_0 \in T$. So $\|x(t_0)\| \le M$. Next assume that $t_0 = 0$. Then we have $r(0) = r(b)$ and $r(0) \le 0$ and $r'(b) \ge 0$. Hence $0 \ge r'(0) = \|x(0)\|^{p-2}(x'(0),$

$x(0))_{R^N} = \|x(b)\|^{p-2}(x'(b), x(b))_{R^N} = r'(b) \geq 0 \Rightarrow r'(0) = 0$ and so the previous argument applies. Similarly if $t_0 = b$. So in all cases we conclude that $\|x\|_\infty \leq M$ for all $x \in \Gamma$. Using this fact together with estimates (15) and (16) in (14), we obtain

$$\begin{aligned} \|x'\|_p^p &\leq \lambda^{p-1} a \|x\|_p^p + \lambda^{p-1}\beta\|x\|_p^r\|x'\|_p^\tau + \lambda^{p-1}M^s\|c\|_1 \\ &\leq aM^b b + \beta M^r b^{r/p}\|x'\|_p^\tau + M^s\|c\|_1 \quad (\tau < p). \end{aligned}$$

From this inequality, we deduce that there exists $M_1 > 0$ such that for all $x \in \Gamma$ we have

$$\begin{aligned} &\|x'\|_p \leq M_1 \\ &\quad \Rightarrow \Gamma \text{ is bounded in } W^{1,p}(T, R^N). \end{aligned}$$

Apply theorem 1 to conclude that there exists $x \in D$ which solves (13). Evidently this is a solution of problem (1).

Remark. Theorem 3 partially extends theorem 1 of Knobloch [9], where $p = 2$ and f is continuous (see also theorem 6.1 and corollary 6.2 of Mawhin [10]). Also partially improves theorems 4.1 and 4.5 of Guo [6] where $N = 1$ and f is continuous. Finally it partially extends the existence result of Zhang [16] (who deals with the Dirichlet problem) to the periodic problem.

4. Nonlinear scalar bvp problems

In this section we deal with the scalar nonlinear boundary value problem (2). We introduce the following hypotheses on the data of (2).

$H(f)_2$: $f : T \times R \times R \to R$ satisfies hypotheses $H(f)_2$ with $N = 1$.
$H(\xi)$: $\xi_1 : R \to 2^R$ and $\xi_2 : R \to 2^R$ are maximal monotone maps with $0 \in \xi_1(0)$ and $0 \in \xi_2(0)$.

Remark. It is well-known from convex analysis that $\xi_1 = \partial\phi_1$ and $\xi_2 = \partial\phi_2$ with ϕ_1, ϕ_2: $R \to \overline{R} = R \cup \{+\infty\}$ being proper (i.e. $\phi_i \not\equiv +\infty$, $i = 1, 2$), convex and lower semicontinuous. In fact $\phi_i(r) = \int_0^r \xi_i^0(s)\,ds$ where $\xi_i^0(s) = \text{proj}\,(0; \xi_i(s))$ (i.e. $\xi_i^0(s)$ is the element of $\xi_i(s)$ of minimal absolute value), $i = 1, 2$. The map $s \to \xi_i^0(s)$ is increasing and $\xi_i(s) = [\xi_i^0(s^-), \xi_i^0(s^+)]$, $i = 1, 2$.

Let $D = \{x \in C^1(T) : |x'|^{p-2}x' \in W^{1,q}(T),\ x'(0) \in \xi_1(x(0)), -x'(b) \in \xi_2(x(b))\}$ and consider the operator $L : D \subseteq L^p(T) \to L^q(T)$ defined by

$$L(x) = -(|x'|^{p-2}x'), \quad x \in D.$$

PROPOSITION 4. *If hypotheses* $H(f)_2$ *and* $H(\xi)$ hold, *then* L *is maximal monotone.*

Proof. From the proof of theorem 3 we know that it suffices to show that $R(L + J) = L^q(T)$ with $J : L^p(T) \to L^q(T)$ defined by $J(x) = |x|^{p-2}x$. This surjective property of $L + J$ is equivalent to saying that for every $g \in L^q(T)$ the problem

$$\left.\begin{aligned} &-(|x'(t)|^{p-2}x'(t))' + |x(t)|^{p-2}x(t) = g(t) \quad \text{a.e. on } T \\ &x'(0) \in \xi_1(x(0)), \ -x'(b) \in \xi_2(x(b)) \end{aligned}\right\} \quad (19)$$

has a solution, where the notion of solution is defined as in § 3. To solve (19) we proceed as in the proof of proposition 2. So let $\nu, w \in R$ and consider the following auxiliary problem

$$\begin{aligned} &-(|x'(t)|^{p-2}x'(t))' + |x(t)|^{p-2}x(t) = g(t) \ \text{ a.e. on } T \\ &x(0) = \nu, \ x(b) = w. \end{aligned} \tag{20}$$

