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# New integral mean estimates for polynomials

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**Abstract.** In this paper we prove some $L^P$ inequalities for polynomials, where $p$ is any positive number. They are related to earlier inequalities due to A Zygmund, N G De Bruijn, V V Arestov, etc. A generalization of a polynomial inequality concerning self-inversive polynomials, is also obtained.

**Keywords.** Polynomials; integral mean estimates; inequalities in the complex domain; self-inversive polynomials.

## 1. Introduction and statement of results

Let $\mathbb{F}_n$ be the class of polynomials $f(z)=\sum_{j=0}^{n} a_j z^j$ of degree at most $n$. For $f \in \mathbb{F}_n$ define

$$\|f\|_p = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^p \,\mathrm{d}\theta\right\}^{1/p}, \quad 0<p<\infty$$

and

$$\|f\|_\infty = \max_{|z|=1} |f(z)|.$$

If $f \in \mathbb{F}_n$, then according to a famous result known as Bernstein's inequality (for reference see [11] or [17])

$$\|f'\|_\infty \le n\|f\|_\infty. \tag{1}$$

Concerning the maximum modulus of $f \in \mathbb{F}_n$ on a large circle $|z| = R > 1$, we have

$$\|f(Rz)\|_\infty \le R^n\|f\|_\infty. \tag{2}$$

Inequality (2) is a simple deduction from the maximum modulus principle (see [16, p. 346] or [12, p. 158, problem III 269]).

Inequalities (1) and (2) can be obtained by letting $p \to \infty$ in the inequalities

$$\|f'\|_p \le n\|f\|_p, \quad p \ge 1 \tag{3}$$

and

$$\|f(Rz)\|_p \le R^n\|f\|_p R > 1, \quad p > 0 \tag{4}$$

respectively. Inequality (3) is due to Zygmund [19] who proved it for all trigonometric polynomials of degree $n$ and not only for those which are of the form $f(e^{i\theta})$, whereas inequality (4) is a simple consequence of a result of Hardy [9] (see also [13, Th. 5.5]). Since inequality (3) was deduced from M Riesz's interpolation formula [15] by means of

Minkowski's inequality, it was not clear whether the restriction on $p$ was indeed essential. This question was open for a long time. Finally, Arestov [2] proved that (3) remains true for $0 < p < 1$.

If $f(z) \neq 0$ for $|z| < 1$, then for each $p > 0$, inequality (3) can be replaced by [7, 10, 14]

$$\|f'\|_p \leq n \frac{\|f\|_p}{\|1 + z^n\|_p} \tag{5}$$

whereas in this case inequality (4) can be replaced by [1, 6, 14]

$$\|f(Rz)\|_p \leq \frac{\|R^n z^n + 1\|_p}{\|1 + z^n\|_p} \|f\|_p, \quad \text{for each } p > 0. \tag{6}$$

Recently the authors [5] have investigated the dependence of $\|f(Rz) - f(z)\|_p$ on $\|f\|_p$ and proved that if $f \in \mathbb{F}_n$, then for every $p \geq 1$ and $R \geq 1$,

$$\|f(Rz) - f(z)\|_p \leq (R^n - 1)\|f\|_p. \tag{7}$$

For other results of the same nature see [3]. Here we first show that the inequality (7) holds for each $p > 0$. In fact we prove

**Theorem 1.** *If $f \in \mathbb{F}_n$, then for every $p > 0$ and $R \geq 1$,*

$$\|f(Rz) - f(z)\|_p \leq (R^n - 1)\|f\|_p. \tag{8}$$

The result is the best possible and equality holds for $f(z) = \alpha z^n$, $\alpha \neq 0$.

*Remark* 1. Dividing both sides of (8) by $R - 1$ and letting $R \to 1$, we obtain inequality (3) for each $p > 0$.

If $f \neq \mathbb{F}_n$ and $f(z) \neq 0$ for $|z| < 1$, then (8) can be sharpened. In this case we prove the following result which is a generalization of inequality (5).

