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Multidimensional modified fractional calculus operators involving a general class of polynomials

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Abstract. In the present work, we introduce and study essentially a class of multidimensional modified fractional calculus operators involving a general class of polynomials in the kernel. These operators are considered in the space of functions $M_\gamma(R^n_+)$. Some mapping properties and fractional differential formulas are obtained. Also images of some elementary and special functions are established.

Keywords and phrases. Fractional calculus operators; Mellin transform; general class of polynomials; H-function.

1. Introduction and preliminaries

Srivastava [6] introduced and studied a general class of polynomials which is defined by

$$S_N^M[x] = \sum_{r=0}^{[N/M]} \frac{(-N)_{Mr}}{r!} A_{N,r} x^r, \quad N = 0, 1, \ldots, \tag{1}$$

where M is an arbitrary positive integer and the coefficients $A_{N,r} (N, r \geq 0)$ are arbitrary constants real or complex.

This general class of polynomials (1.1) unifies and extends a number of classical orthogonal polynomials such as Jacobi polynomials, Hermite polynomials, Laguerre polynomials, Gagenbauer polynomials, Bessel polynomials and several other classes of generalized hypergeometric polynomials.

Throughout this paper we use some notations. As usual R and C represent the fields of real and complex numbers respectively. R^n denotes the set of n-tuple real numbers, R^n_+ non-negative real numbers, and C^n complex numbers. For brevity, we write x^λ for the product $x_1 \cdots x_n$ and x^p for $x_1^p \cdots x_n^p$ with $x = (x_1, \ldots, x_n)$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in C^n$ and $p \in C$. Further we write $|\lambda|$ for the sum $\lambda_1 + \cdots + \lambda_n$. By φ_+ we mean the positive part of a function φ defined by

$$\varphi_+(x) = \begin{cases} \varphi(x), & \text{if} \quad \varphi(x) > 0, \\ 0, & \text{if} \quad \varphi(x) \leq 0. \end{cases} \tag{1.2}$$

2. Modified fractional integrals

The multidimensional modified fractional integral operators $Y^{\mu;N,M}_{+;n}$ and $Y^{\mu;N,M}_{-;n}$ are defined as follows:

$$Y_{+;n}^{\mu;N,M}f(x) = \frac{1}{\Gamma(\mu+1)}\frac{\partial^n}{\partial x_1\ldots\partial x_n}\int_{R_+^n}\left[\min\left(\frac{x_1}{t_1},\ldots,\frac{x_n}{t_n}\right)-1\right]_+^{\mu}$$
$$\times S_N^M\left[z\left\{\min\left(\frac{x_1}{t_1},\ldots,\frac{x_n}{t_n}\right)-1\right\}^a\right]f(t)\mathrm{d}t, \tag{2.1}$$

and

$$Y_{-;n}^{\mu;N,M}f(x) = \frac{(-1)^n}{\Gamma(\mu+1)}\frac{\partial^n}{\partial x_1\cdots\partial x_n}\int_{R_+^n}\left[1-\max\left(\frac{x_1}{t_1},\ldots,\frac{x_n}{t_n}\right)\right]_+^{\mu}$$
$$\times S_N^M\left[z\left\{1-\max\left(\frac{x_1}{t_1},\ldots,\frac{x_n}{t_n}\right)\right\}^a\right]f(t)\mathrm{d}t, \tag{2.2}$$

for $Re(\mu) > 0$.

The fractional integral operators (2.1) and (2.2) have a large number of special cases due to the presence of the general class of polynomials in the kernels of the integrals. We mention below a few of them for the sake of illustration.