Let $\gamma(t) = (1 - \frac{t}{b})\nu + \frac{t}{b}w$, $t \in T$ and set $y(t) = x(t) - \gamma(t)$. Then if we rewrite (20) in terms of the unknown function $y(\cdot)$, we have

$$\begin{aligned} &-\left(\left|y'(t) + \frac{w-\nu}{b}\right|^{p-2}\left(y'(t) + \frac{w-\nu}{b}\right)\right)' \\ &\qquad + |y(t) + \gamma(t)|^{p-2}(y(t) + \gamma(t)) = g(t) \ \text{ a.e. on } T \\ &y(0) = y(b) = 0. \end{aligned} \tag{21}$$

We will solve the homogeneous Dirichlet problem (21). To this end consider the operator $A : W_0^{1,p}(T) \to W^{-1,q}(T)$ defined by

$$\begin{aligned} \langle A(y), z\rangle = &\int_0^b \left|y'(t) + \frac{w-\nu}{b}\right|^{p-2}\left(y'(t) + \frac{w-\nu}{b}\right)z'(t)\,\mathrm{d}t \\ &+ \int_0^b |y(t) + \gamma(t)|^{p-2}(y(t) + \gamma(t))z(t)\,\mathrm{d}t, \end{aligned}$$

whereas before $\langle \cdot, \cdot \rangle$ denotes the duality brackets for the pair $(W_0^{1,p}(T), W^{-1,q}(T))$.

Claim 1. $A(\cdot)$ is monotone, demicontinuous (hence maximal monotone) and weakly coercive.

To establish the monotonicity of $A(\cdot)$, let $y, z \in W_0^{1,p}(T)$. We have

$$\begin{aligned} &\langle A(y) - A(z), y - z\rangle \\ &= \int_0^b \left[\left|y' + \frac{w-\nu}{b}\right|^{p-2}\left(y' + \frac{w-\nu}{b}\right)(y' - z') \right. \\ &\quad \left. - \left|z' + \frac{w-\nu}{b}\right|^{p-2}\left(z' + \frac{w-\nu}{b}\right)(y' - z')\right]\mathrm{d}t \\ &\quad + \int_0^b [|y+\gamma|^{p-2}(y+\gamma)(y-z) - |z+\gamma|^{p-2}(z+\gamma)(y-z)]\,\mathrm{d}t \\ &= \int_0^b \left[\left|y' + \frac{w-\nu}{b}\right|^{p-2}\left(y' + \frac{w-\nu}{b}\right)\left(y' + \frac{w-\nu}{b} - z' - \frac{w-\nu}{b}\right)\right. \\ &\quad \left. - \left|z' + \frac{w-\nu}{b}\right|^{p-2}\left(z' + \frac{w-\nu}{b}\right)\left(y' + \frac{w-\nu}{b} - z' - \frac{w-\nu}{b}\right)\right]\mathrm{d}t \\ &\quad + \int_0^b [|y+\gamma|^{p-2}(y+\gamma)(y+\gamma-z-\gamma) \\ &\quad - |z+\gamma|^{p-2}(z+\gamma)(y+\gamma-z-\gamma)]\mathrm{d}t \\ &\geq \int_0^b \left[\left|y' + \frac{w-\nu}{b}\right|^{p} - \left|y' + \frac{w-\nu}{b}\right|^{p-1}\left|z' + \frac{w-\nu}{b}\right| - \left|z' + \frac{w-\nu}{b}\right|^{p-1}\left|y' + \frac{w-\nu}{b}\right|\right. \end{aligned}$$

$$+\left|z'+\frac{w-\nu}{b}\right|^{p}\Bigg]\,\mathrm{d}t+\int_0^b[|y+\gamma|^p-|y+\gamma|^{p-1}|z+\gamma|$$

$$-|z+\gamma|^{p-1}|y+\gamma|+|z+\gamma|^p]\,\mathrm{d}t$$

$$=\int_0^b\left(\left|y'+\frac{w-\nu}{b}\right|^{p-1}-\left|z'+\frac{w-\nu}{b}\right|^{p-1}\right)\left(\left|y'+\frac{w-\nu}{b}\right|-\left|z'+\frac{w-\nu}{b}\right|\right)\mathrm{d}t$$

$$+\int_0^b(|y+\gamma|^{p-1}-|z+\gamma|^{p-1})(|y+\gamma|-|z+\gamma|)\,\mathrm{d}t\geq 0\quad(\text{see §2}).$$

