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Absolute N_{q_α}-summability of the series conjugate to a Fourier series

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Abstract. The object of the paper is to study the absolute Nq_α-summability of the series conjugate to a Fourier series, generalising a known result.

Keywords. Fourier series; conjugate series; kernel cesàro sum.

1. Introduction

In the year 1921 [6] F Nevanlinna suggested and discussed an interesting method of summation named as N_q-method. In 1932 [4] A F Moursund applied this method for the summation of Fourier series and allied series. In 1933 in a paper [5] he discussed N_{q_p}-method of summation, where p is a positive integer and applied it to p-th derived Fourier series. In his Ph D thesis [10] Samal studied absolute N_{q_p}-method of summation of p-th derived Fourier series. Samal in chapter-V of his Ph D thesis [10] extended the N_{q_p}-method of summation to N_{q_α}-method of summation for any $\alpha \geq 0$ and applied $|N_{q_\alpha}|$-method of summation to the Fourier series. In this paper we shall apply this method of summation to the series conjugate to a Fourier series.

DEFINITION 1 ([5], [10]).

Let $F(w)$ be a function of a continuous parameter w defined for all $w > 0$. The N_{q_α}-method of summation ($\alpha > 0$) consists in forming the N_{q_α}-transform or mean of $F(w)$

$$N_{q_\alpha}F(w) \equiv \int_0^1 q_\alpha(t)F(wt)\mathrm{d}t.$$

If $\lim_{w\to\infty} N_{q_\alpha}F(w) = S$, then we say that $F(w)$ is summable by N_{q_α}-method to the sum S. In short we write

$$\lim_{w\to\infty} F(w) = S(N_{q_\alpha}),$$

where the class of functions $q_\alpha(t)$ is such that when $\alpha \geq 1$, all conditions (i)–(vii) hold and in case $0 \leq \alpha < 1$, all hold except conditions (iii) and (iv). In later case $(0 \leq \alpha < 1)\, q_\alpha(t) \geq 0$ and monotonic increasing (the case $h = 0$ of (vi))

(i) $\quad q_\alpha(t) \geq 0$ for $0 \leq t \leq 1$

(ii) $\quad \int_0^1 q_\alpha(t)\mathrm{d}t = 1$

(iii) $(\mathrm{d}/\mathrm{d}t)^{\beta} q_\alpha(t)$ exists and absolutely continuous for $0 \le t \le 1$, where $[\alpha] = h$ and $\beta = 0, 1, 2, \ldots, h-1$

(iv) $(\mathrm{d}/\mathrm{d}t)^{\beta} q_\alpha(t) = 0$ for $t = 1, \beta = 0, 1, 2, \ldots, h-1$

(v) $(\mathrm{d}/\mathrm{d}t)^{h} q_\alpha(t)$ exists for $0 < t < 1$

(vi) $q^h(t) \ge 0$ and monotonic increasing for $0 < t < 1$, where $q^h(t) = (-1)^h (\mathrm{d}/\mathrm{d}t)^h q_\alpha(t)$

(vii) $\displaystyle\int_0^t \frac{Q_h(u)}{u^{1+\alpha-h}}\mathrm{d}u = 0\left(\frac{Q_h(t)}{t^{\alpha-h}}\right)$ for $0 < t \le \pi$,

where

$$Q_h(t) = \int_{1-t}^{1} q^h(u)\mathrm{d}u. \tag{1}$$

Also we set

$$Q(t) = \int_{1-t}^{1} q_\alpha(u)\mathrm{d}u. \tag{2}$$

The following equivalent definition of N_{q_α}-method is due to Samal ([9], [10]).

DEFINITION 2 ([9], [10]).

Suppose that $q_\alpha(t)$ is the class of functions as defined in Definition 1, and $\sum_{n=0}^{\infty} u_n$ is an infinite series with $S(w) = \sum_{n \le w} u_n$.

If

$$\lim_{w\to\infty} \sum_{n\le w} u_n Q\left(1 - \frac{n}{w}\right) = l. \tag{3}$$

We say that $\sum u_n$ is summable by N_{q_α}-method to the sum l. In short we write

$$\sum u_n = l(N_{q_\alpha}).$$

Further the series $\sum u_n$ is said to be $|N_{q_\alpha}|$-summable if

$$\int_A^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w} n u_n q_\alpha(n/w)\right| < \infty, \tag{4}$$

where A is a positive constant.

For $\alpha = 0$, the method reduces to original N_q-method [6] and if α is any positive integer then the method reduces to the N_{q_p}-method of Moursund [5].

Let $f(t)$ be a periodic function with period 2π and Lebesgue integrable over $(-\pi, \pi)$. Let

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt). \tag{5}$$

The series conjugate to (5) at $t = x$ is given by

$$\sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x). \tag{6}$$

We write

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\}$$
$$\psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\}$$
$$\Psi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \psi(u) \mathrm{d}u \ (\alpha > 0)$$
$$\Psi_0(t) = \psi(t)$$
$$\psi_\alpha(t) = \Gamma(\alpha+1) t^{-\alpha} \Psi_\alpha(t).$$

In his Ph D thesis [10] M Samal proved the following result:

Theorem M. *If $\phi_\alpha(t) \in BV(0,\pi)$, then the Fourier series of $f(t)$ is $|N_{q_\alpha}|$-summable at $t = x$ for $\alpha > 0$.*

2. Theorem and corollary

In the present paper we shall prove the following theorem.

Theorem. *If $\int_0^\pi \frac{|d\Psi_\alpha(t)|}{t^\alpha} < \infty$ and $\Psi_\alpha(+0) = 0$, then the series conjugate to the Fourier series of $f(t)$ is $|N_{q_\alpha}|$-summable at $t = x$ for $\alpha > 0$.*

By taking $q_\alpha(t) = \beta(1-t)^{\beta-1}$, where $0 < \alpha < \beta < h+1$, $[\alpha] = h$, in our theorem we obtain the following corollary.

COROLLARY [2]

If $\int_0^\pi \frac{|d\Psi_\alpha(t)|}{t^\alpha} < \infty$ and $\Psi_\alpha(+0) = 0$ then the series conjugate to the Fourier series of $f(t)$ at $t = x$ is summable $|C,\beta|$ for $\beta > \alpha > 0$.

3. Notations and lemmas

Notations

For our purpose we will use the following notations:

$$[w] = N$$
$$J(n,u) = \int_u^\pi (t-u)^{h-\alpha} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1} \cos nt \, \mathrm{d}t$$
$$S^{i,j}(x,u) = \sum_{n \le x} (x-n)^i \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^j \cos nu$$
$$\bar{S}^{i,j}(x,u) = \sum_{n \le x} (x-n)^i \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^j \sin nu$$
$$p(w,t) = \sum_{n \le w} q_\alpha\left(\frac{n}{w}\right) \cos nt$$
$$p_1(w,t) = \sum_{n \le w-1} q_\alpha\left(\frac{n}{w}\right) \cos nt$$

$$p_2(w,t)=\sum_{n\le w-(\pi/u)} q_\alpha(n/w)\cos nt,\quad 0<t\le u<\pi \text{ and } wu>\pi$$

$$p_3(w,t)=\sum_{[w-(\pi/u)]+1}^{[w-1]} q_\alpha(n/w)\cos nt,\quad 0<t\le u<\pi \text{ and } wu>\pi.$$

Lemmas

The followings are some important lemmas on N_{q_α}-method.

