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Parabolic ample bundles III: Numerically effective vector bundles

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Abstract. In this continuation of [Bi2] and [BN], we define numerically effective vector bundles in the parabolic category. Some properties of the usual numerically effective vector bundles are shown to be valid in the more general context of numerically effective parabolic vector bundles.

Keywords. Parabolic structure; numerically effective; numerically flat.

1. Introduction

Parabolic bundles were introduced in [MS]. It is now known that various notions related to the usual vector bundles actually extend to the context of parabolic vector bundles. In [Bi2] the notion of ampleness of a vector bundle was extended to the parabolic category. In [Bi2] and [BN], various results on usual ample vector bundles were generalized to parabolic ample bundles.

The notion of numerically effectiveness of a vector bundle is very closely related to the notion of ampleness. Defining the notion of numerically effectiveness of a parabolic vector bundle, we generalize some known results on usual numerically effective vector bundles to the more general context of parabolic bundles.

2. Some properties of numerically effective vector bundles

Let X be a connected smooth projective variety over $\mathbb{C}$. For a vector bundle E over X, the projective bundle over X consisting of all hyperplanes in the fibers of E will be denoted by $\mathbb{P}E$. The tautological relative ample line bundle over $\mathbb{P}E$ will be denoted by $\mathcal{O}_{\mathbb{P}E}(1)$. We recall the definition of a numerically effective vector bundle.

DEFINITION 2.1

A line bundle L over X is called *numerically effective* (abbreviated as *nef*) if for any morphism $f: C \longrightarrow X$, where C is a connected smooth projective curve, the inequality

$$\deg(f^*L) \geq 0$$

is valid.

More generally, a vector bundle E over X is called *numerically effective* if the line bundle $\mathcal{O}_{\mathbb{P}E}(1)$ over $\mathbb{P}E$ is nef in the above sense.

For a vector bundle V over a connected smooth curve C, let $d_{\min}(V)$ denote the degree of the final piece of the graded object for the Harder–Narasimhan filtration of V. In other words, if

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_l \subset V_{l+1} = V \tag{2.2}$$

is the Harder–Narasimhan filtration of V, then $d_{\min}(V) := \deg(V/V_l)$.

PROPOSITION 2.3

A vector bundle E over X is nef if and only if for any morphism f from a curve C to X, as in Definition 2.1, *the inequality*

$$d_{\min}(f^*E) \geq 0$$

is valid.

Proof. Let $f : C \longrightarrow X$ be a morphism from a smooth curve.

If $d_{\min}(f^*E) \geq 0$, then it is easy to see that $\deg(L) \geq 0$, where L is any quotient line bundle of f^*E. Indeed, if

$$0 = E'_0 \subset E'_1 \subset \cdots \subset E'_l \subset E'_{l+1} = f^*E \tag{2.4}$$

is the Harder–Narasimhan filtration of f^*E and $\deg(V/V_l) \geq 0$, then for any line bundle L' over C with $\deg(L') < 0$, the following is valid

$$H^0(C, \mathrm{Hom}(E'_{i+1}/E'_i, L')) = 0$$

for all $i \in [0, l]$. This implies that L' cannot be a quotient line bundle of f^*E.

The above observation that $\deg(L) \geq 0$ immediately yields that the vector bundle E is nef [Vi, Proposition 2.9].

To prove the converse, assume that E is nef. Take f as above, and let a filtration as in (2.4) be the Harder–Narasimhan filtration of f^*E. Let r be the rank of f^*E/E'_l.

Since E is nef, the pullback f^*E is also nef, and hence we have that $\wedge^r f^*E$ is nef [Vi, Corollary 2.20]. But the line bundle $\wedge^r(f^*E/E'_l)$ is a quotient of $\wedge^r f^*E$. Thus the inequality $d_{\min}(f^*E) = \deg(\wedge^r(f^*E/E'_l)) \geq 0$ is valid. This completes the proof of the proposition. □

Fix a polarization $\mathcal{L}$ on X. Any vector bundle V over X admits a unique Harder–Narasimhan filtration as in (2.2) [Ko, Ch. V, Theorem 7.15]. As before, define

$$d_{\min}(V) := \deg(V/V_l) := \int_X c_1(V/V_l) \cup c_1(\mathcal{L})^{d-1}$$

where $d = \dim_{\mathbb{C}} X$.

Theorem 2.5. *If E is a nef vector bundle over X, then $d_{\min}(E) \geq 0$. If $\dim_{\mathbb{C}} X = 1$, then any vector bundle E over X, with $d_{\min}(E) \geq 0$, is nef.*

Proof. Let $0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_l \subset E_{l+1} = E$ be the Harder–Narasimhan filtration of E.