If in (2.1) and (2.2), we set $M = 1$, $N = 0$ and $A_{0,0} = 1$, then the general class of polynomials reduces to unity, and we get

$$Y_{+;n}^{\mu;0,1}f(x) = \frac{1}{\Gamma(\mu+1)}\frac{\partial^n}{\partial x_1\cdots\partial x_n}\int_{R_+^n}\left[\min\left(\frac{x_1}{t_1},\ldots,\frac{x_n}{t_n}\right)-1\right]_+^{\mu}f(t)\mathrm{d}t$$
$$= X_+^{\mu}f(x), \tag{2.3}$$

and

$$Y_{-;n}^{\mu;0,1}f(x) = \frac{(-1)^n}{\Gamma(\mu+1)}\frac{\partial^n}{\partial x_1\cdots\partial x_n}\int_{R_+^n}\left[1-\max\left(\frac{x_1}{t_1},\ldots,\frac{x_n}{t_n}\right)\right]_+^{\mu}f(t)\mathrm{d}t$$
$$= X_-^{\mu}f(x), \tag{2.4}$$

where X_+^{μ} and X_-^{μ} are the modified fractional integrals introduced by Brychkov *et al* [1, pp. 246–248] and studied by Tuan and Saigo [10], and Raina [3]. On the other hand, by expressing the general class of polynomials involved in (2.1) and (2.2) by its series form (1.1), and changing order of summation and integration, we get

$$Y_{-;n}^{\mu;N,M}f(x) = \sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}X_+^{\mu+ra}f(x), \tag{2.5}$$

and

$$Y_{-;n}^{\mu;N,M}f(x) = \sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}X_-^{\mu+ra}f(x). \tag{2.6}$$

Now, if we set $n = 1$ in (2.1) and (2.2) (or in (2.5) and (2.6)), we get

$$Y_{+;1}^{\mu;N,M}f(x) = \frac{1}{\Gamma(\mu+1)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x\left(\frac{x}{t}-1\right)^{\mu}S_N^M\left[z\left(\frac{x}{t}-1\right)^a\right]f(t)\mathrm{d}t$$
$$= \sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}I_+^{\mu+ra}x^{-\mu-ra}f(x) \tag{2.7}$$

and

$$Y^{\mu;N,M}_{-;1}f(x)=\frac{-1}{\Gamma(\mu+1)}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^\infty\left(1-\frac{x}{t}\right)^\mu S_N^M\left[z\left(1-\frac{x}{t}\right)^a\right]f(t)\mathrm{d}t$$
$$=\sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}I_-^{\mu+ra}x^{-\mu-ra}f(x) \tag{2.8}$$

where I_+^μ and I_-^μ are the well-known Riemann–Liouville and Weyl integral operators respectively.

Also, on setting $M=1, A_{N,r}=\binom{N+\lambda}{N}\frac{1}{(\lambda+1)_r}$ in (2.1) and (2.2), and using the known result [9, p. 101, equation (5.1.6)] therein, we get

$$Y^{\mu;N,1}_{+;n}f(x)=\frac{1}{\Gamma(\mu+1)}\frac{\partial^n}{\partial x_1\cdots\partial x_n}\int_{R_+^n}\left[\min\left(\frac{x_1}{t_1},\ldots,\frac{x_n}{t_n}\right)-1\right]_+^\mu$$
$$\times L_n^{(\lambda)}\left(z\left\{\min\left(\frac{x_1}{t_1},\ldots,\frac{x_n}{t_n}\right)-1\right\}^a\right)f(t)\mathrm{d}t, \tag{2.9}$$

and

$$Y^{\mu;N,1}_{-;n}f(x)=\frac{(-1)^n}{\Gamma(\mu+1)}\frac{\partial^n}{\partial x_1\ldots\partial x_n}\int_{R_+^n}\left[1-\max\left(\frac{x_1}{t_1},\ldots,\frac{x_n}{t_n}\right)\right]_+^\mu$$
$$\times L_n^{(\lambda)}\left(z\left\{1-\max\left(\frac{x_1}{t_1},\ldots,\frac{x_n}{t_n}\right)\right\}^a\right)f(t)\mathrm{d}t, \tag{2.10}$$

for $Re(\mu)>0$, where $L_N^{(\lambda)}(x)$ stands for generalized Laguerre polynomial [9, p. 101, equation (5.1.6)].