Lemma 1. *The kernel $q_\alpha(t)$ is monotonic decreasing if $\alpha\ge 1$, its derivatives of odd orders less than h are negative and monotonic increasing, its derivatives of even orders less than h are positive and monotonic decreasing and there exists a constant A_h such that*

$$\left|\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} q_\alpha(t)\right| < A_h \quad (\beta=0,1,2,3,\ldots,h-1)$$

and

$$\int_0^1 \left|\frac{\mathrm{d}^h}{\mathrm{d}t^h} q_\alpha(t)\right| \mathrm{d}t < A_h.$$

This Lemma is an analogue of Lemma 2.2 of Moursund [5].

Lemma 2 [10]. *For $\alpha\ge 0, Q_h(t)$ is a continuous and monotonic increasing function of $t, Q(0)=0, Q(1)=1$ and $Q_h(t)\ge 0$.*

This follows directly from the definition of $Q(t)$ and $Q_h(t)$.

Lemma 3 [10].

$$\int_0^1 \frac{q^h(t)}{(1-t)^{\alpha-h}}\mathrm{d}t \text{ exists for all } \alpha\ge 0.$$

Proof. Let $\alpha>h$ (when $\alpha=h$ the result is valid in view of (iii), (iv) and (v). We have

$$\begin{aligned}\int_0^1 \frac{Q_h(t)}{t^{1+\alpha-h}}\mathrm{d}t &= \int_0^1 \frac{\mathrm{d}t}{t^{1+\alpha-h}}\int_{1-t}^1 q^h(u)\mathrm{d}u\\ &= \int_0^1 q^h(u)\mathrm{d}u \int_{1-u}^1 \frac{\mathrm{d}t}{t^{1+\alpha-h}}\\ &= -\frac{1}{\alpha-h}\left[\int_0^1 q^h(u)\mathrm{d}u - \int_0^1 \frac{q^h(u)}{(1-u)^{\alpha-h}}\mathrm{d}u\right].\end{aligned}$$

Since $\int_0^1 \frac{Q_h(t)}{t^{1+\alpha-h}}\mathrm{d}t$ and $\int_0^1 q^h(u)\mathrm{d}u$ exist, it follows that $\int_0^1 \frac{q^h(u)}{(1-u)^{\alpha-h}}\mathrm{d}u$ exists.

Lemma 4 [9]. *For $0<\alpha<1$, $\sum_{k=1}^\infty \frac{Q(1/k)}{K^{1-\alpha}}$ is convergent.*

Lemma 5 [1]. *If $\beta>\alpha>0$, $\Psi_\alpha(t)$ is of $BV(0,\pi)$ and $\Psi_\alpha(+0)=0$, then $\Psi_\beta(t)$ is an integral in $(0,\pi)$ and for almost all values of t,*

$$\Psi'_\beta(t) = \frac{1}{\Gamma(\beta-\alpha)} \int_0^t (t-u)^{\beta-\alpha-1} \mathrm{d}\Psi_\alpha(u).$$

Lemma 6. *For* $0 < \alpha < 1$

$$\int_u^\pi (t-u)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} P_1(w,t)\mathrm{d}t = O(w^{\alpha+1}).$$

Proof.

$$\begin{aligned} |p_1(w,t)| &\le \sum_{n<w-1} q_\alpha(n/w) \\ &< \int_0^w q_\alpha(x/w)\mathrm{d}x \\ &= w\int_0^1 q_\alpha(t)\mathrm{d}t \\ &= w. \end{aligned}$$

Similarly $\left|\frac{\mathrm{d}}{\mathrm{d}t}p_1(w,t)\right| \le w^2$. Now

$$\begin{aligned} \int_u^\pi (t-u)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}p_1(w,t)\mathrm{d}t &= \int_u^{u+(1/w)} (t-u)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}p_1(w,t)\mathrm{d}t \\ &\quad + \int_{u+(1/w)}^\pi (t-u)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}p_1(w,t)\mathrm{d}t \\ &= \int_u^{u+(1/w)} (t-u)^{-\alpha} O(w^2)\mathrm{d}t \\ &\quad + w^\alpha \int_{u+(1/w)}^\eta \frac{\mathrm{d}}{\mathrm{d}t}p_1(w,t)\mathrm{d}t, \end{aligned}$$

for some $u + \frac{1}{w} < \eta < \pi$, by application of mean value theorem.

$$\begin{aligned} &= O\left\{w^2 \int_u^{u+(1/w)} (t-u)^{-\alpha}\mathrm{d}t\right\} + w^\alpha\left\{p_1(w,\eta) - p_1\left(w, u+\frac{1}{w}\right)\right\} \\ &= O(w^{\alpha+1}) + O(w^{\alpha+1}) \\ &= O(w^{\alpha+1}). \end{aligned}$$

Lemma 7. For $0 < \alpha < 1$, $0 < u < \pi$ *and* $wu > \pi$

$$\int_u^\pi (t-u)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t} p_2(w,t)\mathrm{d}t = O\left\{\frac{w^\alpha q_\alpha(1-(\pi/wu))}{u}\right\}.$$

Proof. We have

$$|p_2(w,t)| = \left|\sum_{n\le w(\pi/u)} q_\alpha(n/w)\cos nt\right|$$

$$\leq q_\alpha\Big(1-\frac{\pi}{wu}\Big)\max_{1<L,L'<w-(\pi/u)}\Big|\sum_{L}^{L'}\cos nt\Big|$$

$$<\frac{Kq_\alpha(1-(\pi/wu))}{t}$$

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}p_2(w,t)\Big| = \Big|\sum_{n\leq w-(\pi/u)} nq_\alpha(n/w)\sin nt\Big|$$

$$\leq\Big(w-\frac{\pi}{u}\Big)q_\alpha\Big(1-\frac{\pi}{wu}\Big)\max_{1<L,L'<w-(\pi/u)}\Big|\sum_{L}^{L'}\sin nt\Big|$$

$$<\frac{Kwq_\alpha(1-(\pi/wu))}{t}.$$

Now using the technique used in the second half of the proof of the Lemma 6 it can be proved that

$$\int_u^\pi (t-w)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}p_2(w,t)\mathrm{d}t = O\Big\{\frac{w^\alpha q_\alpha(1-(\pi/wu))}{u}\Big\}.$$

Lemma 8. *For* $0<\alpha<1$, $0<u<\pi$ *and* $wu>\pi$

$$\int_u^\pi (t-u)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}p_3(w,t)\mathrm{d}t = O\Big\{w^{\alpha+1}Q\Big(\frac{\pi}{wu}\Big)\Big\}.$$

Proof.

$$|P_3(w,t)|\leq\sum_{[w-(\pi/u)]+1}^{[w-1]} q_\alpha(n/w)$$

$$<\int_{w-(\pi/u)}^{w} q_\alpha(x/w)\mathrm{d}x$$

$$=w\int_{1-(\pi/wu)}^{1} q_\alpha(t)\mathrm{d}t = wQ(\pi/wu).$$

Similarly, $|\frac{\mathrm{d}}{\mathrm{d}t}p_3(w,t)|<w^2Q(\pi/wu)$.