**Theorem 2.** *If $f \in \mathbb{F}_n$ and $f(z)$ does not vanish in $|z| < 1$, then for every $p > 0$ and $R \geq 1$,*

$$\|f(Rz) - f(z)\|_p \leq \frac{(R^n - 1)}{\|1 + z^n\|_p} \|f\|_p. \tag{9}$$

The result is best possible and equality holds for $f(z) = az^n + b$, $|a| = |b|$.

*Remark* 2. Dividing both sides of (9) by $R - 1$ and letting $R \to 1$, we obtain inequality (4) for each $p > 0$.

A polynomial $f \in \mathbb{F}_n$ is said to be self-inversive if $f(z) = Q(z)$ where $Q(z) = z^n \overline{f(1/\bar{z})}$. It is known [4, 8] that if $f \in \mathbb{F}_n$ is self-inversive polynomial, then for every $p \geq 1$,

$$\|f'\|_p \leq n \frac{\|f\|_p}{\|1 + z^n\|_p}. \tag{10}$$

Finally we present the following result which extends (10) to $p \in (0, 1)$.

**Theorem 3.** *If $f \in \mathbb{F}_n$ is self-inversive polynomial, then for every $p > 0$ and $R \geq 1$,*

$$\|f(Rz) - f(z)\|_p \leq \frac{(R^n - 1)}{\|1 + z^n\|_p} \|f\|_p. \tag{11}$$

The result is sharp and equality holds for $f(z) = z^n + 1$.

*Remark* 3. Dividing both sides of (11) by $R-1$ and letting $R \to 1$, we obtain inequality (10) for each $p > 0$.

## 2. Lemmas

For the proofs of these theorems we need the following lemmas.

*Lemma* 1. *If $f \in \mathbb{F}_n$ and $f(z)$ has all its zeros in $|z| \le k$ where $k \le 1$, then for every $R > 1$,*

$$|f(Rz)| \ge \left(\frac{R+k}{1+k}\right)^n |f(z)| \quad \textit{for } |z| = 1.$$

*Proof of Lemma* 1. Since all the zeros of $f(z)$ lie in $|z| \le k \le 1$, we write

$$f(z) = C\prod_{j=1}^{n}(z - r_j e^{i\theta_j})$$

where $r_j \le k$, $j = 1, 2, \dots, n$, so that, for each $\theta, 0 \le \theta < 2\pi$ and $R > 1$, it can be easily verified that

$$\begin{aligned}\left|\frac{f(Re^{i\theta})}{f(e^{i\theta})}\right| &= \prod_{j=1}^{n}\left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{e^{i\theta} - r_j e^{i\theta_j}}\right| \\ &\ge \prod_{j=1}^{n}\left(\frac{R+r_j}{1+r_j}\right) \ge \prod_{j=1}^{n}\left(\frac{R+k}{1+k}\right) \\ &= \left(\frac{R+k}{1+k}\right)^n.\end{aligned}$$

This implies

$$|f(Re^{i\theta})| \ge \left(\frac{R+k}{1+k}\right)^n |f(e^{i\theta})| \quad \text{for } R > 1, \quad 0 \le \theta < 2\pi$$

Hence

$$|f(Rz)| \ge \left(\frac{R+k}{1+k}\right)^n |f(z)| \quad \text{for } |z| = 1 \quad \text{and } R > 1.$$

This completes the proof of Lemma 1.

*Lemma* 2. *If $f \in \mathbb{F}_n$ and $f(z)$ does not vanish in $|z| < 1$, then*

$$|f(Rz) - f(z)| \le |Q(Rz) - Q(z)| \quad \textit{for } |z| \ge 1 \quad \textit{and } R \ge 1 \tag{12}$$

*where $Q(z) = z^n f(1/\bar{z})$.*

*Proof of Lemma* 2. For $R = 1$, there is nothing to prove. Henceforth we assume that $R > 1$. Since the polynomial $f(z)$ has all its zeros in $|z| \ge 1$, therefore, for every complex number $\beta$ such that $|\beta| > 1$ the polynomial $g(z) = f(z) - \beta Q(z)$, where $Q(z) = z^n \overline{f(1/\bar{z})}$, has all its zeros in $|z| \le 1$. Applying Lemma 1 to the polynomial $g(z)$ with $k = 1$, we get

$$|g(Rz)| \ge \left(\frac{R+1}{2}\right)^n |g(z)| \quad \text{for } |z| = 1 \quad \text{and } R > 1. \tag{13}$$