Now by dividing R_+^n for a fixed $x\in R_+^n$ into n subdomains with zero-measure intersection,

$$R_+^n=\bigcup_{k=1}^n\left\{t\in R_+^n\middle|\frac{x_k}{t_k}\le\frac{x_j}{t_j},\ j=1,\ldots,n;\ j\ne k\right\}, \tag{2.11}$$

the multidimensional fractional integral operator $Y^{\mu,N,M}_{+;n}$ can be expressed as a finite sum of single integrals

$$Y^{\mu;N,M}_{+;n}f(x)=\frac{1}{\Gamma(\mu+1)}\sum_{k=1}^n\frac{\partial}{\partial x_k}\left[x_k\int_0^1(1-t)^\mu t^{n-\mu-1}\right.$$
$$\left.\times S_N^M\left[z\left(\frac{1}{t}-1\right)^a\right]f(x_1t,\ldots,x_nt)\mathrm{d}t\right]. \tag{2.12}$$

Similarly by dividing R_+^n into subdomains with zero-measure intersection,

$$R_+^n=\bigcup_{k=1}^n\left\{t\in R_+^n\middle|\frac{x_k}{t_k}\ge\frac{x_j}{t_j},\ j=1,\ldots,n;\ j\ne k\right\}, \tag{2.13}$$

we obtain

$$Y^{\mu;N,M}_{-;n}f(x)=\frac{-1}{\Gamma(\mu+1)}\sum_{k=1}^n\frac{\partial}{\partial x_k}\left[x_k\int_1^\infty(t-1)^\mu t^{n-\mu-1}\right.$$
$$\left.\times S_N^M\left[z\left(1-\frac{1}{t}\right)^a\right]f(x_1t,\ldots,x_nt)\mathrm{d}t\right]. \tag{2.14}$$

3. Modified fractional integrals of some special functions

We shall require a known result due to Raina [3] contained in the following:

Lemma If $s=(s_1,\ldots,s_n)\in C^n$, $h=(h_1,\ldots,h_n)\in R_+^n$ *and* $t^{(|s/h|)-1}f(t)\in L_1(R_+)$,

$$\int_{R_+^n} x^{s-1} f(\max(x^{h_1},\ldots,x^{h_n})\mathrm{d}x = \frac{|s/h|}{s_1\ldots s_n} f^*(|s/h|) \tag{3.1}$$

$Re(s_j)>0(j=1,\ldots,n)$, *and*

$$\int_{R_+^n} x^{s-1} f(\min(x_1^{h_1},\ldots,x_n^{h_n})\mathrm{d}x = \frac{(-1)^{n-1}|s/h|}{s_1\ldots s_n} f^*(|s/h|) \tag{3.2}$$

$Re(s_j)<0(j=1,\ldots,n)$

where $f^*(t)$ *denotes the one-dimensional Mellin transform of* $f(x)$.

Now we may calculate the modified fractional integrals of some elementary and special functions.

(i) Set $f(x)=x^d$ for $d\in C^n$. Making use of the known formulas [10, p. 257, equations (3.5) and (3.6)], we get

$$Y_{+;n}^{\mu;N,M}x^d = \sum_{r=0}^{[N/M]} \frac{(-N)_{Mr}}{r!} A_{N,r}z^r \frac{\Gamma(\mu+ra+1)}{\Gamma(\mu+1)} \frac{\Gamma(n-\mu-ra+|d|)}{\Gamma(n+|d|)} x^d, \tag{3.3}$$

provided that $Re(d_j)>-1(j=1,\ldots,n)$ and $n+Re(|d|)>Re(\mu+ra)$ $(r=0,1,\ldots,[N/M])$, and

$$Y_{-;n}^{\mu;N,M}x^d = \sum_{r=0}^{[N/M]} \frac{(-N)_{Mr}}{r!} A_{N,r}z^r \frac{\Gamma(\mu+ra+1)}{\Gamma(\mu+1)} \frac{\Gamma(1-n-|d|)}{\Gamma(1+\mu+ra-n-|d|)} x^d, \tag{3.4}$$

provied that $Re(d_j)<-1(j=1,\ldots,n)$ and $n+Re(|d|)<Re(\mu+ra)+1(r=0,1,\ldots,[N/M])$.