Using the technique similar to that used in the second half of the proof of the Lemma 6, it can be proved that

$$\int_u^\pi (t-u)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}p_3(w,t)\mathrm{d}t = 0\{w^{\alpha+1}Q(\pi/wu)\}.$$

Lemma 9. *For* $0<\alpha<1$,

$$\int_u^\pi (t-u)^{-\alpha}\sin nt\,\mathrm{d}t = O(n^{\alpha-1})$$

Proof. By application of mean value theorem for some $u+(1/n)<\eta<\pi$

$$\int_u^\pi (t-u)^{-\alpha}\sin nt\,dt = \int_u^{u+(1/n)} (t-u)^{-\alpha}\sin nt\,dt + \int_{u+(1/n)}^{\pi} (t-u)^{-\alpha}\sin nt\,dt$$
$$= O\left\{\int_u^{u+(1/n)} (t-u)^{-\alpha}dt\right\} + n^\alpha \int_{u+(1/n)}^{n} \sin nt\,dt$$
$$= O(n^{\alpha-1}) + n^\alpha \frac{\cos u(n+(1/n)) - \cos n\eta}{n}$$
$$= O(n^{\alpha-1}).$$

Lemma 10. For $\alpha \geq 1$.

$$\sum_{n\leq w} q_\alpha\left(\frac{n}{w}\right) J(n,u) = O(w^{\alpha+1}).$$

Proof.

$$\left|\left(\frac{d}{dt}\right)^h p(w,t)\right| \leq w^h \sum_{n\leq w} q_\alpha\left(\frac{n}{w}\right)$$
$$\leq w^h\left\{q_\alpha(o) + \int_o^w q_\alpha\left(\frac{x}{w}\right)dx\right\}$$
$$= w^h\left\{q_\alpha(o) + w\int_o^1 q_\alpha(t)dt\right\}$$
$$= O(w^{h+1}),\ \text{as } q_\alpha(o) \text{ is finite.}$$

Similarly, $\left(\frac{d}{dt}\right)^{h+1} p(w,t) = O(w^{h+2})$.

Now proceeding as in the second half of the proof of the Lemma 6 it can be easily proved that

$$\sum_{n\leq w} q_\alpha\left(\frac{n}{w}\right) J(n,u) = O(w^{\alpha+1}).$$

Lemma 11 [3]. *Let $\lambda = \{\lambda_n\}$ be a positive monotonic increasing sequence with $\lambda_n \to \infty$ as $n \to \infty$.*

$$A_\lambda(x) = A_\lambda^o(x) = \sum_{\lambda_n\leq x} a_n$$

and

$$A_\lambda^r(x) = \sum_{\lambda_n\leq x} (x-\lambda_n)^r a_n \quad (r>0).$$

Then if k is a positive integer,

$$A_\lambda(x) = \frac{1}{k!}\left(\frac{d}{dx}\right)^k A_\lambda^k(x).$$

Lemma 12 [[7], [8]]. *Let $C_n^{(i)}, S_n^{(i)}, \bar{S}_n^{(i)}$ denote the nth Cesaro sums of order $i \geq 0$ corresponding to the series $\sum_1^\infty (-1)^n n^j$, $\sum_1^\infty \left(\frac{d}{du}\right)^j \cos nu$, $\sum_1^\infty (d/du)^j \sin nu$ respectively.*

Then

(i) $$S_n^{(i)} = O(n^{i+j+1}), \quad 0 < u \le \frac{1}{n}$$

(ii) $$S_n^{(i)} = O(n^j u^{-i-1}) + O(n^{i-1} u^{-j-1}), \quad \frac{1}{n} < u \le \pi$$

(iii) $$\bar{S}_n^{(i)} = O(n^{i+j+1}), \quad O < u \le \frac{1}{n} .$$

(iv) $$\bar{S}_n^{(i)} = O(n^j u^{-i-1}) + O(n^i u^{-j-1}), \quad \frac{1}{n} < u \le \pi$$

(v) $$C_n^{(i)} = O(n^{\max j, i-1}) \quad \textit{if } j \textit{ is even and } \ge 2.$$

Lemma 13. *Let* $x > 0$,

(i) *If* $\frac{1}{x} < u \le \pi$, *then*

$$S^{i,j}(x,u) = \begin{cases} O(x^i u^{-j-1}) & \text{for} \quad 0 \le j \le i \\ O(x^j u^{-i-1}) & \text{for} \quad j > i \ge 0. \end{cases}$$

(ii) *If* $\frac{1}{x} \ge u > 0$

$$S^{i,j}(x,u) = O(x^{i+j+1}).$$

(iii) *If* $\frac{1}{x} < u \le \pi$, *then*

$$\bar{S}^{i,j}(x,u) = \begin{cases} O(x^i u^{-j-1}) & \text{for} \quad 0 \le j \le i \\ O(x^j u^{-i-1}) & \text{for} \quad j > i \ge 0. \end{cases}$$

Proof. If $[x] = m$, then by repeated application of Abel's transformation, we have

$$\begin{aligned} S^{i,j}(x,u) &= \sum_{n \le x} (x-n)^i \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^j \cos nu \\ &= \sum_{n=1}^{m-1} \Delta (x-n)^i S_n^{(o)} + (x-m)^i S_m^{(o)} \\ &= \sum_{n=1}^{m-2} \Delta^2 (x-n)^i S_n^{(1)} + [\Delta (x-n)^i]_{n=m-1} S_{m-1}^{(1)} + (x-m)^i S_m^{(o)} \\ &= \sum_{n=1}^{m-3} \Delta^3 (x-n)^i S_n^{(2)} + [\Delta^2 (x-n)^i]_{n=m-2} S_{m-2}^{(2)} \\ &\quad + [\Delta (x-n)^i]_{n=m-1} S_{m-1}^{(1)} + (x-m)^i S_m^{(o)} \\ &= \sum_{n=1}^{m-i} \Delta^i (x-n)^i S_n^{(i-1)} + \sum_{k=0}^{i-1} [\Delta^k (x-n)^i]_{n=m-k} S_{m-k}^{(k)}. \end{aligned} \tag{7}$$

(i) Now for $k = 0, 1, 2, \ldots, (i-1)$ and $\frac{1}{x} < u \le \pi$

$$[\Delta^k (x-n)^i]_{n=m-k} S_{m-k}^{(k)} = O(x^j u^{-k-1}) + O(x^{k-1} u^{-j-1}) \tag{8}$$

by Lemma 12 (ii).