Clearly $g(Re^{i\theta}) \neq 0$ for every $R > 1$ and $0 \leq \theta < 2\pi$, which implies

$$|g(Rz)| > 0 \quad \text{for } |z| = 1 \quad \text{and } R > 1. \tag{14}$$

Now for points $e^{i\theta}, 0 \leq \theta < 2\pi$, which are not the zeros of $g(z)$, we get from (13)

$$|g(Re^{i\theta})| > |g(e^{i\theta})| \quad \text{for every } R > 1. \tag{15}$$

Since by (14), the inequality (15) is trivially true for those points $e^{i\theta}, 0 \leq \theta < 2\pi$, which are the zeros of $g(z)$, it follows that

$$|g(z)| < |g(Rz)| \quad \text{for } |z| = 1 \quad \text{and } R > 1.$$

Using Rouche's theorem and noting that all the zeros of $g(Rz)$ lie in $|z| \leq (1/R) < 1$, we conclude that the polynomial

$$\begin{aligned} h(z) &= (g(Rz) - g(z)) \\ &= (f(Rz) - f(z) - \beta(Q(Rz) - Q(z)) \end{aligned} \tag{16}$$

has all its zeros in $|z| < 1$ for every $\beta$ with $|\beta| > 1$ and $R > 1$. This implies

$$|f(Rz) - f(z)| \leq |Q(Rz) - Q(z)| \quad \text{for } |z| \geq 1 \quad \text{and } R > 1. \tag{17}$$

If inequality (17) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$ such that

$$|f(Rz_0) - f(z_0)| > |Q(Rz_0) - Q(z_0)|.$$

Since all the zeros of $Q(z)$ lie in $|z| \leq 1$, it follows (as in the case of $g(z)$) that all the zeros of $Q(Rz) - Q(z)$ lie in $|z| < 1$ for every $R > 1$. Hence $Q(Rz_0) - Q(z_0) \neq 0$ with $|z_0| \geq 1$. We take

$$\beta = \frac{f(Rz_0) - f(z_0)}{Q(Rz_0) - Q(z_0)}$$

so that $\beta$ is a well defined real or complex number with $|\beta| > 1$ and with this choice of $\beta$, from (16) we get

$$h(z_0) = 0 \quad \text{where } |z_0| \geq 1.$$

This is clearly a contradiction to the fact that all the zeros of $g(z)$ lie in $|z| < 1$. Thus

$$|f(Rz) - f(z)| \leq |Q(Rz) - Q(z)| \quad \text{for } |z| \geq 1 \quad \text{and } R > 1.$$

This proves Lemma 2.

Next we describe a result of Arestov.

$$\text{For} \quad \gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n) \in \mathbb{C}^{n+1} \quad \text{and} \quad f(z) = \sum_{j=0}^{n} a_j z^j \in \mathbb{F}_n,$$

we define

$$\Lambda_\gamma f(z) = \sum_{j=0}^{n} \gamma_j a_j z^j.$$

The operator $\Lambda_\gamma$ is said to be admissible if it preserves one of the following properties:

(i) $f(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \leq 1\}$,
(ii) $f(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \geq 1\}$.

The result of Arestov may now be stated as follows.

*Lemma* 3 [2, Theorem 4]. *Let $\phi(x)=\psi(\log x)$ where $\psi$ is a convex nondecreasing function on $\mathbb{R}$. Then for all $f \in \mathbb{F}_n$ and each admissible operator $A_\gamma$,*

$$\int_0^{2\pi} \phi(|A_\gamma f(e^{i\theta})|)\mathrm{d}\theta \leq \int_0^{2\pi} \phi(C(\gamma,n)|f(e^{i\theta})|)\mathrm{d}\theta,$$

where $C(\gamma,n)=\max(|\gamma_0|,|\gamma_n|)$.

In particular, Lemma 2 applies with $\phi : x \to x^p$ for every $p \in (0,\infty)$. Therefore, we have

$$\|A_\gamma f\|_p \leq C(\gamma,n)\|f\|_p, \quad 0<p<\infty.$$

Next we use Lemma 3 to prove the following interesting result.