(ii) Set $f(x)=\frac{x^{-d-1}}{\Gamma(|d|+1)}[\min(x_1,\ldots,x_n)-1]_+^{|d|}$ for $d\in C^n$, then making use of the known formulas [10, p. 259, equations (3.9) & (3.10)], we get for $Re(|d|)>0$

$$Y_{+;n}^{\mu;N,M}\frac{x^{-d-1}}{\Gamma(|d|+1)}[\min(x_1,\ldots,x_n)-1]_+^{|d|} = \sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r \times \frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}\cdot\frac{x^{-d-1}[\min(x_1,\ldots,x_n)-1]_+^{\mu+ra+|d|}}{\Gamma(\mu+ra+|d|+1)} \tag{3.5}$$

and

$$Y_{-;n}^{\mu;N,M}\frac{x^{-d-1}}{\Gamma(|d|+1)}[1-\max(x_1,\ldots,x_n)]_+^{|d|} = \sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r \times \frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}\frac{x^{-d-1}[1-\max(x_1,\ldots,x_n)]_+^{\mu+ra+|d|}}{\Gamma(\mu+ra+|d|+1)} \tag{3.6}$$

(iii) Set $f(x) = x^{-1}H^{m_1,n_1}_{p_1,q_1}[\min(x_1^h,\ldots,x_n^h)]$, for $h \in R_+$, where

$$H^{m_1,n_1}_{p_1,q_1}[t] = H^{m_1,n_1}_{p_1,q_1}\left[t\,\middle|\,\begin{matrix}(a_j,A_j)_{1,p_1}\\(b_j,B_j)_{1,q_1}\end{matrix}\right] = \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty}\theta(\sigma)t^{-\sigma}\mathrm{d}\sigma \tag{3.7}$$

$$\theta(\sigma) = \frac{\prod_{j=1}^{m_1}\Gamma(b_j+B_j\sigma)\prod_{j=1}^{n_1}\Gamma(1-a_j-A_j\sigma)}{\prod_{j=m_1+1}^{q_1}\Gamma(1-b_j-B_j\sigma)\prod_{j=n_1+1}^{p_1}\Gamma(a_j+A_j\sigma)} \tag{3.8}$$

is well-known Fox's H-function [2]. For details of this function, one can refer to Srivastava *et al* [7, Chap 2].

To evaluate the modified fractional integral of this function, we use the multi-dimensional Mellin inversion formula [1]:

$$f(x) = \frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty} f^*(s)x^{-s}\mathrm{d}s \quad \text{for} \quad f^*(s) = \int_{R_+^n} x^{s-1}f(x)\mathrm{d}x = M\{f(x)\},$$

where, and in what follows, the notation

$$\int_{(\gamma)-i\infty}^{(\gamma)+i\infty} \text{ means } \int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\cdots\int_{\gamma_n-i\infty}^{\gamma_n+i\infty}, \quad \text{for } \gamma = (\gamma_1,\ldots,\gamma_n)\in R^n.$$

Now, making use of (3.2), we get

$$Y^{\mu;N,M}_{+;n}x^{-1}H^{m_1,n_1}_{p_1,q_1}[\min(x_1^h,\ldots,x_n^h)] = \sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r\,\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}$$
$$\times\frac{(-1)^{n-1}}{h(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty}\frac{\Gamma(-\mu-ra-|s|)}{\Gamma(-|s|)}H^*(s)\frac{|s|}{s_1\cdots s_n}x^{-s-1}\mathrm{d}s, \tag{3.9}$$

where $H^*(s) = \theta(\frac{|s|}{h})$

Now, interpreting (3.9) as the H-function by using the known formula [3, p. 158, equation (3.7)], we get