Since $\Delta^i(x-n)^i$ is a constant $\sum_{n=1}^{m-i}\Delta^i(x-n)^i S_n^{(i-1)}$ is of order $S_n^{(i)}$. Thus using Lemma 12 (ii)

$$\sum_{n=1}^{m-i}\Delta^i(x-n)^i S_n^{(i-1)} = O(x^j u^{-i-1}) + O(x^{i-1}u^{-j-1}). \tag{9}$$

Since $x^j u^{-k-1}$ and $x^{k-1}u^{-j-1}$ are respectively dominated by $x^j u^{-i-1}$ and $x^{i-1}u^{-j-1}$, we obtain using (8) and (9) in (7)

$$S^{i,j}(x,u) = O(x^j u^{-i-1}) + O(x^{i-1}u^{-j-1}).$$

So for $\frac{1}{x} < u \leq \pi$,

$$S^{i,j}(x,u) = \begin{cases} O(x^i u^{-j-1}) & \text{for} \quad 0 \leq j \leq i \\ O(x^j u^{-i-1}) & \text{for} \quad j > i \geq 0. \end{cases}$$

(ii) Now for $k = 0, 1, 2, \ldots, (i-1)$ and $\frac{1}{x} \geq u\, 0$

$$[\Delta^k(x-n)^i]_{n=m-k} S_{m-k}^{(k)} = O(x^{j+k+1}) \tag{10}$$

by Lemma 12 (i).

Also using Lemma 12 (i)

$$\sum_{n=1}^{m-i}\Delta^i(x-n)^i S_n^{(i-1)} = O(x^{i+j+1}). \tag{11}$$

Since x^{i+j+1} dominates x^{j+k+1}, we obtain after using (10) and (11) in (7)

$$S^{i,j}(x,u) = O(x^{i+j+1}).$$

(iii) The proof of (iii) is similar to that of (i).

Lemma 14. *For* $0 < u \leq \pi$, $wu > 2\pi$ *and* $\alpha > 0$

$$\int_u^\pi (t-u)^{h-\alpha}\mathrm{d}t \int_1^{w-\pi/u} (-1)^h \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x$$
$$= O\left\{\frac{w^{\alpha-h}q^h(1-(\pi/wu))}{u^{h+1}}\right\}.$$

Proof.

$$\int_u^\pi (t-u)^{h-\alpha}\mathrm{d}t \int_1^{w-(\pi/u)} (-1)^h \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x$$
$$= \int_u^{u+(1/w)} (t-u)^{h-\alpha}\mathrm{d}t \int_1^{w-(\pi/u)} (-1)^h \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x$$
$$+ \int_{u+(1/w)}^\pi (t-u)^{h-\alpha}\mathrm{d}t \int_1^{w-(\pi/u)} (-1)^h \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x$$
$$= I_1 + I_2 \text{ say.}$$

As $q^h\left(\frac{x}{w}\right)$ is monotonic increasing in x by the mean value theorem for some $1<\xi< w-\frac{\pi}{u}$,

$$
\begin{aligned}
I_1 &= \int_u^{u+(1/w)} (t-u)^{h-\alpha}\mathrm{d}t \int_1^{w-(\pi/u)} (-1)^h \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right) \frac{\mathrm{d}}{\mathrm{d}x} S^{h,h+1}(x,t)\mathrm{d}x \\
&= \int_u^{u+(1/w)} (t-u)^{h-\alpha}\mathrm{d}t \frac{1}{w^h} \\
&\quad \times \int_1^{w-(\pi/u)} \left[(-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^h q_\alpha(\theta)\right]_{\theta=(x/w)} \frac{\mathrm{d}}{\mathrm{d}x} S^{h,h+1}(x,t)\mathrm{d}x \\
&= \int_u^{u+(1/w)} (t-u)^{h-\alpha}\mathrm{d}t \frac{1}{w^h} \int_1^{w-(\pi/u)} q^h\left(\frac{x}{w}\right) \frac{\mathrm{d}}{\mathrm{d}x} S^{h,h+1}(x,t)\mathrm{d}x \\
&= w^{-h} \int_u^{u+(1+w)} (t-u)^{h-\alpha}\mathrm{d}t\, q^h\left(1-\frac{\pi}{wu}\right) \int_\xi^{w-(\pi/u)} \frac{\mathrm{d}}{\mathrm{d}x} S^{h,h+1}(x,t)\mathrm{d}x \\
&= w^{-h} q^h\left(1-\frac{\pi}{wu}\right) \int_u^{u+(1/w)} (t-u)^{h-\alpha}\left\{S^{h,h+1}\left(w-\frac{\pi}{u},t\right) - S^{h,h+1}(\xi,t)\right\}\mathrm{d}t \\
&= w^{-h} q^h\left(1+\frac{\pi}{wu}\right) \int_u^{u+(1/w)} (t-u)^{h-\alpha} O(w^{h+1}t^{-h-1})\mathrm{d}t,
\end{aligned}
$$

using Lemma 13(i), (ii)

$$
\begin{aligned}
&= O\left\{\frac{wq^h(1-(\pi/wu))}{u^{h+1}} \int_u^{u+(1/w)} (t-u)^{h-\alpha}\mathrm{d}t\right\} \\
&= O\frac{w^{\alpha-h}q^h(1-(\pi/wu))}{u^{h+1}}.
\end{aligned}
$$

For some $1<\zeta<w-\frac{\pi}{u}$, by application of mean value theorem

$$
\begin{aligned}
I_2 &= \int_{u+(1/w)}^{\pi} (t-u)^{h-\alpha}\mathrm{d}t \frac{1}{w^h} \\
&\quad \times \int_1^{w-(\pi/u)} \left[(-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^h q_\alpha(\theta)\right]_{\theta=(x/w)} \frac{\mathrm{d}}{\mathrm{d}x} S^{h,h+1}(x,t)\mathrm{d}x \\
&= w^{-h} \int_{u+(1/w)}^{\pi} (t-u)^{h-\alpha}\mathrm{d}t\, q^h\left(1-\frac{\pi}{wu}\right) \int_\zeta^{w-(\pi/u)} \frac{\mathrm{d}}{\mathrm{d}x} S^{h,h+1}(x,t)\mathrm{d}x \\
&= w^{-h} q^h\left(1+\frac{\pi}{u}\right) \int_{u+(1/w)}^{\pi} (t-u)^{h-\alpha}\left\{S^{h,h+1}\left(w-\frac{\pi}{u},t\right) - S^{h,h+1}(\zeta,t)\right\}\mathrm{d}t \\
&= w^{-h} q^h\left(1-\frac{\pi}{u}\right) w^{\alpha-h} \int_{u+(1/w)}^{\eta} \left\{S^{h,h+1}\left(w-\frac{\pi}{u},t\right) - S^{h,h+1}(\zeta,t)\right\}\mathrm{d}t,
\end{aligned}
$$

(for some $u+\frac{1}{u}<\eta<\pi$, by application of mean value theorem)

$$
\begin{aligned}
&= w^{\alpha-2h} q^h\left(1-\frac{\pi}{wu}\right)\left\{S^{h,h}\left(w-\frac{\pi}{u},\eta\right) - S^{h,h}(\zeta,\eta)\right. \\
&\qquad \left. - S^{h,h}\left(w-\frac{\pi}{u},u+\frac{1}{w}\right) + S^{h,h}\left(\zeta,u+\frac{1}{w}\right)\right\}
\end{aligned}
$$

$$= w^{\alpha-2h} q^h \left(1 - \frac{\pi}{wu}\right) O(w^h u^{-h-1}), \text{ by Lemma 13(i) (ii)}$$

$$= O\left\{\frac{w^{\alpha-h} q^h (1 - (\pi/wu))}{u^{h+1}}\right\}.$$

This completes the proof of Lemma 14.