*Lemma* 4. *If $f \in \mathbb{F}_n$ and $f(z)$ does not vanish in $|z|<1$, then for each $p>0$, $R \geq 1$ and $\alpha$ real,*

$$\|(f(Rz)-f(z))+e^{i\alpha}(R^n f(z/R)-f(z))\|_p \leq (R^n-1)\|f\|_p. \tag{18}$$

The result is the best possible and equality holds for $f(z)=az^n+b$, $|a|=|b|$.

*Proof of Lemma* 4. The result is obvious for $R=1$, so we assume $R>1$. First we show that for every $R>1$ and $\alpha$ real, all the zeros of the polynomial

$$R(z)=\sum_{j=0}^{n}\binom{n}{j}\{(R^j-1)+e^{i\alpha}(R^{n-j}-1)\}z^j$$

lie on the unit circle. Let

$$\begin{aligned} H(z)&=\sum_{j=0}^{n}\binom{n}{j}(R^j-1)z^j \\ &=(Rz+1)^n-(z+1)^n. \end{aligned}$$

The zeros $z_k, k=1,2,\ldots,n$ of $H(z)$ are given by

$$z_k=\frac{1-e^{(2k\pi i/n)}}{e^{(2k\pi i/n)}-R}.$$

Since $R>1$, it can be easily seen that $|z_k|<1$, $k=1,2\ldots,n$. Hence all the zeros of $H(z)$ lie in $|z|<1$ for every $R>1$. If now

$$\begin{aligned} G(z)&=z^n\overline{H(1/\overline{z})}=z^nH(1/z) \\ &=\sum_{j=0}^{n}\binom{n}{j}(R^{n-j}-1)z^{n-j} \\ &=\sum_{j=0}^{n}\binom{n}{j}(R^{n-j}-1)z^j, \end{aligned}$$

then all the zeros of $G(z)$ lie in $|z|>1$ and it follows that (see [12, Prob. 26, P.108]) the polynomial

$$\begin{aligned} R(z) &= H(z) + e^{i\alpha} z^n \overline{H(1/\bar{z})} \\ &= \sum_{j=0}^{n} \binom{n}{j} \{(R^j - 1) + e^{i\alpha}(R^{n-j} - 1)\} z^j \end{aligned}$$

has all its zeros on the circle $|z| = 1$ for every $R > 1$ and $\alpha$ real. Now by hypothesis $f(z)$ has all its zeros in $|z| \geq 1$ therefore, by Szegö's convolution theorem [18], if $f \in \mathbb{F}_n$, then

$$\begin{aligned} \Lambda f(z) &= (f(Rz) - f(z)) + e^{i\alpha}((R^n f(z/R) - f(z)) \\ &= (R^n - 1)a_n z^n + \{(R^{n-1} - 1) + e^{i\alpha}(R - 1)\} a_{n-1} z^{n-1} \\ &\quad + \cdots + \{(R - 1) + e^{i\alpha}(R^{n-1} - 1)\} a_1 z + (R^n - 1)a_0, \end{aligned}$$

does not vanish in $|z| < 1$ for every $R > 1$ and $\alpha$ real. Therefore, $\Lambda$ is an admissible operator. Applying Lemma 3, we obtain for each $p > 0$, $R > 1$ and $\alpha$ real,

$$\begin{aligned} &\int_0^{2\pi} |(f(Re^{i\theta}) - f(e^{i\theta})) + e^{i\alpha}(R^n f(e^{i\theta}/R) - f(e^{i\theta}))|^p \, d\theta \\ &\leq (R^n - 1)^p \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta, \end{aligned}$$

which is equivalent to the desired result and this completes the proof of Lemma 4.

*Lemma* 5. *If $f \in \mathbb{F}_n$, then for every $p > 0$, $R \geq 1$ and $\alpha$ real,*

$$\|(f(Rz) - f(z)) + e^{i\alpha}(R^n f(z/R) - f(z))\|_p \leq (R^n - 1)\|f\|_p. \tag{19}$$

The result is the best possible and equality holds for $f(z) = az^n + b$, $|a| = |b|$.

