Proc. Indian Acad. Sci. (Math. Sci.), Vol. 109, No. 2, May 1999, pp. 203–209.

On a new mean and functions of bounded deviation

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MS received 1 October 1997; revised 15 December 1998

Abstract. The main purpose of the present paper is to introduce a new type of mean of a function and to study some of its special properties. In the last section we make use of this mean to show that a function of bounded deviation is not necessarily a function of bounded variation.

Keywords. Fractional integral mean; bounded deviation; bounded variation; bounded variation in the mean.

1. Preliminaries

First we recall some standard notations for several of the function classes of interest.

$L^1 = L^1\,[0, 2\pi]$ is the space of 2π-periodic and Lebesgue integrable functions over $[0, 2\pi]$,
$L^\infty = L^\infty\,[0, 2\pi]$ consists of these L^1 functions which are essentially bounded,
$\mathrm{BV} = \mathrm{BV}\,[0, 2\pi]$ consists of those L^1 functions which are of bounded variation on $[0, 2\pi]$.

Bounded deviation (*BD*) (see [7], p. 229; [3]). The class BD of functions of bounded deviation is the subset of L^1 consisting of those functions f such that a relation

$$\left| n\int_a^b f(t)\mathrm{e}^{-int}\mathrm{d}t \right| \leq C \tag{1.1}$$

holds, where C is a constant independent of the sub-interval (a, b) of $[0, 2\pi]$. If in addition f is even, the interval $[0, 2\pi]$ in the above definition can be replaced by $[0, \pi]$.

Bounded variation in the mean (V_1). V_1 denotes the subset of L^1 consisting of those functions f for which there exists a constant Q such that if $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_m, \beta_m)$ is any set of non-overlapping intervals in $[0, 2\pi]$ then

$$\int_0^{2\pi} \left| \sum_{k=1}^{m} [f(\beta_k - t) - f(\alpha_k - t)] \right| \mathrm{d}t \leq Q. \tag{1.2}$$

In [3] it is shown that a function belongs to V_1, if and only if, its sequence of Fourier co-efficients is $0(\frac{1}{n})$. Obviously if a function belongs to BD then its Fourier co-efficients satisfy this order relation and hence $\mathrm{BD} \subseteq V_1$. Also it is easy to see that $\mathrm{BV} \subseteq \mathrm{BD}$. It has been observed by Zygmund ([7], p. 229) that a function of bounded deviation is not necessarily a function of bounded variation.

Introducing a new mean. Let $f \in L^1$. Let the Fourier series of f at $t = x$ be given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{1.3}$$

We write

$$\phi(t) = \phi_x(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}$$
$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\phi(u)\mathrm{d}u, \quad \alpha > 0$$
$$\Phi_0(t) = \phi(t),$$
$$\phi_\alpha(t) = \Gamma(\alpha+1)t^{-\alpha}\Phi_\alpha(t).$$

$\Phi_\alpha(t)$ and $\phi_\alpha(t)$ are respectively known as the α-th integral and α-th integral mean of $\phi(t)$.

As far as the authors are aware, writing $P(t) = \phi(t) - \phi_1(t)$, it was Chandra [4] who first studied some properties of $P(t)$. He made extensive use of this mean (see [4], [5], [6]) for studying the summability and convergence problems of Fourier series. Chandra's work on $P(t)$ motivated us to introduce a new mean as follows:

We define

$$p(k,t) = \sum_{\nu=0}^{k}(-1)^\nu \binom{k}{\nu}\phi_\nu(t), \quad k \in N$$
$$p(0,t) = \phi(t),$$
$$p(1,t) = P(t), \text{ the mean studied by Chandra.}$$

At this stage we wish to acknowledge that our present work has been greatly influenced by the works of Prem Chandra (see [4], [5], [6]).

Let $\alpha \geq 0$, we write $G_\alpha(k,t)$ and $g_\alpha(k,t)$ respectively for the α-th integral and α-th integral mean of $p(k,t)$.

Purpose of the present work. In § 2 of the present paper we first study some special properties of the mean $p(k,t)$. In § 3 we apply Chandra's mean (that is $p(k,t)$ with $k = 1$) to give another demonstration of the known fact ([7], p. 229) that BD(a,b) is not contained in BV(a,b).