Lemma 15.

$$\sum_{n \le x} (x-n)^r (-1)^n n^r = O(x^r) \text{ and } \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n \le x} (x-n)^r (-1)^n n^r\right) = O(x^r).$$

Proof. If r is even

$$\left|\sum_{n \le x} (x-n)^r (-1)^n n^r\right| = \left|\sum_{n \le x} (x-n)^r \left[\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^r \cos nu\right]_{u=\pi}\right|$$

$$= [S^{r,r}(x,u)]_{u=\pi}$$

$$= O(x^r) \text{ by Lemma 13 (i).}$$

If r is odd

$$\left|\sum_{n \le x} (x-n)^r (-1)^n n^r\right| = \left|\left[\sum_{n \le x} (x-n)^r \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^r \sin nu\right]_{\pi/2}^{\pi}\right|$$

$$= [S^{-r,r}(x,u)]_{u=\pi}$$

$$= O(x^r) \text{ by Lemma 13 (iii).}$$

Similarly it can be proved that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n \le x} (x-n)^r (-1)^n n^r\right) = O(x^r).$$

Lemma 16. *For* $0 < u \le \pi, wu > 2\pi$ *and* $\alpha > 0$

$$\int_u^{\pi} (t-u)^{h-\alpha} \mathrm{d}t \int_{w-(\pi/u)}^{w} (-1)^h \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right) \frac{\mathrm{d}}{\mathrm{d}x} s^{h,h+1}(x,t) \mathrm{d}x$$

$$= O\left\{\frac{w^{\alpha-h+1} Q_h(\pi/wu)}{u^h}\right\}.$$

Proof.

$$\int_u^{\pi} (t-u)^{h-\alpha} \mathrm{d}t \int_{w-(\pi/u)}^{w} (-1)^h \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right) \frac{\mathrm{d}}{\mathrm{d}x} S^{h,h+1}(x,t) \mathrm{d}x$$

$$= \int_u^{u+(1/w)} (t-u)^{h-\alpha} \mathrm{d}t \int_{w-(\pi/u)}^{w} (-1)^h \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right) \frac{\mathrm{d}}{\mathrm{d}x} S^{h,h+1}(x,t) \mathrm{d}x$$

$$+\int_{u+(1/w)}^{\pi}(t-u)^{h-\alpha}\mathrm{d}t\int_{w-(\pi/u)}^{w}(-1)^{h}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{h}q_{\alpha}(x/w)\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x$$
$$=J_1+J_2 \text{ say.}$$

$$J_1=h\int_{u}^{u+(1/w)}(t-u)^{h-\alpha}\mathrm{d}t\int_{w-(\pi/u)}^{w}(-1)^{h}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h}q_{\alpha}\left(\frac{x}{w}\right)S^{h-1,h+1}(x,t)\mathrm{d}x$$
$$=h\int_{w-(\pi/u)}^{w}(-1)^{h}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{h}q_{\alpha}\left(\frac{x}{w}\right)\mathrm{d}x\int_{u}^{u+(1/w)}(t-u)^{h-\alpha}S^{h-1,h+1}(x,t)\mathrm{d}t$$
$$=h\int_{w-(\pi/u)}^{w}(-1)^{h}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{h}q_{\alpha}(x/w)\int_{u}^{u+(1/w)}(t-u)^{h-\alpha}O(x^{h+1}t^{-h})\,\mathrm{d}t,$$

by Lemma 13 (i)

$$=h\int_{w-(\pi/u)}^{w}(-1)^{h}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{h}q_{\alpha}\left(\frac{x}{w}\right)\mathrm{d}xO\left\{\frac{x^{h+1}}{u^{h}}\int_{u}^{u+(1/w)}(t-u)^{h-\alpha}\mathrm{d}t\right\}$$
$$=h\int_{w-(\pi/u)}^{w}(-1)^{h}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{h}q_{\alpha}\left(\frac{x}{w}\right)\mathrm{d}xO(u^{-h}x^{h+1}w^{\alpha-h-1})$$
$$=O\left\{u^{-h}w^{\alpha-h-1}\int_{w-(\pi/u)}^{w}\left[(-1)^{h}\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{h}q_{\alpha}(\theta)\right]_{\theta=x/w}x^{h+1}w^{-h}\,\mathrm{d}x\right\}$$
$$=O\left\{u^{-h}w^{\alpha-h}\int_{w-(\pi/u)}^{w}q^{h}(x/w)\mathrm{d}x\right\}$$
$$=O\left\{u^{-h}w^{\alpha-h+1}\int_{1-(\pi/wu)}^{1}q^{h}(\theta)\mathrm{d}\theta\right\}$$
$$=O\left\{u^{-h}w^{\alpha-h+1}Q_{h}\left(\frac{\pi}{wu}\right)\right\}$$

by application of mean value theorem for some $u+\frac{1}{w}<\zeta<\pi$,

$$J_2=h\int_{u+(1/w)}^{\pi}(t-u)^{h-\alpha}\mathrm{d}t\int_{w-(\pi/u)}^{w}(-1)^{h}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{h}q_{\alpha}\left(\frac{x}{w}\right)S^{h-1,h+1}(x,t)\mathrm{d}x$$
$$=h\int_{w-(\pi/u)}^{w}(-1)^{h}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{h}q_{\alpha}\left(\frac{x}{w}\right)\mathrm{d}x\int_{u+(1/w)}^{\pi}(t-u)^{h-\alpha}S^{h-1,h+1}(x,t)\mathrm{d}t$$
$$=h\int_{w-(\pi/u)}^{w}(-1)^{h}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{h}q_{\alpha}\left(\frac{x}{w}\right)\mathrm{d}x\,w^{\alpha-h}\int_{u+(1/w)}^{\zeta}S^{h-1,h+1}(x,t)\mathrm{d}t$$
$$=hw^{\alpha-2h}\int_{w-(\pi/u)}^{w}q^{h}(x/w)O(x^{h}u^{-h})\,\mathrm{d}x,\ \text{by lemma 13 (i)}$$
$$=O\left(w^{\alpha-h+1}u^{-h}\int_{1-(\pi/wu)}^{1}q^{h}(\theta)\mathrm{d}\theta\right)$$
$$=O\left(\frac{w^{\alpha-h+1}Q_{h}(\pi/wu)}{u^{h}}\right).$$

Hence

$$\int_u^{\pi}(t-u)^{h-\alpha}\mathrm{d}t\int_{w-(\pi/u)}^{w}(-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x$$

$$=O\left\{\frac{w^{\alpha-h+1}Q_h(\pi/wu)}{u^h}\right\}.$$

Lemma 17. *For* $\alpha\geq 1$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1}p(w,t)=\frac{1}{h!}\int_1^w(-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x.$$

Proof.