Let $f : C \longrightarrow X$ be a smooth irreducible complete intersection curve such that the restriction of all E_{i+1}/E_i, $0 \leq i \leq l$, to C is a semistable vector bundle. The existence of such a curve C is ensured by the main theorem of [MR] which says that a vector bundle over a smooth projective variety is semistable if and only if its restriction to the general complete intersection curve of sufficiently large degree is semistable. Thus the restriction

$$0 = E_0 \subset E_1|_C \subset E_2|_C \subset \cdots \subset E_l|_C \subset E_{l+1}|_C = f^*E$$

of the above filtration of E to C is actually the Harder–Narasimhan filtration of f^*E. Indeed, if $F \subset f^*E$ is the maximal semistable subbundle, then $\mu(F) \geq \mu(E_1|_C)$, and hence F does not admit any nonzero homomorphism to the quotient vector bundle $(E_{i+1}|_C)/(E_i|_C)$ for any $i \geq 1$. Thus F must coincide with $E_1|_C$. Now the above claim is established by using induction on the length of the Harder–Narasimhan filtration of the vector bundle E.

If E is nef, then from Proposition 2.3 it follows that $d_{\min}(f^*E) \geq 0$. This, in turn, implies that $d_{\min}(E) \geq 0$.

Now let X be a Riemann surface, and let E be a vector bundle over X with $d_{\min}(E) \geq 0$. This implies that $d_{\min}(S^k(E)) \geq 0$, where $S^k(E)$ is the k-fold symmetric tensor power of E.

Let L be a line bundle over X with $\deg(L) > 0$. A consequence of the inequality $d_{\min}(S^k(E)) \geq 0$ is that any quotient of $S^k(E)$ is of nonnegative degree. Hence any quotient of $S^k(E) \otimes L$ is of strictly positive degree. Now from a theorem of [Ha] we conclude that $S^k(E) \otimes L$ is ample. The theorem of [Ha] in question says that a vector bundle over a complete curve is ample if and only if its degree and also the degrees of all its quotient bundles are all strictly positive.

For a nonconstant morphism of curves $f : C \longrightarrow X$, if V is an ample vector bundle over X, then f^*V is ample. So from [Vi, Proposition 2.9] we conclude that E must be nef. This completes the proof of the theorem. □

We remark that the second part of Theorem 2.5 can also be deduced using Proposition 2.3.

A vector bundle E over a projective manifold X is called *numerically flat* if both E and its dual E^* are nef [DPS, Definition 1.17].

PROPOSITION 2.6

A vector bundle E is numerically flat if and only if E is semistable with $c_1(E) = 0 = c_2(E)$.

Proof. Let E be a numerically flat vector bundle over X. Since the Harder–Narasimhan filtration of E^* is simply the dual of the Harder–Narasimhan filtration of E, the conditions obtained from Theorem 2.5, namely $d_{\min}(E) \geq 0$ and $d_{\min}(E^*) \geq 0$, immediately imply that E is semistable with $c_1(E) = 0$. That $c_2(E) = 0$ follows, of course, from Corollary 1.19 of [DPS].

Conversely, let E be a semistable vector bundle over X with $c_1(E) = 0 = c_2(E)$. From [Si, Theorem 2, page 39] we know that E admits a filtration

$$F_1 \subset F_2 \subset \cdots \subset F_k \subset F_{k+1} = E$$

and a flat connection ∇ on E which preserves each F_i and the induced connection on each F_{i+1}/F_i is a unitary flat connection.

Thus for any map $f : C \longrightarrow X$ from a curve C, the vector bundle f^*E over C has a flat connection, namely $f^*\nabla$, such that on each f^*F_{i+1}/f^*F_i it induces a unitary flat connection. This implies that f^*E is a semistable vector bundle over the curve C with $\deg(f^*E) = 0$. Now from the second part of Theorem 2.5 we conclude that f^*E is nef. Thus E must be nef. The same argument shows that E^* is nef. This completes the proof of the proposition. □

In the next section we will define the notion of nefness in the context of parabolic sheaves.

3. Parabolic nef bundles

Let D be an effective divisor on X. Let E_* be a parabolic vector bundle over X with parabolic structure over D. In [Bi2] parabolic ample bundles were defined; this generalizes the notion of ample vector bundles to the parabolic context.

DEFINITION 3.1

A parabolic vector bundle E_* is called *parabolic nef* if there is an ample line bundle L over X such that $S^k(E_*)\otimes L$ is parabolic ample for every k, where $S^k(E_*)$ denotes the k-fold parabolic symmetric tensor power of the parabolic bundle E_*. (See [Bi2] for the definition of parabolic tensor product.)

If the parabolic structure of E_* is trivial, i.e., zero is the only parabolic weight, then from Proposition 2.9 of [Vi] it follows that E_* is parabolic nef if and only if the underlying vector bundle is nef in the usual sense.

Henceforth, we will assume that the parabolic divisor D on X is a normal crossing divisor. By this we mean that D is reduced, each irreducible component of D is smooth, and furthermore, the irreducible components intersect transversally.

The parabolic structure of a parabolic bundle E_* is defined as follows: for each irreducible component D_i of the parabolic divisor D, a filtration by coherent subsheaves of the vector bundle $E|_{D_i}$ over D_i is given, together with a system of parabolic weights corresponding to the filtration [MS], [Bi1].