2. Properties of $p(k,t)$

We prove the following

Theorem 1. *For each* $k \in N$

(a) $\phi(t) \in \mathrm{BV}[0,\pi] \Leftrightarrow p(k,t) \in \mathrm{BV}[0,\pi]$ *and* $t^{-1}p(k,t) \in L(0,\pi)$

(b) $p(k,t) \in \mathrm{BV}[0,\pi] \Rightarrow p(k+1,t) \in \mathrm{BV}[0,\pi]$

(c) $t^{-1}p(k,t) \in L[0,\pi] \Rightarrow t^{-1}p(k+1,t) \in L[0,\pi]$.

Remarks. The case $k = 1$ of part (a) is due to Chandra [4]. For $k = 0$, the results at (b) and (c) are special cases of known results (see Lemma 1).

We need the following.

Lemma 1. *For* $\beta > \alpha \geq 0$

(i) [1] $\phi_\alpha(t) \in \mathrm{BV}[0,\pi] \Rightarrow \phi_\beta(t) \in \mathrm{BV}[0,\pi]$

(ii) [2] $\phi_\alpha(t)/t \in L[0,\pi] \Rightarrow \phi_\beta(t)/t \in L[0,\pi]$.

Proof of Theorem 1. Part (a) By a simple computation, we get

$$\mathrm{d}(t^k p(k,t)) = t^k \mathrm{d}\phi(t) \tag{2.1}$$

which further ensures that

$$\mathrm{d}\phi(t) = kt^{-1}p(k,t) + \mathrm{d}p(k,t). \tag{2.2}$$

By Lemma 1(i), $p(k,t) \in \mathrm{BV}[0,\pi]$ for every $k \in N$ whenever $\phi(t) \in \mathrm{BV}[0,\pi]$ and hence part (a) follows from (2.2).

Part (b) and (c) For any $k \in N$, we have

$$\begin{aligned} p(k+1,t) &= \sum_{\nu=0}^{k+1} (-1)^\nu \left(\binom{k}{\nu} + \binom{k}{\nu-1} \right) \phi_\nu(t), \\ &= p(k,t) - \sum_{s=0}^{k} (-1)^s \binom{k}{s} \phi_{s+1}(t). \end{aligned} \tag{2.3}$$

Integrating by parts k times in succession, we get

$$\begin{aligned} \int_0^t u^k p(k,u)\mathrm{d}u &= \sum_{s=0}^{k} (-1)^s \binom{k}{s} s!\, t^{k-s} G_{s+1}(k,t), \\ &= t^{k+1} \sum_{s=0}^{k} (-1)^s \binom{k}{s} (s+1)^{-1} g_{s+1}(k,t). \end{aligned} \tag{2.4}$$

In the like manner, we have

$$\begin{aligned} \int_0^t u^{k-\nu}\Phi_\nu(u)\mathrm{d}u &= \sum_{s=0}^{k-\nu} (-1)^s \binom{k-\nu}{s} s!\, t^{k-\nu-s} \Phi_{\nu+s+1}(t) \\ &= t^{k+1} \sum_{s=\nu}^{k} (-1)^{s-\nu} \frac{(k-\nu)!}{(k-s)!(s+1)!} \phi_{s+1}(t). \end{aligned} \tag{2.5}$$

On the other hand using (2.5) in the left side of (2.4), we obtain

$$\begin{aligned} \int_0^t u^k p(k,u)\mathrm{d}u &= \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} \nu! \int_0^t u^{k-\nu}\Phi_\nu(u)\mathrm{d}u \\ &= t^{k+1} \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} \nu! \sum_{n=\nu}^{k} (-1)^{s-\nu} \frac{(k-\nu)!}{(k-s)!(s+1)!} \phi_{s+1}(t) \\ &= t^{k+1} \sum_{\nu=0}^{k} \sum_{s=\nu}^{k} (-1)^s \binom{k}{s} (s+1)^{-1} \phi_{s+1}(t) \end{aligned}$$

$$= t^{k+1}\sum_{s=0}^{k}(-1)^s\binom{k}{s}\phi_{s+1}(t)$$
$$= t^{k+1}[p(k,t) - p(k+1,t)], \tag{2.6}$$

using (2.3).