This proves the monotonicity of $A(\cdot)$. For the demicontinuity, let $y_n\to y$ in $W^{1,p}(T)$ as $n\to\infty$. For every $z\in W_0^{1,p}(T)$, we have

$$|\langle A(y_n)-A(y),z\rangle|$$

$$\leq\left|\int_0^b\left(\left|y_n'+\frac{w-\nu}{b}\right|^{p-2}\left(y_n'+\frac{w-\nu}{b}\right)z'-\left|y'+\frac{w-\nu}{b}\right|^{p-2}\left(y'+\frac{w-\nu}{b}\right)z'\right)\mathrm{d}t\right|$$

$$+\left|\int_0^b(|y_n-\gamma|^{p-2}(y_n+\gamma)z-|y+\gamma|^{p-2}(y+\gamma)z)\,\mathrm{d}t\right|.$$

By passing to a subsequence if necessary, we may assume that $y_n'(t)\to y'(t)$ a.e. on T and $y_n(t)\to y(t)$ for all $t\in T$. So via the extended dominated convergence theorem (see Ash [1], theorem 7.5.2, p. 295), we have that

$$|\langle A(y_n)-A(y),z\rangle|\to 0\quad\text{as } n\to\infty.$$

Since $z\in W_0^{1,p}(T)$ was arbitrary, we conclude that $A(y_n)\xrightarrow{w}A(y)$ in $W^{-1,q}(T)$ as $n\to\infty$. So $A(\cdot)$ is demicontinuous.

Finally we will show the weak coercivity of $A(\cdot)$. Indeed we have

$$\langle A(y),y\rangle=\int_0^b\left|y'+\frac{w-\nu}{b}\right|^{p-2}\left(y'+\frac{w-\nu}{b}\right)y'\,\mathrm{d}t+\int_0^b|y+\gamma|^{p-2}(y+\gamma)y\,\mathrm{d}t.$$

Note that

$$\int_0^b\left|y'+\frac{w-\nu}{b}\right|^{p-2}\left(y'+\frac{w-\nu}{b}\right)y'\,\mathrm{d}t$$

$$=\int_0^b\left|y'+\frac{w-\nu}{b}\right|^{p-2}\left(y'+\frac{w-\nu}{b}\right)\left(y'+\frac{w-\nu}{b}\right)\mathrm{d}t$$

$$-\int_0^b\left|y'+\frac{w-\nu}{b}\right|^{p-2}\left(y'+\frac{w-\nu}{b}\right)\left(\frac{w-\nu}{b}\right)\mathrm{d}t$$

$$\geq\left\|y'+\frac{w-\nu}{b}\right\|_p^p-c_1\left\|y'+\frac{w-\nu}{b}\right\|_p^{p-1}\quad\text{for some } c_1>0.\tag{22}$$

Similarly we have

$$\int_0^b|y+\gamma|^{p-2}(y+\gamma)y\,\mathrm{d}t\geq\|y+\gamma\|_p^p-c_2\|y+\gamma\|_p^{p-1}\quad\text{for some } c_2>0.\tag{23}$$

From (22) and (23) it follows that

$$\langle A(y),y\rangle\geq\|y+\gamma\|_{1,p}^p-c_3\|y+\gamma\|_{1,p}^{p-1}\quad\text{with } c_3=\max\{c_1,c_2\}.$$

This then gives us the desired weak coercivity of $A(\cdot)$. So we have proved the claim.