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1}p(w,t)=\sum_{n\leq w}q_\alpha\left(\frac{n}{w}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1}\cos nt$$

$$=q_\alpha(1)\sum_{n\leq w}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1}\cos nt-\int_1^w\frac{\mathrm{d}}{\mathrm{d}x}q_\alpha\left(\frac{x}{w}\right)\left\{\sum_{n\leq x}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1}\cos nt\right\}\mathrm{d}x$$

$$=-\int_1^w\frac{\mathrm{d}}{\mathrm{d}x}q_\alpha\left(\frac{x}{w}\right)\left\{\sum_{n\leq x}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1}\cos nt\right\}\mathrm{d}x,\ \text{since } q_\alpha(1)=0 \text{ for } \alpha\geq 1$$

$$=-\frac{1}{h!}\int_1^w\frac{\mathrm{d}}{\mathrm{d}x}q_\alpha\left(\frac{x}{w}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h\left\{\sum_{n\leq x}(x-n)^h\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1}\cos nt\right\}\mathrm{d}x,\ \text{by Lemma 11}$$

$$=-\frac{1}{h!}\int_1^w\frac{\mathrm{d}}{\mathrm{d}x}q_\alpha\left(\frac{x}{w}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h S^{h,h+1}(x,t)\mathrm{d}x$$

$$=\frac{1}{h!}\left[\sum_{k=1}^{h-1}(-1)^k\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k q_\alpha\left(\frac{x}{w}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{h-k}S^{h,h+1}(x,t)\right]_1^w$$

$$+\frac{1}{h!}\int_1^w(-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x,$$

integrating by parts $(h-1)$ times,

$$=\frac{1}{h!}\int_1^w(-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x,$$

as the integrated part vanishes for $x=1$ and $x=w$.

Lemma 18 [10]. *For* $\alpha\geq 1$

$$\sum_{n\leq w}(-1)^n n^h q_\alpha\left(\frac{n}{w}\right)=O\left\{q^h\left(1-\frac{1}{w}\right)\right\}+O\left\{wQ_h\left(\frac{1}{w}\right)\right\}.$$

Proof. Using the technique used in the proof of Lemma 17 we get

$$\sum_{n\le w}(-1)^n n^h q_\alpha(n/w)$$

$$= \frac{1}{h!}\int_1^w (-1)^h \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{n\le x}(x-n)^h(-1)^n n^h\right\}\mathrm{d}x$$

$$= \frac{1}{h!}\int_1^{w-1} (-1)^h \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{n\le x}(x-n)^h(-1)^n n^h\right\}\mathrm{d}x$$

$$+ \frac{1}{h!}\int_{w-1}^w (-1)^h \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{\mathrm{d}}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{n\le x}(x-n)^h(-1)^n n^h\right\}\mathrm{d}$$

$$= I_1 + I_2 \text{ say.}$$

$$I_1 = \left[(-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\right]_{x=w-1}\int_\xi^{w-1}\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{n\le x}(x-n)^h(-1)^n n^h\right.$$

by mean value theorem, for $1 < \zeta < w-1$,

$$= \frac{1}{w^h}q^h\left(1-\frac{1}{w}\right)\left[\sum_{n\le x}(x-n)^h(-1)^n n^h\right]_\zeta^{w-1}$$

$$= O\left\{q^h\left(1-\frac{1}{w}\right)\right\} \text{ by Lemma 15.}$$

$$I_2 = O\left(\int_{w-1}^w (-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)x^h\mathrm{d}x\right), \text{ by Lemma 15}$$

$$= O\left(w\int_{1-(1/w)}^1 \theta^h(-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^h q_\alpha(\theta)\mathrm{d}\theta\right)$$

$$= O\left(wQ_h\left(\frac{1}{w}\right)\right).$$

Hence

$$\sum_{n\le w}(-1)^n n^h q_\alpha\left(\frac{n}{w}\right) = O\left\{q^h\left(1-\frac{1}{w}\right)\right\} + O\left\{wQ_h\left(\frac{1}{w}\right)\right\}.$$

Lemma 19 [10]. *Let* $\alpha \ge 1$. *For* $r = 0, 1, 2, \ldots, h-1$

$$\sum_{n\le w}(-1)^n n^r q_\alpha\left(\frac{n}{w}\right) = O(1).$$

Proof.

$$\sum_{n\le w}(-1)^n n^r q_\alpha\left(\frac{n}{w}\right) = -\int_1^w \frac{\mathrm{d}}{\mathrm{d}x}q_\alpha\left(\frac{x}{w}\right)\left\{\sum_{n\le x}(-1)^n n^r\right\}\mathrm{d}x.$$

Proceeding as in Lemma 17, we get

$$\sum_{n\le w}(-1)^n n^r q_\alpha\left(\frac{n}{w}\right)$$

$$= \frac{(-1)^{r+1}}{r!}\int_1^w \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{r+1} q_\alpha\left(\frac{x}{w}\right)\left\{\sum_{n\le x}(x-n)^r(-1)^n n^r\right\}\mathrm{d}x$$

$$= \int_1^w O\left\{(-1)^{r+1}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{r+1} q_\alpha\left(\frac{x}{w}\right)x^r\right\}\mathrm{d}x, \text{ by Lemma 15}$$

$$= O\left\{\int_0^1 \frac{x^r}{w^r}(-1)^{r+1}\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{r+1} q_\alpha(\theta)\mathrm{d}\theta\right\}$$

$$= O\left\{\int_0^1 (-1)^{r+1}\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{r+1} q_\alpha(\theta)\mathrm{d}\theta\right\}$$

$$= O(1), \text{ by Lemma 1.}$$

4. Proof of the theorem

We shall prove our theorem in two cases namely Case I for $O < \alpha < 1$ and Case II for $\alpha \ge 1$.

Case I. For $O < \alpha < 1$.

We know $B_n(x) = \frac{2}{\pi}\int_0^\pi \psi(t)\sin nt\,\mathrm{d}t$.

By the use of Lemma 5, we have

$$\begin{aligned} B_n(x) &= \frac{2}{\pi}\int_0^\pi \sin\, nt\,\psi(t)\mathrm{d}t \\ &= \frac{2}{\pi}\int_0^\pi \sin nt\,\mathrm{d}t\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-u)^{-\alpha}\mathrm{d}\Psi_\alpha(u) \\ &= \frac{2}{\alpha\Gamma(1-\alpha)}\int_0^\pi \mathrm{d}\Psi_\alpha(u)\int_u^\pi (t-u)^{-\alpha}\sin nt\,\mathrm{d}t \end{aligned}$$

By Definition 2 the series $\sum B_n(x) \in |N_{q_\alpha}|$ if and only if

$$I = \int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w} nB_n(x)q_\alpha\left(\frac{n}{w}\right)\right| < \infty.$$

Now

$$\begin{aligned} I &= \int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w} q_\alpha\left(\frac{n}{w}\right)\frac{2n}{\pi\Gamma(1-\alpha)}\int_0^\pi \mathrm{d}\Psi_\alpha(u)\int_u^\pi (t-u)^{-\alpha}\sin nt\,\mathrm{d}t\right| \\ &\le k\int_0^\pi |\mathrm{d}\Psi_\alpha(u)|\int_1^\infty \left|\int_u^\pi (t-u)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}p(w,t)\mathrm{d}t\right|\frac{\mathrm{d}w}{w^2}. \end{aligned}$$