Now, combining the results of (2.4) and (2.6), we get

$$p(k+1,t) = p(k,t) - \sum_{s=0}^{k}(-1)^s\binom{k}{s}(s+1)^{-1}g_{s+1}(k,t). \tag{2.7}$$

As $g_{s+1}(k,t)$ is the $(s+1)$-th integral mean of $p(k,t)$ by Lemma 1, we obtain for $s = 0,1,2,\ldots$

(A) $p(k,t)\in \mathrm{BV}[0,\pi] \Rightarrow g_{s+1}(k,t)\in \mathrm{BV}[0,\pi]$
(B) $t^{-1}p(k,t)\in L[0,\pi] \Rightarrow t^{-1}g_{s+1}(k,t)\in L[0,\pi]$.

Hence parts (b) and (c) follow at once from (2.7).

COROLLARY

(I) *Let r and s be any pair of positive integers. Then* $\phi(t)\in \mathrm{BV}[0,\pi] \Leftrightarrow p(r,t)\in \mathrm{BV}[0,\pi]$ *and* $p(s,t)/t\in L[0,\pi]$
(II) *Let* $p(k,t)\in \mathrm{BV}[0,\pi]$ *for some* $k\in N$. *Then for* $r,s\in N\, t^{-1}p(r,t)\in L[0,\pi] \Leftrightarrow t^{-1}p(s,t)\in L[0,\pi]$
(III) *Let* $t^{-1}p(k,t)\in L[0,\pi]$ *for some* $k\in N$. *Then for* $r,s\in N\, p(r,t)\in \mathrm{BV}[0,\pi] \Leftrightarrow p(s,t)\in \mathrm{BV}[0,\pi]$.

Proof of (I). By Theorem 1(a), 1(b) and 1(c) for any $r,s\in N\, \phi(t)\in \mathrm{BV}[0,\pi] \Rightarrow p(r,t)\in \mathrm{BV}[0,\pi]$ and $t^{-1}p(s,t)\in L[0,\pi]$. Let $m = \max(r,s)$. By Theorem 1(b) and (c), we have

$$p(r,t)\in \mathrm{BV}[0,\pi] \Rightarrow p(m,t)\in \mathrm{BV}[0,\pi]$$

and

$$t^{-1}p(s,t)\in L[0,\pi] \Rightarrow t^{-1}p(m,t)\in L[0,\pi].$$

Hence by Theorem 1(a)

$$p(r,t)\in \mathrm{BV}[0,\pi] \quad \text{and} \quad t^{-1}p(s,t)\in L[0,\pi] \Rightarrow \phi(t)\in \mathrm{BV}[0,\pi]$$

and this completes the proof of (I).

Clearly (II) and (III) follow from (I) and Theorem 1(a).

The conditions imposed on $p(k,t)$ in Theorem 1(a) are mutually exclusive one being of the Jordan type and other one being of the Dini type. Hence the bounded variation of $p(k,t)$ need not ensure the bounded variation of $\phi(t)$.

3. BD class strictly includes BV class

We prove

Theorem 2. *There exists a function in BD class which is not necessarily in BV class.*

We need the following.

Lemma 2. *There exists a function f such that*

(i) $f \in L^\infty$,
(ii) $p(1,t) \in \mathrm{BV}[0,\pi]$,
(iii) *for* $n \in N$ *and* $0 < 1/n < c \leq \pi$ *there is a positive constant B independent of n and c so that*

$$\left| \int_{1/n}^{c} \frac{p(1,t)}{t} \mathrm{d}t \right| \leq B,$$

(iv) $\lim \int_{1/n}^{c} (p(1,u)/u)\, \mathrm{d}u$ *exists an* $n \to \infty$ *for every c with* $0 < c \leq \pi$, *and*
(v) $p(1,t)/t \notin L(0,\pi)$.

Proof. If f is an even function and $x = 0$, then $\phi(t) = f(t)$. Let $0 < \epsilon < 1/2$ and let us write

$$h(t) = \frac{\cos \sqrt{\log \frac{2\pi}{t}}}{t(\log \frac{2\pi}{t})^{(1/2)+\epsilon}}, \quad 0 < t \leq \pi, \text{ and}$$

$$H(t) = \int_0^t h(u)\mathrm{d}u, \quad 0 < t \leq \pi.$$

We define

$$f(t) = \phi(t) = \frac{\mathrm{d}}{\mathrm{d}t}[tH(t)], \quad 0 < t \leq \pi$$

and elsewhere by periodicity, as an even function.