Recall that a monotone, demicontinuous and weakly coercive operator is surjective. So we can find $y \in W_0^{1,p}(T)$ such that $A(y) = g$. For every $\psi \in C_0^\infty(T)$ we have

$$\int_0^b \left|y' + \frac{w-\nu}{b}\right|^{p-2}\left(y' + \frac{w-\nu}{b}\right)\psi' \,\mathrm{d}t = \int_0^b (g(t)\psi(t) - |y+\gamma|^{p-2}(y+\gamma)\psi) \,\mathrm{d}t.$$

From the definition of the distributional derivative it follows that

$$-\left(\left|y' + \frac{w-\nu}{b}\right|^{p-2}\left(y' + \frac{w-\nu}{b}\right)\right)'(t) = g(t) - (|y+\gamma|^{p-2}(y+\gamma))(t) \quad \text{a.e. on } T$$

$\Rightarrow y(\cdot)$ is a solution of (21).

Moreover, if x, y are two solutions of (21), we have

$$\begin{aligned} 0 &= \int_0^b (|x'|^{p-2}x' - |y'|^{p-2}y')(x' - y') \,\mathrm{d}t + \int_0^b (|x|^{p-2}x - |y|^{p-2}y)(x - y) \,\mathrm{d}t \\ &\geq 2^{2-p} \int_0^b \big| |x'| - |y'| \big|^p \,\mathrm{d}t + 2^{2-p} \int_0^b \big| |x| - |y| \big|^p \,\mathrm{d}t. \end{aligned}$$

From this inequality as in the proof of proposition 2, we obtain $x = y$. So the solution of (21) is unique, hence so is that of (20). Denote the unique solution of (20) by $\theta(\nu, w)$. By definition $\theta(\nu, w)(\cdot) \in C^1(T)$. So we can define $\rho : R \times R \to R \times R$ by

$$\rho(\nu, w) = (-|\theta(\nu, w)'(0)|^{p-2}\theta(\nu, w)'(0),\ |\theta(\nu, w)'(b)|^{p-2}\theta(\nu, w)'(b)).$$

Claim 2. $\rho(\cdot, \cdot)$ is monotone.

Let $x = \theta(a, \beta)$ and $x_1 = \theta(a_1, \beta_1)$. Using Green's formula, we have

$$\begin{aligned} &\left(\rho(a, \beta) - \rho(a_1, \beta_1), \begin{pmatrix} a - a_1 \\ \beta - \beta_1 \end{pmatrix}\right)_{R^2} = -(|x'(0)|^{p-2}x'(0) \\ &\quad - |x_1'(0)|^{p-2}x_1'(0))(a - a_1) + (|x'(b)|^{p-2}x'(b) - |x_1'(b)|^{p-2}x_1'(b))(\beta - \beta_1) \\ &= \int_0^b |x'(t)|^{p-2}x'(t)(x'(t) - x_1'(t)) \,\mathrm{d}t - \int_0^b |x_1'(t)|^{p-2}x_1'(t)(x'(t) - x_1'(t)) \,\mathrm{d}t \\ &\quad + \int_0^b ((|x'(t)|^{p-2}x'(t))' - (|x_1'(t)|^{p-2}x_1'(t))')(x(t) - x_1(t)) \,\mathrm{d}t. \end{aligned}$$

Note that by the elementary inequality mentioned in § 2, we have

$$\int_0^b |x'(t)|^{p-2}x'(t)(x'(t) - x_1'(t)) \,\mathrm{d}t - \int_0^b |x_1'(t)|^{p-2}x_1'(t)(x'(t) - x_1'(t)) \,\mathrm{d}t \geq 0.$$

Also since x, x_1 are solutions of (20), we have

$$\begin{aligned} &\int_0^b ((|x'(t)|^{p-2}x'(t))' - (|x_1'(t)|^{p-2}x_1'(t))')(x(t) - x_1(t)) \,\mathrm{d}t \\ &= \int_0^b (|x(t)|^{p-2}x(t) - |x_1(t)|^{p-2}x_1(t))(x(t) - x_1(t)) \,\mathrm{d}t \geq 0. \end{aligned}$$

So finally we obtain

$$\left(\rho(a,\beta)-\rho(a_1,\beta_1),\begin{pmatrix}a-a_1\\ \beta-\beta_1\end{pmatrix}\right)_{R^2}\geq 0$$

$$\Rightarrow \rho(\cdot,\cdot) \text{ is monotone.}$$

Claim 3. $\rho(\cdot,\cdot)$ is continuous.