$$Y^{\mu;N,M}_{+;n}x^{-1}H^{m_1,n_1}_{p_1,q_1}[\min(x_1^h,\ldots,x_n^h)] = \sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}x^{-1}$$
$$\times H^{m_1,n_1+1}_{p_1+1,q_1+1}\left[\min(x_1^h,\ldots,x_n^h)\,\middle|\,\begin{matrix}(1+\mu+ra,h),(a_j,A_j)_{1,p_1}\\(b_j,B_j)_{1,q_1},(1,h)\end{matrix}\right], \tag{3.10}$$

provided that $\gamma_j < 0 (j = 1,\ldots,n)$, $-|\gamma| > Re(\mu + Ra) > 0$ $(r = 0,1,\ldots,[N/M])$; and

$$\text{(i)}\quad \Delta = \sum_{j=1}^{n_1}A_j - \sum_{j=n_1+1}^{p_1}A_j + \sum_{j=1}^{m_1}B_j - \sum_{j=m_1+1}^{q_1}B_j > 0, \quad \text{or}$$

$$\text{(ii)}\quad \Delta = 0,\quad Re\left(\sum_{j=1}^{p_1}a_j - \sum_{j=1}^{q_1}b_j\right) - \frac{p_1-q_1}{2} + |\gamma|\left(\sum_{j=1}^{p_1}A_j - \sum_{j=1}^{q_1}B_j\right) > 1;$$

$$Re(a_j) < 1 - \frac{|\gamma|}{h}A_j(j = 1,\ldots,n_1), \quad Re(b_j) > -\frac{|\gamma|}{h}B_j(j = 1,\ldots,m_1).$$

Similarly, we have

$$Y_{-;n}^{\mu;N,M}x^{-1}H_{p_1,q_1}^{m_1,n_1}[\max(x_1^h,\ldots,x_n^h)] = \sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}x^{-1}$$

$$\times H_{p_1+1,q_1+1}^{m_1+1,n_1}\left[\max(x_1^h,\ldots,x_n^h)\middle|\begin{matrix}(a_j,A_j)_{1,p_1},(1+\mu+ra,h)\\(1,h),(b_j,B_j)_{1,q_1}\end{matrix}\right], \tag{3.11}$$

provided that $\gamma_j > 0 (j = 1,\ldots,n)$, $Re(\mu+ra) > 0$, $(r = 0,1,\ldots,[N/M])$; $Re(a_j) < 1-\frac{|\gamma|}{h}A_j(j = 1,\ldots,n_1)$, $Re(b_j) > -\frac{|\gamma|}{h}B_j(j = 1,\ldots,m_1)$ and conditions (i) and (ii) stated with (3.10) hold.

Note that on setting $N = 0$, $M = 1$ and $A_{0,0} = 1$ in pairs of equations (3.3) and (3.4) and (3.10) and (3.11), we get the results established by Tuan and Saigo [10, p. 257, equations (3.5) and (3.6)] and by Raina [3, pp. 158-159, equations (3.11) and (3.12)] respectively. Morever, if $A_j = 1(j = 1,\ldots,p_1)$, $B_j = 1(j = 1,\ldots,q_1)$ and $h = 1$, then (3.10) and (3.11) yield the corresponding formulas for Meijer's G-function obtained by Tuan and Saigo [10, p. 259, equations (3.13) and (3.14)]. Further if we set $n = 1$ in (3.10) and (3.11), then we get the formulas obtained by Raina and Koul [4, p. 99, equation (7)] and [5, p. 277, equation (2.5)] respectively.

4. Modified fractional operators on space $M_\gamma(R_+^n)$

Following [10], let $M_\gamma(R_+^n)$ denote the space of functions f which are defined on R_+^n, where $\gamma = (\gamma_1,\ldots,\gamma_n) \in R^n$. It is proved there that $f \in M_\gamma(R_+^n)$, if and only if, f can be represented as the inverse Mellin transform,

$$f(x) = \frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty} f^*(s)x^{-s}\mathrm{d}s, \tag{4.1}$$

of a function $f^*(s)$ infinitely differentiable and with compact support on $((\gamma)-i\infty, (\gamma)+i\infty)$.