Then $\Phi_1(t) = tH(t)$, $\phi_1(t) = H(t)$ and

$$p(1,t) = t\,h(t) = \frac{\cos \sqrt{\log \frac{2\pi}{t}}}{(\log \frac{2\pi}{t})^{(1/2)+\epsilon}}, \quad 0 < t \leq \pi.$$

It is easy to verify that f satisfies conditions (i)–(v) of the lemma.

Proof of Theorem 2. Let f be an even 2π-periodic function and $x = 0$. Then $f(t) = \phi(t)$. Suppose that f satisfies (i)–(v) of Lemma 2.

By Theorem 1(a) conditions (i) and (v) taken together ensure that $\phi(t) \notin \mathrm{BV}[0,\pi]$. Now it remains to prove that

$$\left| \int_a^b \phi(t) \mathrm{e}^{-int} \mathrm{d}t \right| \leq C/n \tag{3.1}$$

for every subinterval (a,b) in $[0,\pi]$ where C is some positive constant independent of a and b.

For every c with $0 < c \leq \pi$, we have

$$\int_0^c \phi(t) \frac{1 - \mathrm{e}^{-int}}{t} \mathrm{d}t = \frac{\Phi_1(c)}{c}(1 - \mathrm{e}^{-inc}) - \int_0^c \Phi_1(t) \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1 - \mathrm{e}^{-int}}{t}\right] \mathrm{d}t$$

$$= \phi_1(c)(1-e^{-inc}) - \int_0^c t\phi_1(t)\left[\frac{in\,e^{-int}}{t} - \frac{1-e^{-int}}{t^2}\right]dt$$

$$= \phi_1(c)(1-e^{-inc}) - in\int_0^c \phi_1(t)e^{-int}dt$$

$$+ \int_0^c \phi_1(t)\left(\frac{1-e^{-int}}{t}\right)dt$$

which ensures that

$$\int_0^c \phi(t)e^{-int}dt = \int_0^c p(1,t)e^{-int}dt - \int_0^c p(1,t)\frac{1-e^{-int}}{in\,t}dt + \phi_1(c)\left(\frac{1-e^{-inc}}{in}\right)$$

$$= I - \frac{1}{in}J + \frac{1}{in}K, \quad \text{say.} \tag{3.2}$$

We write

$$J = \int_0^{1/n} p(1,t)\frac{1-e^{-int}}{t}dt + \int_{1/n}^c \frac{p(1,t)}{t}dt - \int_{1/n}^c p(1,t)\frac{e^{-int}}{t}dt$$

$$= J_1 + J_2 - J_3, \quad \text{say.} \tag{3.3}$$

As $p(1,t) \in \mathrm{BV}[0,\pi]$ there exists a positive constant C_1 so that $|p(1,t)| \le \frac{1}{2}C_1$ for all $0 < t \le \pi$ and hence

$$|J_1| \le \frac{1}{2}C_1 \int_0^{1/n} \frac{|1-e^{-int}|}{t}dt$$

$$\le C_1 n \int_0^{1/n} dt = C_1. \tag{3.4}$$

As $p(1,t)$ satisfies condition (iii) there exists a positive constant C_2 independent of c so that

$$|J_2| \le C_2 \text{ for all } n \in N \text{ and } 0 < c \le \pi. \tag{3.5}$$

Now using the familiar technique employed in proving the convergence tests ([7], p. 57 and p. 59) for Fourier series (Jordan's test) and conjugate series (analogue of Jordan's test) it can be proved that

$$|J_3| \le C_3, \tag{3.6}$$

where C_3 is a constant independent of c.

By routine argument $p(1,t) \in \mathrm{BV}[0,\pi]$ implies that

$$|I| \le C_4/n, \tag{3.7}$$

where C_4 is a constant independent of c. Lastly since $f \in L^1$ there exists a positive constant C_5 independent of c such that

$$|K| \le C_5. \tag{3.8}$$

Collecting the results of (3.2)–(3.8), we have

$$\left|\int_0^c \phi(t)e^{-int}dt\right| \le A/n, \tag{3.9}$$

where A is a constant independent of c. As we can write

$$\int_a^b \phi(t)\mathrm{e}^{-int}\mathrm{d}t = \int_0^b \phi(t)\mathrm{e}^{-int}\mathrm{d}t - \int_0^a \phi(t)\mathrm{e}^{-int}\mathrm{d}t$$

the inequality (3.1) follows by an appeal to (3.9) and this completes the proof of Theorem 2.

Acknowledgement

The authors are thankful to the referee for some valuable suggestions which led to the improvement of the paper.

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