Assume $\nu_n \to \nu$ and $w_n \to w$ in R as $n\to\infty$. Set $x_n=\theta(\nu_n,w_n)$, $n\geq 1$, and $x=\theta(\nu,w)$. As before we introduce $\gamma_n(t)=(1-\frac{t}{b})\nu_n+\frac{t}{b}w_n$, $n\geq 1$ and $\gamma(t)=(1-\frac{t}{b})\nu+\frac{t}{b}w$ and define $y_n=x_n-\gamma_n$, $n\geq 1$, and $y=x-\gamma$. We have

$$-\left(\left|y_n'(t)+\frac{w-\nu}{b}\right|^{p-2}\left(y_n'(t)+\frac{w-\nu}{b}\right)\right)'+|y_n+\gamma_n|^{p-2}(y_n+\gamma_n)=g(t)\quad \text{a.e. on } T$$

$$y_n(0)=y_n(b)=0.$$

Multiplying with $y_n(t)$ and then integrating over T, we obtain

$$\int_0^b -(|(y_n+\gamma_n)'|^{p-2}(y_n+\gamma_n)')'y_n\,dt$$
$$+\int_0^b |y_n+\gamma_n|^{p-2}(y_n+\gamma_n)y_n\,dt=\int_0^b g(t)y_n(t)\,dt.$$

Using Green's formula, we have

$$\int_0^b -(|(y_n+\gamma_n)'|^{p-2}(y_n+\gamma_n)')'y_n\,dt=\int_0^b |y_n'+\gamma_n'|^{p-2}(y_n'+\gamma_n')y_n'\,dt$$
$$=\|y_n'+\gamma_n'\|_p^p-\int_0^b |y_n'+\gamma_n'|(y_n'+\gamma_n')\gamma_n'\,dt$$
$$\geq \|y_n'+\gamma_n'\|_p^p-c_4\|y_n'+\gamma_n'\|_p^{p-1}\quad \text{for some}\quad c_4>0.$$

Also we have that

$$\int_0^b |y_n+\gamma_n|^{p-2}(y_n+\gamma_n)\gamma_n\,dt\geq \|y_n+\gamma_n\|_p^p-\|y_n+\gamma_n\|_p^{p-1}\|\gamma_n\|_p.$$

Thus finally we have

$$\|y_n'+\gamma_n'\|_p^p+\|y_n+\gamma_n\|_p^p\leq \|g\|_q(\|y_n+\gamma_n\|_p+\|\gamma_n\|_p)$$
$$+\|y_n+\gamma_n\|_p^{p-1}\|\gamma_n\|_p+c_4\|y_n'+\gamma_n'\|_p^{p-1}.$$

Since $\sup_{n\geq 1}\|\gamma_n\|_p<\infty$, from this last inequality it follows that $\{x_n=y_n+\gamma_n\}_{n\geq 1}$ is bounded in $W^{1,p}(T)$. Then $\{|x_n|^{p-2}x_n\}_{n\geq 1}$ and $\{|x_n'|^{p-2}x_n'\}_{n\geq 1}$ are both bounded in $L^q(T)$ and using eq. (20) we see that $\{|x_n'|^{p-2}x_n'\}_{n\geq 1}$ is bounded in $W^{1,q}(T)$. Thus we may assume that $x_n \xrightarrow{w} u$ in $W^{1,p}(T)$ and $|x_n'|^{p-2}x_n' \xrightarrow{w} \nu$ in $W^{1,q}(T)$ as $n\to\infty$. In particular $(|x_n'|^{p-2}x_n')' \xrightarrow{w} \nu'$ in $L^q(T)$ and $x_n\to u$ in $C(T)$ as $n\to\infty$. Moreover, as in the proof of proposition 2, we can check that $|x_n(\cdot)|^{p-2}x_n(\cdot)\to |u(\cdot)|^{p-2}u(\cdot)$ in $C(T)$ as $n\to\infty$. Hence in the limit as $n\to\infty$, we have

$$-\nu'(t)+|u(t)|^{p-2}u(t)=g(t)\quad \text{a.e. on } T$$
$$u(0)=\nu,\quad u(b)=\nu.$$