Theorem 1. *(a) Let* $Re(\mu) > 0$; $\gamma_j + Re(d_j) < 1 (j = 1,\ldots,n)$; $Re(\mu+ra) + Re(|d|) + |\gamma| < n$ $(r = 0,1,\ldots,[N/M])$ *for* $d \in C^n$ *and* $\gamma \in R^n$, *then the operator* $x^dY_{+;n}^{\mu;N,M}.x^{-d}$ *is a homeomorphism of the space,* $M_\gamma(R_+^n)$ *onto itself.*

Moreover, it can be written in the form

$$x^dY_{+;n}^{\mu,N,M}x^{-d}f(x) = \sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}$$

$$\cdot\frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty}\frac{\Gamma(-\mu-ra-|d|-|s|+n)}{\Gamma(-|d|-|s|+n)}f^*(s)x^{-s}\mathrm{d}s \tag{4.2}$$

(b) Let $Re(\mu) > 0$; $\gamma_j + Re(d_j) > 0$ $(j = 1,\ldots,n)$;

$1 + Re(\mu+ra) + Re(|d|) + |\gamma| > n (r = 0,1,\ldots,[N/M])$ *for* $d \in C^n$ *and* $\gamma \in R^n$, *then the operator* $x^dY_{-;n}^{\mu,N,M}x^{-d}$ *is a homeomorphism of the space* $M_\gamma(R_+^n)$ *onto itself, and*

$$x^dY_{-;n}^{N,M}x^{-d}f(x) = \sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}$$

$$\cdot\frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty}\frac{\Gamma(1+|d|+|s|-n)}{\Gamma(1+\mu+ra+|d|+|s|-n)}f^*(s)x^{-s}\mathrm{d}s. \tag{4.3}$$

Proof. (a) Making use of (3.3) and (4.1), we get

$$x^d Y^{N,M}_{+;n} x^{-d} f(x) = x^d Y^{\mu,N,M}_{+;n} x^{-d} \cdot \frac{1}{(2\pi i)^n} \int_{(\gamma)-i\infty}^{(\gamma)+i\infty} f^*(s) x^{-s} \mathrm{d}s$$

$$= \frac{x^d}{(2\pi i)^n} \int_{(\gamma)-i\infty}^{(\gamma)+i\infty} f^*(s) Y^{\mu,N,M}_{+;n} x^{-s-d} \mathrm{d}s$$

$$= \frac{1}{(2\pi i)^n} \int_{(\gamma)-i\infty}^{(\gamma)+i\infty} f^*(s) \sum_{r=0}^{[N/M]} \frac{(-N)_{Mr}}{r!} A_{N,r} z^r$$

$$\times \frac{\Gamma(\mu + ra + 1)}{\Gamma(\mu + 1)} \cdot \frac{\Gamma(-\mu - ra - |d| - |s| + n)}{\Gamma(-|d| - |s| + n)} x^{-s} \mathrm{d}s.$$

Changing the order of summation and integration in the above expression, we easily get (4.2). The interchange of order of integration is possible since $f^*(s)$ has a compact support. The function

$$\sum_{r=0}^{[N/M]} \frac{(-N)_{Mr}}{r!} A_{N,r} z^r \frac{\Gamma(ra + \mu + 1)}{\Gamma(\mu + 1)} \cdot \frac{\Gamma(-\mu - ra - |d| - |s| + n)}{\Gamma(-|d| - |s| + n)} f^*(s)$$

has compact support and is infinitely differentiable on $((\gamma) - i\infty, (\gamma) + i\infty)$ if so does $f^*(s)$. Hence $x^d Y^{\mu,N,M}_{+;n} x^{-d}$ belongs to $M_\gamma(R^n_+)$. The continuity of the mapping $f \to x^d Y^{\mu,N,M}_{+;n} x^{-d} f$ in $M_\gamma(R^n_+)$ is obvious.