*Proof of Lemma* 5. The result is trivial for $R = 1$. Henceforth we assume $R > 1$. Since $f(z)$ is a polynomial of degree at most $n$, we can write

$$f(z) = f_1(z) f_2(z) = \prod_{j=1}^{k} (z - z_j) \prod_{j=k+1}^{n} (z - z_j), \quad k \geq 1$$

where all the zeros of $f_1(z)$ lie in $|z| \geq 1$ and all the zeros of $f_2(z)$ lie in $|z| < 1$. First we suppose that $f_1(z)$ has no zero on $|z| = 1$ so that all the zeros of $f_1(z)$ lie in $|z| > 1$. Let $Q_2(z) = z^{n-k} \overline{f_2(1/\bar{z})}$, then all the zeros of $Q_2(z)$ lie in $|z| > 1$ and $|Q_2(z)| = |f_2(z)|$ for $|z| = 1$. Now consider the polynomial

$$F(z) = f_1(z) Q_2(z) = \prod_{j=1}^{k} (z - z_j) \prod_{j=k+1}^{n} (1 - z\bar{z}_j),$$

then all the zeros of $F(z)$ lie in $|z| > 1$ and for $|z| = 1$,

$$|F(z)| = |f_1(z)||Q_2(z)| = |f_1(z)||f_2(z)| = |f(z)|. \tag{20}$$

Since $f(z)/F(z)$ is not a constant, by maximum modulus principle

$$|f(z)| < |F(z)| \quad \text{for } |z| = r < 1.$$

Applying Rouche's theorem, it follows that the polynomial $G(z) = f(z) + \lambda F(z)$ does not vanish in $|z| \leq 1$ for every $\lambda$ with $|\lambda| > 1$. Using Szegö's convolution theorem [18], it follows as in the proof of Lemma 4 that the polynomial

$$(G(Rz)-G(z))+e^{i\alpha}(R^nG(z/R)-G(z))$$

does not vanish in $|z|\leq 1$ for every $R>1$ and $\alpha$ real. Replacing $G(z)$ by $f(z)+\lambda F(z)$, it follows that the polynomial

$$\begin{aligned}T(z)= &\{(f(Rz)-f(z))+e^{i\alpha}(R^nf(z/R)-f(z))\}\\ &+\lambda\{(F(Rz)-F(z))+e^{i\alpha}(R^nF(z/R)-F(z))\}\end{aligned} \tag{21}$$

does not vanish in $|z|\leq 1$ for every $\lambda$ with $|\lambda|>1$. This implies

$$\begin{aligned}&|(f(Rz)-f(z))+e^{i\alpha}(R^nf(z/R)-f(z))|\\ &\quad\leq |(F(Rz)-F(z))+e^{i\alpha}(R^nF(z/R)-F(z))|\end{aligned} \tag{22}$$

for $|z|\leq 1, R>1$ and $\alpha$ real. If inequality (22) is not true, then there is a point $z=z_0$ with $|z_0|\leq 1$ such that

$$\begin{aligned}&|(f(Rz_0)-f(z_0))+e^{i\alpha}(R^nf(z_0/R)-f(z_0))|\\ &\qquad> |(F(Rz_0)-F(z_0))+e^{i\alpha}(R^nF(z_0/R)-F(z_0))|.\end{aligned}$$

Since all the zeros of polynomial $F(z)$ lie in $|z|>1$, it follows that all the zeros of the polynomial

$$(F(Rz)-F(z))+e^{i\alpha}(R^nF(z/R)-F(z))$$

lie in $|z|>1$ for every $R>1$ and $\alpha$ real. Hence

$$(F(Rz_0)-F(z_0))+e^{i\alpha}(R^nF(z_0/R)-F(z_0))\neq 0 \quad \text{with } |z_0|\leq 1.$$