Since $|x_n'|^{p-2}x_n' \xrightarrow{w} \nu$ in $W^{1,q}(T)$, we have $|x_n'|^{p-2}x_n' \to \nu$ in $C(T)$ as $n \to \infty$ (recall that $W^{1,q}(T)$ is embedded compactly in $C(T)$). So $\sigma^{-1}(|x_n'|^{p-2}x_n') = x_n' \to \sigma^{-1}(\nu)$ in $C(T)$ as $n \to \infty$ (recall $\sigma(z) = |z|^{p-2}z$) and so $\sigma^{-1}(\nu) = u'$, hence $\nu = |u'|^{p-2}u'$. Thus in the limit as $n \to \infty$, we have

$$
\begin{aligned}
&-(|u'(t)|^{p-2}u'(t))' + |u(t)|^{p-2}u(t) = g(t) \quad \text{a.e. on } T\\
&u(0) = \nu, \quad u(b) = w\\
&\Rightarrow u = \theta(\nu, w) = x.
\end{aligned}
$$

Also $|x_n'|^{p-2}x_n' \to |x'|^{p-2}x'$ in $C(T)$ as $n \to \infty$ and so finally $\theta(\nu_n, w_n)'(t) \to \theta(\nu, w)'(t)$ for all $t \in T$ as $n \to \infty$. Therefore we conclude that $\rho(\nu_n, w_n) \to \rho(\nu, w)$ which proves the continuity of $\rho(\cdot,\cdot)$.

Claim 4. $\rho(\cdot,\cdot)$ is weakly coercive.

We have

$$
\frac{\left(\rho(\nu,w), \begin{pmatrix}\nu\\ w\end{pmatrix}\right)_{R^2}}{\left\|\begin{pmatrix}\nu\\ w\end{pmatrix}\right\|} = \frac{|x'(b)|^{p-2}x'(b)w - |x'(0)|^{p-2}x'(0)\nu}{\left\|\begin{pmatrix}\nu\\ w\end{pmatrix}\right\|},
$$

where $x = \theta(\nu, w)$. From Green's formula we have

$$
\begin{aligned}
\frac{\left(\rho(\nu,w), \begin{pmatrix}\nu\\ w\end{pmatrix}\right)_{R^2}}{\left\|\begin{pmatrix}\nu\\ w\end{pmatrix}\right\|} &= \frac{\int_0^b(|x'(t)|^{p-2}x'(t))'x(t)\,\mathrm{d}t + \|x'\|_p^p}{\left\|\begin{pmatrix}\nu\\ w\end{pmatrix}\right\|}\\
&= \frac{\|x\|_p^p + \|x'\|_p^p - \int_0^b g(t)x(t)\mathrm{d}t}{\left\|\begin{pmatrix}\nu\\ w\end{pmatrix}\right\|}
\end{aligned}
$$

$$
(\text{since} - (|x'(t)|^{p-2}x'(t))' + |x(t)|^{p-2}x(t) = g(t) \text{ a.e. on } T).
$$

Using the mean value theorem for integrals (see Hewitt–Stromberg [8], theorem 21.69, p. 420), we can find $t_0 \in T$ such that $|x(t_0)|b = \int_0^b |x(t)|\,\mathrm{d}t$. So for every $t \in T$ we have

$$
\begin{aligned}
&|x(t)| \le |x(t_0)| + \int_{t_0}^t |x'(s)|\,\mathrm{d}s \le \frac{1}{b}\|x\|_1 + b^{1/q}\|x'\|_p \le b^{-1/p}\|x\|_p\\
&\quad + b^{1/q}\|x'\|_p \le c_5\|x\|_{1,p} \quad \text{for some } c_5 > 0\\
&\Rightarrow \left\|\begin{pmatrix}\nu\\ w\end{pmatrix}\right\| \le c_6\|x\|_{1,p} \quad \text{for some } c_6 > 0.
\end{aligned}
$$

Thus finally we can write that

$$
\frac{\left(\rho(\nu,w), \begin{pmatrix}\nu\\ w\end{pmatrix}\right)_{R^2}}{\left\|\begin{pmatrix}\nu\\ w\end{pmatrix}\right\|} \ge \frac{\|x\|_{1,p}^p - \|g\|_q\|x\|_{1,p}}{c_6\|x\|_{1,p}}
$$

$$
\Rightarrow \rho(\cdot,\cdot) \quad \text{is weakly coercive.}
$$

Because $\rho(\cdot,\cdot)$ is monotone and continuous, is maximal monotone. Set $\widehat{\xi}_1=\sigma\circ\xi_1$ and $\widehat{\xi}_2=\sigma\circ\xi$ and define $\xi: R\times R\to 2^{R\times R}$ by $\xi(a,\beta)=[\widehat{\xi}_1(a),\widehat{\xi}_2(\beta)]$.