Since $\int_0^\pi \frac{|\mathrm{d}\Psi_\alpha(u)|}{u^\alpha} < \infty$ it is enough to show that

$$\int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}p(w,t)\mathrm{d}t\right| = O\left(\frac{1}{u^\alpha}\right). \tag{12}$$

Now

$$\begin{aligned}
&\int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}p(w,t)\mathrm{d}t\right| \\
&= \int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}\left\{\frac{\mathrm{d}}{\mathrm{d}t}p_1(w,t) - q_\alpha\left(\frac{N}{w}\right)N\sin Nt\right\}\mathrm{d}t\right| \\
&\le \int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}p_1(w,t)\mathrm{d}t\right| \\
&\quad + \int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}q_\alpha(N/w)N\sin Nt\,\mathrm{d}t\right|.
\end{aligned} \tag{13}$$

And

$$\begin{aligned}
&\int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}p_1(w,t)\mathrm{d}t\right| \\
&= \int_1^{\pi/u} \frac{\mathrm{d}w}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}p_1(w,t)\mathrm{d}t\right| \\
&\quad + \int_{\pi/u}^\infty \frac{\mathrm{d}w}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}p_1(w,t)\mathrm{d}t\right| \\
&= J_1 + J_2 \text{ say.}
\end{aligned} \tag{14}$$

By the use of Lemma 6

$$\begin{aligned}
J_1 &= \int_1^{\pi/u} O(w^{\alpha+1})\frac{\mathrm{d}w}{w^2} = O\left(\int_1^{\pi/u} w^{\alpha-1}\mathrm{d}w\right) \\
&= O(1/u^\alpha)
\end{aligned} \tag{15}$$

$$\begin{aligned}
J_2 &= \int_{\pi/u}^\infty \frac{\mathrm{d}w}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}\left\{\frac{\mathrm{d}}{\mathrm{d}t}p_2(w,t) + \frac{\mathrm{d}}{\mathrm{d}t}p_3(w,t)\right\}\mathrm{d}t\right| \\
&\le \int_{\pi/u}^\infty \frac{\mathrm{d}w}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}p_2(w,t)\mathrm{d}t\right| \\
&\quad + \int_{\pi/u}^\infty \frac{\mathrm{d}w}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}p_3(w,t)\mathrm{d}t\right| \\
&= J_{21} + J_{22} \text{ say.}
\end{aligned} \tag{16}$$

By the use of Lemma 7

$$\begin{aligned}
J_{21} &= \int_{\pi/u}^\infty \frac{\mathrm{d}w}{w^2} O\left\{\frac{w^\alpha q_\alpha(1-(\pi/wu))}{u}\right\} \\
&= O\left\{\int_{\pi/u}^\infty \frac{w^\alpha q_\alpha(1-(\pi/wu))}{u}\frac{\mathrm{d}w}{w^2}\right\} \\
&= O\left\{u^{-\alpha}\int_0^1 \frac{q_\alpha(t)}{(1-t)^\alpha}\mathrm{d}t\right\} \\
&= O(u^{-\alpha}) \text{ by Lemma 3.}
\end{aligned} \tag{17}$$

By the use of Lemma-8

$$J_{22} = \int_{\pi/u}^{\infty} \frac{\mathrm{d}w}{w^2} O\Big\{w^{\alpha+1} Q\Big(\frac{\pi}{wu}\Big)\Big\}$$

$$= O\Big\{\int_{\pi/u}^{\infty} Q\Big(\frac{\pi}{wu}\Big) w^{\alpha-1}\mathrm{d}w\Big\}$$

$$= O\Big\{u^{-\alpha}\int_0^1 \frac{Q(t)}{t^{1+\alpha}}\mathrm{d}t\Big\}$$

$$= O(u^{-\alpha}) \text{ by (vii), since for } O < \alpha < 1, Q_h(t) = Q(t). \tag{18}$$

Now (14), (15), (16), (17) and (18) together imply

$$\int_1^{\infty} \frac{\mathrm{d}w}{w^2}\left|\int_u^{\pi} (t-u)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t} p_1(w,t)\mathrm{d}t\right| = O(u^{-\alpha}). \tag{19}$$

By the use of Lemma 9

$$\int_1^{\infty} \frac{\mathrm{d}w}{w^2}\left|\int_u^{\pi} (t-u)^{-\alpha} N q_\alpha(N/w)\sin Nt\,\mathrm{d}t\right|$$

$$= \int_1^{\infty} \frac{\mathrm{d}w}{w^2} O\{N^\alpha q_\alpha(N/w)\}$$

$$= O\Big\{\int_1^{\alpha} N^\alpha q_\alpha(N/w)\frac{\mathrm{d}w}{w^2}\Big\}$$

$$= O\Big\{\sum_{k=1}^{\infty}\int_k^{k+1} N^\alpha q_\alpha(N/w)\frac{\mathrm{d}w}{w^2}\Big\}$$

$$= O\Big\{\sum_{k=1}^{\infty} k^\alpha \int_k^{k+1} q_\alpha(k/w)\frac{\mathrm{d}w}{w^2}\Big\} \quad \text{as} \quad k < w < k+1$$

$$= O\Big\{\sum_{k=1}^{\infty}\frac{k^\alpha}{k}\int_{1-(1/k+1)}^{1} q_\alpha(\theta)\mathrm{d}\theta\Big\}$$

$$= O\Big\{\sum_{k=1}^{\infty}\frac{Q(1/k+1)}{k^{1-\alpha}}\Big\}$$

$$= O\Big\{\sum_{k=1}^{\infty}\frac{Q(1/k)}{k^{1-\alpha}}\Big\}$$

$$= O(1) \text{ by Lemma 4.} \tag{20}$$

Now (19) and (20) together imply (12) and the theorem is proved for $O < \alpha < 1$.

Case II. For $\alpha \geq 1$.

Now

$$nB_n(x) = -\frac{2}{\pi}\int_0^{\pi} \psi(t)\frac{\mathrm{d}}{\mathrm{d}t}\cos nt\,\mathrm{d}t$$

$$=\frac{2}{\pi}\left[\sum_{k=1}^{h}(-1)^{k}\Psi_{k}(t)\left(\frac{d}{dt}\right)^{k}\cos nt\right]_{0}^{\pi}$$
$$+\frac{2}{\pi}(-1)^{h+1}\int_{0}^{\pi}\Psi_{h}(t)\left(\frac{d}{dt}\right)^{h+1}\cos nt\,dt.$$

By the use of Lemma 5,

$$\int_{0}^{\pi}\Psi_{h}(t)\left(\frac{d}{dt}\right)^{h+1}\cos nt\,dt$$
$$=\frac{1}{\Gamma(1+h-\alpha)}\int_{0}^{\pi}\left(\frac{d}{dt}\right)^{h+1}\cos nt\,dt\int_{0}^{t}(t-u)^{h-\alpha}d\Psi_{\alpha}(x)$$
$$=\frac{1}{\Gamma(1+h-\alpha)}\int_{0}^{\pi}d\Psi_{\alpha}(u)\int_{u}^{\pi}(t-u)^{h-\alpha}\left(\frac{d}{dt}\right)^{h+1}\cos nt\,dt.$$