We take

$$\lambda=\frac{(f(Rz_0)-f(z_0))+e^{i\alpha}(R^nf(z_0/R)-f(z_0))}{(F(Rz_0)-F(z_0))+e^{i\alpha}(R^nF(z_0/R)-F(z_0))}$$

so that $\lambda$ is a well defined real or complex number with $|\lambda|>1$ and with this choice of $\lambda$ from (21), we get

$$T(z_0)=0 \quad \text{where } |z_0|\leq 1.$$

This is clearly a contradiction to the fact that $T(z)$ does not vanish in $|z|\leq 1$. Thus for every $R>1$ and $\alpha$ real, we have

$$\begin{aligned}&|(f(Rz)-f(z))+e^{i\alpha}(R^nf(z/R)-f(z))|\\ &\quad\leq |(F(Rz)-F(z))+e^{i\alpha}(R^nF(z/R)-F(z))|\end{aligned}$$

for $|z|\leq 1$, which in particular gives for $|z|=1$,

$$\begin{aligned}&|(f(Rz)-f(z))+e^{i\alpha}(R^nf(z/R)-f(z))|\\ &\quad\leq |(F(Rz)-F(z))+e^{i\alpha}(R^nF(z/R)-F(z))|.\end{aligned}$$

Hence for each $p>0$ and $0\leq\theta<2\pi$, we obtain

$$\int_0^{2\pi} |(f(Re^{i\theta})-f(e^{i\theta}) + e^{i\alpha}(R^n f(e^{i\theta}/R)-f(e^{i\theta/R}))|^p \, d\theta$$
$$\leq \int_0^{2\pi} |(F(Re^{i\theta})-F(e^{i\theta}))+e^{i\alpha}(R^n F(e^{i\theta}/R)-F(e^{i\theta}))|^p \, d\theta.$$

Using Lemma 4 and (20), it follows that for each $p > 0$ and $0 \leq \theta < 2\pi$,

$$\int_0^{2\pi} |(f(Re^{i\theta})-f(e^{i\theta})) + e^{i\alpha}(R^n f(e^{i\theta}/R)-f(e^{i\theta}))|^p \, d\theta$$
$$\leq (R^n-1)^p \int_0^{2\pi} |F(e^{i\theta})|^p \, d\theta,$$
$$= (R^n-1)^p \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta.$$

This implies for each $p > 0,\ R > 1$ and $\alpha$ real,

$$\|(f(Rz)-f(z))+e^{i\alpha}(R^n f(z/R)-f(z))\|_p \leq (R^n-1)\|f\|_p. \tag{23}$$

Now if $f_1(z)$ has a zero on $|z|=1$, then applying (23) to the polynomial $f^*(z)=f_1(tz)f_2(z)$ where $t<1$, we get for each $p>0, R>1$ and $\alpha$ real,

$$\|(f^*(Rz)-f^*(z))+e^{i\alpha}(R^n f^*(z/R)-f^*(z))\|_p \leq (R^n-1)\|f^*\|_p. \tag{24}$$

Letting $t \to 1$ in (24) and using continuity, we obtain for each $p > 0$, $R > 1$ and $\alpha$ real,

$$\|(f(Rz)-f(z))+e^{i\alpha}(R^n f(z/R)-f(z))\|_p \leq (R^n-1)\|f\|_p.$$

This completes the proof of Lemma 5.

## 3. Proofs of the Theorems

*Proof of Theorem* 1. Since $f \in \mathbb{F}_n$, by Lemma 5, we have for each $p > 0, R \geq 1$, and $\alpha$ real

$$\int_0^{2\pi} |(f(Re^{i\theta})-f(e^{i\theta}))+e^{i\alpha}(R^n f(e^{i\theta}/R)-f(e^{i\theta}))|^p \, d\theta$$
$$\leq (R^n-1)^p \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta, \tag{25}$$

Integrating both sides of (25) w.r.t $\alpha$ from 0 to $2\pi$, we get for each $p > 0$,

$$\int_0^{2\pi}\int_0^{2\pi} |(f(Re^{i\theta})-f(e^{i\theta}))+e^{i\alpha}(R^n f(e^{i\theta}/R)-f(e^{i\theta}))|^p \, d\theta \, d\alpha$$
$$\leq 2\pi(R^n-1)^p \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta.$$

Now using in (25) the fact that for any $p > 0$,

$$\int_0^{2\pi} |a+be^{i\alpha}|^p \, d\alpha \geq 2\pi \max\{|a|^p, |b|^p\},$$

(see [7, in eq. (19)]) we obtain

$$\int_0^{2\pi} |(f(Re^{i\theta})-f(e^{i\theta}))|^p \, d\theta \le (R^n-1)^p \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta,$$

which implies

$$\|f(Rz)-f(z)\|_p \le (R^n-1)\|f\|_p$$

for each $p > 0$ and $R \ge 1$. This completes the proof of Theorem 1.