Claim 5. $\xi(\cdot,\cdot)$ is maximal monotone.

First we check the monotonicity of $\xi(\cdot,\cdot)$. It suffices to show the monotonicity of $\widehat{\xi}_i(\cdot)$, $i=1,2$. Let $a\leq\beta$ if $a'\in\xi_1(a)$, $\beta'\in\xi_i(\beta)$, then $a'\leq\beta'$ (since $\xi_i(\cdot)$ is monotone) and so $\sigma(a')\leq\sigma(\beta')$ (since $\sigma(\cdot)$ is monotone). Hence $(\sigma(\beta')-\sigma(a'))(\beta-a)\geq 0$ which proves the monotonicity of $\widehat{\xi}_i(\cdot)$, hence the monotonicity of $\xi(\cdot,\cdot)$ too.

To check the maximality of $\xi(\cdot,\cdot)$ we proceed as follows. Suppose that for all $[a,\beta]\in\operatorname{dom}\xi=\operatorname{dom}\xi_1\times\operatorname{dom}\xi_2$ and all $[a',\beta']\in\xi(a,\beta)=[\widehat{\xi}_1(a),\widehat{\xi}_2(\beta)]$ we have

$$(\sigma(a')-\nu)(a-y)+(\sigma(\beta')-\nu_1)(\beta-y_1)\geq 0.$$

Let $u,u_1\in R$ such that $\nu=\sigma(u),\nu_1=\sigma(u_2)$ (recall that $\sigma(\cdot)$ is surjective). Since ξ_i+I ($I=$ identity map), $i=1,2$, are surjective, we can find $\gamma,\delta\in R$ and $\gamma'\in\xi_1(\gamma)$, $\delta'\in\xi_2(\delta)$ such that

$$\gamma'+\gamma=u+y \text{ and } \delta'+\delta=u_1+y_1.$$

Therefore we have

$$\sigma(\gamma')=\sigma(u+y-\gamma) \text{ and } \sigma(\delta')=\sigma(u_1+y_1-\delta). \tag{24}$$

So take $a=\gamma,\beta=\delta,a'=\gamma'$ and $\beta'=\delta'$. With such choices we have

$$(\sigma(u+y-\gamma)-\sigma(u))(\gamma-y)+(\sigma(u_1+y_1-\delta)-\sigma(u_1))(\delta-y_1)\geq 0.$$

But by virtue of the monotonicity of $\sigma(\cdot)$, each term in the lhs of the above inequality is nonpositive. So we infer that

$$\begin{aligned}&(\sigma(u+y-\gamma)-\sigma(u))(y-\gamma)=0 \quad\text{and } (\sigma(u_1+y_1-\delta)-\sigma(u_1))(y_1-\delta)=0\\ &\Rightarrow y=\gamma,y_1=\delta \quad (\text{since } \sigma \text{ is strictly monotone})\\ &\Rightarrow \gamma'=u,\delta'=u_1 \quad (\text{from (24) and the strict monotonicity of } \sigma).\end{aligned}$$

Thus $\nu\in\widehat{\xi}_1(\gamma)$ and $\nu_1\in\widehat{\xi}_2(\delta)$, which prove the maximality of ξ.

Now set $\mu(\nu,w)=\xi(\nu,w)+\rho(\nu,w)$. Then from Zeidler [15] (theorem 21.I, p. 888), we have that μ is maximal monotone. Moreover, since $(0,0)\in\xi(0,0)$ (see hypothesis H(ξ)) and using claim 4, we can easily see that μ is weakly coercive. Therefore μ is surjective and so we can find $[\nu,w]\in R\times R$ such that $[0,0]\in\mu(\nu,w)$, hence $[\theta(\nu,w)'(0),-\theta(\nu,w)'(b)]=[\widehat{\xi}_1(\nu),\widehat{\xi}_2(w)]$. Then if $x=\theta(\nu,w)$, we can see that it solves problem (19). This proves the maximality of $L(\cdot)$.

This proposition leads us to the following existence theorem for problem (2).

Theorem 5. *If hypotheses* $H(f)_2$ *and* H(ξ) hold, *then problem* (2) *has a solution.*

Proof. By proposition 4, L is maximal monotone. Also $J: L^p(T)\to L^q(T)$ defined by $J(x)=\|x\|_p^{p-2}x$ is strictly monotone and continuous. So $L+J$ is maximal monotone (see Zeidler [15], theorem 32.I, p. 888) and strictly monotone. Therefore $K=(L+J)^{-1}$: $L^q(T)\to D\subseteq W^{1,p}(T)$ is a well-defined maximal monotone operator.