We will henceforth consider only those parabolic bundles E_* for which the filtration over any D_i, defining the quasi-parabolic structure, is by subbundles of $E|_{D_i}$.

Let E_* be a parabolic vector bundle with rational parabolic weights. Then there is a Galois covering map

$$p : Y \longrightarrow X$$

and an orbifold vector bundle V on Y, such that the parabolic bundle E_* is obtained by taking invariants of the direct image of the twists of V using the irreducible components of D [Bi1].

PROPOSITION 3.2

A parabolic vector bundle E_* with rational parabolic weights is parabolic nef if and only if the underlying vector bundle for the corresponding orbifold vector bundle V on Y is nef in the usual sense.

Proof. Let L be an ample line bundle over X. Since the above covering map p is finite, the line bundle p^*L over Y is also ample.

Assume that the vector bundle V is nef. So $S^k(V)\otimes p^*L$ is ample for sufficiently large k [Vi, Proposition 2.9(c)]. Since the orbifold bundle $S^k(V)$ corresponds to the parabolic bundle $S^k(E_*)$ [Bi2], and furthermore, from the definition of parabolic amplitude it is immediate that the parabolic bundle corresponding to an orbifold bundle whose underlying vector bundle is ample, is actually parabolic ample, we conclude that E_* is parabolic nef.

Now assume that E_* is parabolic nef. Lemma 4.6 of [Bi2] says that $S^k(V) \otimes L$ is ample if $S^k(E_*) \otimes L$ is parabolic ample. So we conclude that V must be nef. This completes the proof of the proposition. □

As the tensor product of a nef vector bundle and an ample vector bundle is ample, the above proposition has the following corollary:

COROLLARY 3.3

Let E_* and F_* be two parabolic vector bundles with rational parabolic weights and with parabolic structure over a normal crossing divisor D. Assume that E_* is parabolic nef and F_* is parabolic ample. Then the parabolic tensor product $E_* \otimes F_*$ is parabolic ample.

Fix a polarization over X to define the parabolic degree of a parabolic bundle. A parabolic vector bundle admits a canonical filtration of parabolic subsheaves with each subsequent quotient parabolic semistable, which is a natural generalization of the Harder–Narasimhan filtration to the parabolic context. Following the definition of $d_{\min}$ in § 2, we make the following definition.

For a parabolic sheaf E_*, define $d^{\mathrm{par}}_{\min}(E_*)$ to be the parabolic degree of the minimal parabolic semistable subquotient of E_*, or in other words, $d^{\mathrm{par}}_{\min}(E_*)$ is the parabolic degree of the final piece of the graded object for the Harder–Narasimhan filtration of E_*.

Now, as for Corollary 3.3, the Proposition 3.2 combines with Theorem 2.5 to give the following corollary:

COROLLARY 3.4

Let E_* be a parabolic vector bundle with rational parabolic weights. If E_* is parabolic nef, then $d^{\mathrm{par}}_{\min}(E_*) \geq 0$. If $\dim X = 1$, then the converse is also true; namely, if the inequality $d^{\mathrm{par}}_{\min}(E_*) \geq 0$ is valid, then E_* must be parabolic nef.

A parabolic vector bundle E_* will be called *numerically flat* if both E_* and its parabolic dual E_*^* are parabolic nef.

Let E_* be a numerically flat parabolic bundle over X with rational parabolic weights and with parabolic structure over a normal crossing divisor D. Let $V \longrightarrow Y$ be the orbifold bundle corresponding to E_* for a suitable Galois covering map

$$p : Y \longrightarrow X$$

with Galois group G. Proposition 3.2 says that V is numerically flat, i.e., both V and V^* are nef. Now Proposition 2.6 says that V is semistable with $c_1(V) = 0 = c_2(V)$.

Since V is semistable, from [Bi1] it follows that E_* is parabolic semistable. Since the first and the second Chern class of V vanish, from [Bi3] it follows that the first and the second parabolic Chern class of E_* vanish.

Conversely, if E_* is parabolic semistable with its first and the second parabolic Chern class zero, then from [Bi1] and [Bi3] we know that the corresponding orbifold bundle V is semistable with the first and the second Chern class of V being zero. So Proposition 2.6 yields that V is numerically flat. Now Proposition 3.2 says that the parabolic bundle E_* is numerically flat.

Thus we have proved the following theorem:

Theorem 3.5. *A parabolic bundle E_* with rational parabolic weights is numerically flat if and only if E_* is parabolic semistable with vanishing first and second parabolic Chern classes.*

Using [Bi3], from the above theorem it is easy to deduce that a parabolic vector bundle E_* over X, with rational parabolic weights, is numerically flat if and only if the following condition is valid: the underlying vector bundle E for the parabolic vector bundle E_* has a filtration by subbundles of E such that each subsequent quotient vector bundle equipped with the induced parabolic structure, induced by E_*, corresponds to a unitary representation of the fundamental group of the complement $X - D$, where D is the divisor on X over which the parabolic structure of E_* is defined. The above statement can also be deduced using Proposition 3.2 together with [DPS, Theorem 1.18].

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