*Proof of Theorem* 2. Since the polynomial $f(z)$ does not vanish in $|z| < 1$, it follows from Lemma 2 that

$$|f(Rz)-f(z)| \le |R^n f(z/R)-f(z)| \quad \text{for } |z|=1 \quad \text{and } R \ge 1. \tag{26}$$

By Lemma 5, we have for each $p > 0$, $R \ge 1$ and $\alpha$ real,

$$\int_0^{2\pi} |A(\theta)+e^{i\alpha}B(\theta)|^p \, d\theta \le (R^n-1)^p \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \tag{27}$$

where

$$A(\theta) = f(Re^{i\theta}) - f(e^{i\theta}) \text{ and } B(\theta) = R^n f(e^{i\theta}/R) - f(e^{i\theta}).$$

Integrating both sides of (27) with respect to $\alpha$ from 0 to $2\pi$, we get for each $p > 0$, $R \ge 1$ and $\alpha$ real,

$$\int_0^{2\pi} |A(\theta)+e^{i\alpha}B(\theta)|^p \, d\theta \, d\alpha \le 2\pi(R^n-1)^p \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta. \tag{28}$$

Now for every real $\alpha$ and $t \ge 1$, we have $|t+e^{i\theta}| \ge |1+e^{i\alpha}|$, which implies

$$\int_0^{2\pi} |t+e^{i\alpha}|^p \, d\alpha \ge \int_0^{2\pi} |1+e^{i\alpha}|^p d\alpha, \quad p > 0.$$

If $A(\theta) \neq 0$, we take $t = |B(\theta)|/|A(\theta|$, then by (26), $t \ge 1$ and we get

$$\begin{aligned}
\int_0^{2\pi} |A(\theta)+e^{i\alpha}B(\theta)|^p \, d\alpha &= |A(\theta)|^p \int_0^{2\pi} \left|1+\frac{B(\theta)}{A(\theta)}e^{i\alpha}\right|^p d\alpha \\
&= |A(\theta)|^p \int_0^{2\pi} \left|\frac{B(\theta)}{A(\theta)}+e^{i\alpha}\right|^p d\alpha \\
&= |A(\theta)|^p \int_0^{2\pi} \left|\left|\frac{B(\theta)}{A(\theta)}\right|+e^{i\alpha}\right|^p d\alpha \\
&\ge |A(\theta)|^p \int_0^{2\pi} |1+e^{i\alpha}|^p \, d\alpha \\
&= |f(Re^{i\theta})-f(e^{i\theta})|^p \int_0^{2\pi} |1+e^{i\alpha}|^p \, d\alpha.
\end{aligned}$$

For $A(\theta) = 0$, this inequality is trivially true. Using this in (28), we conclude that for each $p > 0$, $R \ge 1$ and $\alpha$ real,

$$\int_0^{2\pi} |1+e^{i\alpha}|^p \,\mathrm{d}\alpha \int_0^{2\pi} |f(Re^{i\theta})-f(e^{i\theta})|^p \,\mathrm{d}\theta$$
$$\leq 2\pi(R^n-1)^p \int_0^{2\pi} |f(e^{i\theta})|^p \,\mathrm{d}\alpha,$$

which immediately leads to (9) and this completes the proof of Theorem 2.

*Proof of Theorem* 3. Since $f(z)$ is a self-inversive polynomial, we have $f(z)=Q(z)$ where $Q(z)=z^n\overline{f(1/\bar{z})}$. Therefore, for each $R\geq 1$,

$$|f(Rz)-f(z)|=|Q(Rz)-Q(z)| \quad \text{for all } z\in\mathbb{C},$$

so that

$$|B(\theta)/A(\theta)|=\left|\frac{R^nf(e^{i\theta}/R)-f(e^{i\theta})}{f(Re^{i\theta})-f(e^{i\theta})}\right|=1.$$

Using this in (28) and proceeding similarly as in the proof of Theorem 2, we get (11) and this proves Theorem 3.

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