Arguing as in the proof of claim 2 of theorem 3 and using the fact that $0\in\xi_i(0)$, $i=1,2$ (see hypothesis H(ξ)) and since $\operatorname{Gr}\xi_i$ is closed, $i=1,2$ (see § 2), we can check that K is compact. Having this we continue as in the proof of theorem 3 and via the Leray–Schauder principle we obtain a solution for (2).

Remark. A careful reading of the proof of proposition 4 reveals that in the vector case (i.e. $N > 1$) the proof of the maximality of ξ fails for $p > 2$. So it is an open problem whether theorem 5 also holds for vector equations with $p > 2$. It will be interesting to know the answer to this.

Some important special cases of theorem 5 are presented in the corollaries that follow:

(a) Let $K_1, K_2 \subseteq R$ be nonempty closed intervals containing the origin. Let $\xi_i = \partial\delta_{K_i}$, $i = 1, 2$, where $\delta_{K_i}(x) = \begin{cases} 0 \text{ if } x \in K_i \\ +\infty \text{ otherwise} \end{cases}$ and $\partial\delta_{K_i}(\cdot)$ its subdifferential. Then problem (2) takes the following form:

$$\begin{aligned} &-(|x'(t)|^{p-2}x'(t))' + f(t, x(t), x'(t)) = 0 \text{ a.e. on } T \\ &x(0) \in K_1,\ x(b) \in K_2 \\ &x'(0)x(0) = \sup_{v \in K_1} x'(0)v,\ -x'(b)x(b) = \sup_{w \in K_2}(-x'(b))w. \end{aligned} \tag{25}$$

COROLLARY 6

If hypotheses H(f)$_2$ *hold and* ξ_1, ξ_2 *are as above, then problem* (25) *has a solution.*

(b) Let $K_1, K_2 = \{0\}$, in the above case. Then $\xi_i(x) = R$ for all $x \in R$ and so problem (2) becomes the Dirichlet problem

$$\begin{aligned} &-(|x'(t)|^{p-2}x'(t))' + f(t, x(t), x'(t)) = 0 \text{ a.e. on } T \\ &x(0) = x(b) = 0. \end{aligned} \tag{26}$$

COROLLARY 7

If hypotheses H(f)$_2$ *hold, then problem* (26) *has a solution.*

Remark. This corollary partially extends the works of Boccardo *et al* [2], Drabek [4] and DelPino–Elgueta–Manasevich [3], where f is independent of x'. Also it is related to theorems 1 and 2 of Zhang [16] with the nonresonance conditions replaced by the Nagumo–Hartman condition H(f)$_2$ (iv).

(c) Let $K_1, K_2 = R$. Then $\xi_i = \partial\delta_{K_i} = \{0\}$ for $i = 1, 2$. Then problem (2) becomes the homogeneous Neumann problem

$$\begin{aligned} &-(|x'(t)|^{p-2}x'(t))' + f(t, x(t), x'(t)) = 0 \text{ a.e. on } T \\ &x'(0) = x'(b) = 0. \end{aligned} \tag{27}$$

COROLLARY 8

If hypotheses H(f)$_2$ *hold, then problem* (27) *has a solution.*

Remark. This work partially extends theorem 5.2 of Guo [6], where f is independent of x'.

(d) Let $u_1, u_2 : R \to R$ be two contractions. Then $\xi_1 = u_1 - I, \xi_2 = u_2 - I$ are maximal monotone maps (I = identity map). Then problem (2) takes the following form:

$$\begin{aligned} &-(|x'(t)|^{p-2}x'(t))' + f(t, x(t), x'(t)) = 0 \text{ a.e. on } T \\ &x(0) + x'(0) = u_1(x(0)),\quad x(b) - x'(b) = u_2(x(b)). \end{aligned} \tag{28}$$

COROLLARY 9

If hypotheses $\mathrm{H(f)}_2$ *hold and* $u_1, u_2 : R \to R$ *are contractions, then problem* (28) *has a solution.*

Remark. In Gaines–Mawhin [5] (p. 88) we can find semilinear problems (i.e. $p = 2$) with nonlinear boundary conditions.

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