Letting $d = 0$ in the theorem, we obtain

COROLLARY

(a) *Let* $Re(\mu) > 0$, $\gamma_j < 1 (j = 1, \ldots, n)$ *for* $\gamma \in R^n$; $Re(\mu + ra) + |\gamma| < n(r = 0, 1, \ldots, [N/M])$, *then* $Y^{\mu,N,M}_{+;n}$ *is a homeomorphism of* $M_\gamma(R^n_+)$ *onto itself, and*

$$Y^{\mu,N,M}_{+;n} f(x) = \sum_{r=0}^{[N/M]} \frac{(-N)_{Mr}}{r!} A_{N,r} z^r \frac{\Gamma(ra + \mu + 1)}{\Gamma(\mu + 1)} \cdot \frac{1}{(2\pi i)^n}$$

$$\times \int_{(\gamma)-i\infty}^{(\gamma)+i\infty} \frac{\Gamma(-\mu - ra - |s| + n)}{\Gamma(-|s| + n)} f^*(s) x^{-s} \mathrm{d}s. \tag{4.4}$$

(b) *Let* $Re(\mu) > 0$, $\gamma_j > 0 \ (j = 1, \ldots, n)$ *for* $\gamma \in R^n$; $1 + Re(\mu + ra) + |\gamma| > n(r = 0, 1, \ldots, [N/M])$, *then* $Y^{\mu,N,M}_{-;n}$ *is a homeomorphism of* $M_r(R^n_+)$ *onto itself, and*

$$Y^{\mu,N,M}_{=;n} f(x) = \sum_{r=0}^{[N/M]} \frac{(-N)_{Mr}}{r!} A_{N,r} z^r \frac{\Gamma(ra + \mu + 1)}{\Gamma(\mu + 1)} \cdot \frac{1}{(2\pi i)^n}$$

$$\times \int_{(\gamma)-i\infty}^{(\gamma)+i\infty} \frac{\Gamma(1 + |s| - n)}{\Gamma(1 + \mu + ra + |s| - n)} f^*(s) X^{-s} \mathrm{d}s. \tag{4.5}$$

Note that if the general class of polynomials is reduced to unity by setting $N = 0$, $M = 1$ and $A_{0,0} = 1$ in (4.2) and (4.3) then we get the results established by Tuan and Saigo [10, pp. 262–263, equations (5.2) and (5.5)]

5. Modified fractional differentials

Since $Y_{+;n}^{\mu,N,m}$ is a homeomorphism of $M_\gamma(R_+^n)$ onto itself under the conditions stated with (4.4), then there exists its inverse operator which we shall define as $(Y_{+;n}^{\mu,N,M})^{-1}$. This operator is also a homeomorphism of $M_\gamma(R_+^n)$ onto itself and is defined by

$$(Y_{+;n}^{\mu,N,M})^{-1}f(x) = \frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty}\left[\sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}\times\frac{\Gamma(-\mu-ra-|s|+n)}{\Gamma(-|s|+n)}\right]^{-1} f^*(s)x^{-s}\mathrm{d}s. \tag{5.1}$$

Similarly the inverse operator $(Y_{-;n}^{\mu,N,M})^{-1}$ is defined by

$$(Y_{-;n}^{\mu,N,M})^{-1}f(x) = \frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty}\left[\sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r\times\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}\cdot\frac{\Gamma(1+|s|-n)}{\Gamma(1+\mu+ra+|s|-n)}\right]^{-1} f^*(s)x^{-s}\mathrm{d}s. \tag{5.2}$$

It is easy to prove that

$$Y_{+;n}^{\mu,N,M}(Y_{+;n}^{\mu,N,M})^{-1}f(x) = (Y_{+;n}^{\mu,N,M})^{-1}Y_{+;n}^{\mu,N,M}f(x) = f(x);$$
$$Y_{-;n}^{\mu,N,M}(Y_{-;n}^{\mu,N,M})^{-1}f(x) = (Y_{-;n}^{\mu,N,M})^{-1}Y_{-;n}^{\mu,N,M}f(x) = f(x). \tag{5.3}$$