Hence

$$nB_{n}(x)=\frac{2}{\pi}\left[\sum_{k=1}^{h}(-1)^{k}\Psi_{k}(t)\left(\frac{d}{dt}\right)^{k}\cos nt\right]_{0}^{\pi}$$
$$+\frac{2}{\pi\Gamma(1+h-\alpha)}(-1)^{h+1}\int_{0}^{\pi}d\Psi_{\alpha}(u)\int_{u}^{\pi}(t-u)^{h-\alpha}\left(\frac{d}{dt}\right)^{h+1}\cos nt\,dt.$$

Since for $k=1,2,\ldots,h$, $\Psi_{k}(+O)=O$, $\Psi_{k}(u)=0(1)$ and $\int_{0}^{\pi}\frac{|d\Psi_{\alpha}(u)|}{u^{\alpha}}<\infty$, for the proof of theorem it is enough to show that (a)

$$\int_{1}^{\infty}\frac{dw}{w^{2}}\left|\sum_{n\le w}q_{\alpha}\left(\frac{n}{w}\right)\left[\left(\frac{d}{dt}\right)^{k}\cos nt\right]_{t=\pi}\right|<\infty,$$

for $k=1,2,3,\ldots,h$. and (b)

$$\int_{1}^{\infty}\frac{dw}{w^{2}}\left|\sum_{n\le w}q_{\alpha}\left(\frac{n}{w}\right)J(n,u)\right|=O\left(\frac{1}{u^{\alpha}}\right).$$

Proof of (a). $k=1,2,3,\ldots,h$.

If k is odd, then $[(\frac{d}{dt})^{k}\cos nt]_{t=\pi}=0$.

If k is even i.e. $k=2m(m=1,2,3,\ldots)$ then

$$\left[\left(\frac{d}{dt}\right)^{k}\cos nt\right]_{t=\pi}=\left[\left(\frac{d}{dt}\right)^{2m}\cos nt\right]_{t=\pi}$$
$$=(-1)^{m}n^{k}\cos n\pi$$
$$=(-1)^{m}(-1)^{n}n^{k}.$$

Hence for k is even and $k\le h-1$

$$\int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w} q_\alpha(n/w)\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k \cos nt\right]_{t=\pi}\right|$$
$$= \int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w} q_\alpha(n/w)(-1)^n n^k\right|$$
$$= \int_1^\infty \frac{\mathrm{d}w}{w^2}O(1), \text{ by Lemma 19}$$
$$= O\left(\int_1^\infty \frac{\mathrm{d}w}{w^2}\right)$$
$$= O(1).$$

Also if h is even then

$$\int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w} q_\alpha\left(\frac{n}{w}\right)\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^h \cos nt\right]_{t=\pi}\right|$$
$$= \int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w} q_\alpha\left(\frac{n}{w}\right)(-1)^n n^h\right|$$
$$= \int_1^\infty \frac{\mathrm{d}w}{w^2}O\left\{\left(q^h\left(1-\frac{1}{w}\right)\right) + O\left(wQ_h\left(\frac{1}{w}\right)\right)\right\}, \text{ by Lemma 18}$$
$$= O\left(\int_1^\infty q^h\left(1-\frac{1}{w}\right)\frac{\mathrm{d}w}{w^2}\right) + O\left(\int_1^\infty Q_h\left(\frac{1}{w}\right)\frac{\mathrm{d}w}{w}\right)$$
$$= O\left(\int_0^1 q^h(t)\mathrm{d}t + O\left(\int_0^1 \frac{Q_h(t)}{t}\mathrm{d}t\right)\right)$$
$$= O(1). \quad \text{(This follows from (vii) and Lemma 3).}$$

Hence the proof of (a) is over.

Proof of (b).

$$\int_1^{2\pi/u} \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w} q_\alpha\left(\frac{n}{w}\right)J(n,u)\right|$$
$$= \int_1^{2\pi/u} \frac{\mathrm{d}w}{w^2}O(w^{\alpha+1}), \text{ by Lemma 10}$$
$$= O\left(\int_1^{2\pi/u} w^{\alpha-1}\mathrm{d}w\right)$$
$$= O(1/u^\alpha). \tag{21}$$

For $O < u \le \pi, wu > 2\pi$

$$\left|\sum_{n\le w} q_\alpha\left(\frac{n}{w}\right)J(n,u)\right|$$
$$= \left|\sum_{n\le w} q_\alpha\left(\frac{n}{w}\right)\int_u^\pi (t-u)^{h-\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1}\cos nt\,\mathrm{d}t\right|$$

$$= \left| \int_u^\pi (t-u)^{h-\alpha} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1} p(w,t)\mathrm{d}t \right|$$

$$= \frac{1}{h!} \left| \int_u^\pi (t-u)^{h-\alpha} \mathrm{d}t \int_1^w (-1)^h \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right) \frac{\mathrm{d}}{\mathrm{d}x} S^{h,h+1}(x,t)\mathrm{d}x \right|,$$

by Lemma 17

$$\leq \frac{1}{h!} \left| \int_u^\pi (t-u)^{h-\alpha} \mathrm{d}t \int_1^{w-\pi/u} (-1)^h \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right) \frac{\mathrm{d}}{\mathrm{d}x} S^{h,h+1}(x,t)\mathrm{d}x \right|$$

$$+ \frac{1}{h!} \left| \int_u^\pi (t-u)^{h-\alpha} \mathrm{d}t \int_{w-\pi/u}^{w} (-1)^h \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right) \frac{\mathrm{d}}{\mathrm{d}x} S^{h,h+1}(x,t)\mathrm{d}x \right|$$

$$= O\left(\frac{w^{\alpha-h} q^h(1-(\pi/wu))}{u^{h+1}}\right) + O\left(\frac{w^{\alpha-h+1} Q_h(\pi/wu)}{u^h}\right)$$

by the use of Lemma 14 and Lemma 16 respectively.

So

$$\int_{2\pi/u}^\infty \frac{\mathrm{d}w}{w^2} \left| \sum_{n\leq w} q_\alpha\left(\frac{n}{w}\right) J(n,u) \right|$$

$$= O\left(\int_{2\pi/u}^\infty \frac{w^{\alpha-h} q^h(1-(\pi/wu))}{u^{h+1}} \frac{\mathrm{d}w}{w^2} \right) + O\left(\int_{2\pi/u}^\infty \frac{w^{\alpha-h+1} Q_h(\pi/wu)}{u^h} \frac{\mathrm{d}w}{w^2} \right)$$

$$= O\left(\frac{1}{u^\alpha} \int_0^1 \frac{q^h(t)}{(1-t)^{\alpha-h}} \mathrm{d}t \right) + O\left(\frac{1}{u^\alpha} \int_0^1 \frac{Q_h(t)}{t^{1+\alpha-h}} \mathrm{d}t \right)$$

$$= O\left(\frac{1}{u^\alpha}\right), \text{ by Lemma 3 and (vii).} \tag{22}$$

Now (21) and (22) together imply

$$\int_1^\infty \frac{\mathrm{d}w}{w^2} \left| \sum_{n\leq w} q_\alpha\left(\frac{n}{w}\right) J(n,u) \right| = O\left(\frac{1}{u^\alpha}\right).$$

This completes the proof of (b) and hence the proof of the theorem is over.

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