Now the series involved in (5.1) and (5.2) can be reciprocated [8], and can be written as

$$(Y_{+;n}^{\mu,N,M})^{-1}f(x) = \frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty}\Gamma(\mu+1)\Gamma(-|s|+n)\sum_{e=0}^{\infty}\beta_e z^e f^*(s)x^{-s}\mathrm{d}s, \tag{5.4}$$

$$(Y_{-;n}^{\mu,N,M})^{-1}f(x) = \frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty}\frac{\Gamma(\mu+1)}{\Gamma(1+|s|-n)}\sum_{e=0}^{\infty}\eta_e z^e f^*(s)x^{-s}\mathrm{d}s, \tag{5.5}$$

where β_e are given by the recursion formula

$$\beta_0\alpha_0 = 1, \quad \sum_{l=0}^{q}\beta_l\alpha_{q-l} = 0, \quad q = 1,2,3,\ldots; \quad \alpha_0 \neq 0, \tag{5.6}$$

or explicitly by

$$\beta_e = (-1)^e(\alpha_0)^{-e-1}\det\begin{bmatrix}\alpha_1 & \alpha_0 & 0 & 0\ldots 0\\ \alpha_2 & \alpha_1 & \alpha_0 & 0\ldots 0\\ \vdots & \vdots & & \\ \alpha_e & \alpha_{e-1} & & \ldots\alpha_1\end{bmatrix}, \tag{5.7}$$

where

$$\alpha_r = \begin{cases}\dfrac{(-N)_{Mr}}{r!}A_{N,r}\Gamma(ra+\mu+1)\Gamma(-\mu-ra-|s|+n), & \text{if } 0\le r\le [N/M],\\ 0, & \text{if } r > [N/M],\end{cases} \tag{5.8}$$

and η_e are given by the recursion formula

$$\eta_0\delta_0 = 1, \quad \sum_{l=0}^{q} \eta_l\delta_{q-l} = 0, \quad q = 1,2,3,\ldots; \quad \delta_0 \neq 0 \tag{5.9}$$

or explicitly by

$$\eta_e = (-1)^e(\delta_0)^{-e-1} \det \begin{bmatrix} \delta_1 & \delta_0 & 0 & 0\ldots 0 \\ \delta_2 & \delta_1 & \delta_0 & 0\ldots 0 \\ \vdots & \vdots & & \\ \delta_e & \delta_{e-1} & & \ldots \delta_1 \end{bmatrix}, \tag{5.10}$$

where

$$\delta_r = \begin{cases} \dfrac{(-N)_{Mr}}{r!} A_{N,r} \dfrac{\Gamma(ra+\mu+1)}{\Gamma(1+\mu+ra+|s|+n)}, & \text{if } 0 \le r \le [N/M], \\ 0, & \text{if } r > [N/M]. \end{cases} \tag{5.11}$$

Now assuming that $\mu = k$, where k is a positive integer and setting $A_{0,0} = 1$ in (5.4) and (5.5), and making use of the known formulas [10, p. 266, equations (7.5) and (7.6)], we get

$$(Y_{+;n}^{k;N,M})^{-1} f(x) = \prod_{j=1}^{k}\left(n - j + x_1\frac{\partial}{\partial x_1} + \cdots + x_n\frac{\partial}{\partial x_n}\right) f(x) + \frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty} k!\,\Gamma(-|s|+n)\sum_{e=1}^{\infty}\beta_e z^e f^*(s)x^{-s}\mathrm{d}s, \tag{5.12}$$

and

$$(Y_{-;n}^{k;N,M})^{-1} f(x) = (-1)^k \prod_{j=1}^{k}\left(n - j + x_1\frac{\partial}{\partial x_1} + \cdots + x_n\frac{\partial}{\partial x_n}\right) f(x) + \frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty} \frac{k!}{\Gamma(1+|s|-n)}\sum_{e=1}^{\infty}\eta_e z^e f^*(s)x^{-s}\mathrm{d}s. \tag{5.13}$$

In the above results (5.12) and (5.13), if we set $N = 0$ and $M = 1$ then we obtain the results of Tuan and Saigo [10, p. 266, equations (7.5) and (7.6)] and Raina [3, p. 161, equations (5.5) and (5.6)]

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