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Rational curves on moduli spaces of vector bundles

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Abstract. We identify the spaces $\mathrm{Hom}_i(\mathbb{P}^1, M)$ for $i = 1, 2$, where M is the moduli space of vector bundles of rank 2 and determinant isomorphic to $\mathcal{O}(x_0), x_0 \in X$, on a compact Riemann surface of genus $g \geq 2$.

Keywords. Riemann surfaces; determinant bundles; Hilbert scheme; jumping divisor.

1. Introduction

Let X be a compact Riemann surface of genus $g \geq 2$ and M be the moduli space of stable vector bundles of rank 2 and fixed determinant on X. We study the structure of the space $\mathrm{Hom}_i(\mathbb{P}^1, M)$, $i = 1, 2$ of maps of degree one and two from $\mathbb{P}^1$ into M. This study draws its origin from attempts to compute the quantum cohomology of the moduli space M which has recently become an important topic for research (see for example [VM] and [D]).

We now outline the main results of the paper. Consider the natural vector bundle of extensions on the Picard variety $\mathrm{Pic}^0(X)$. The content of Theorem 8 and the discussion following it is that the space $\mathrm{Hom}_1(\mathbb{P}^1, M)$ is the total space of a principal $PGL(2)$-bundle on the Grassmannization of this bundle. In the case of degree two maps, one of the components of the scheme $\mathrm{Hom}_2(\mathbb{P}^1, M)$ is shown to be naturally related to the moduli space of stable bundles of rank two and determinant isomorphic to $\mathcal{O}(x - x_0), x \in X$ (Theorem 9). It is only this component which will figure in the definition of the quantum cohomology of M.

The paper is laid out as follows § 2 lists notation and terminology for subsequent use. In § 3 we collect together various lemmas of technical nature on the determinant of cohomology of a family of vector bundles. Section 4 consists of constructions related to elementary transformations of vector bundles along divisors. These form the main technical framework of the paper. The next two sections deal with the study of maps of degree one and two respectively.

2. Notation

- If E is a vector bundle on $X \times Y$, we denote by $E_x (x \in X)$ the bundle over Y gotten by restricting E to $\{x\} \times Y$.
- If E is a rank n bundle on $X, \lambda(F) := \Lambda^n F$.
- If X and Y are varieties then p and q denote the projections of $X \times Y$ on X and Y respectively.

- $\mathcal{SU}(n, d)$ denotes the moduli space of stable bundles of rank n and degree d.
- For Y smooth, T_Y denotes the tangent bundle of Y and K_Y denotes λT_Y^*.
- $\mathrm{Pic}(X)$ denotes the Picard group of X.
- If $x \in X$, $k(x)$ stands for the torsion sheaf of height 1 supported at x.

3. Preliminaries

Let X be a compact Riemann surface of genus $g, g \geq 2$. Fix a point $x_0 \in X$. Let $M := \mathcal{SU}(2, \mathcal{O}(x_0))$ denote the moduli space of stable bundles on X of rank 2 with determinant isomorphic to $\mathcal{O}(x_0)$. It is well-known that there are Poincaré families on $X \times M$. It is also known that $\mathrm{Pic}(M)$ is $\mathbb{Z}$ and let u denote the ample generator of $\mathrm{Pic}(M)$.

By [R], there is a unique Poincaré bundle E on $X \times M$ with the property that $\det E_{x_0}$ is isomorphic to the line bundle u.

We call such an E *the rigidified Poincaré bundle* and note that $\det E_x$ is independent of $x \in X$.

In the following we recall the definition of the determinant line bundle on M associated to E and a couple of lemmas of technical nature.

Let $\mathcal{F} = \{\mathcal{F}_t\}_{t \in T}$ be a family of vector bundles on X parametrised by a complex manifold T.

If $q : X \times T \to T$ is the natural projection, $R^i q * \mathcal{F}$ (f or $i = 0, 1$) is given globally on T by the cohomology of a complex $j : K^0 \to K^1$ where K^i are locally free and coherent.

DEFINITION 1.

The line bundle $D(T, \mathcal{F})$ on T, called the determinant of cohomology of $\mathcal{F}$ with respect to T, is defined as $(\lambda K^0)^{-1} \otimes (\lambda K^1)$ where λ is the highest exterior power.

Since $\mathrm{Pic}(M) \simeq Z$, $D(M, E) \simeq u^l$ for some $l \in \mathbb{Z}$. It can be shown that ([R] proof of theorem 1) $D(M, E \otimes p^* V) \simeq u$ where V is any rank 2 degree $2g - 3$ bundle on X.

Remark. The Poincaré bundle E has the property that $D(M, E \otimes q^* V) \simeq \det E_{x_0}$ where V is as above.

Suppose $\mathcal{F}$ is a bundle on $X \times T$, where T is a complex manifold, such that $\mathcal{F} \simeq p^* V \otimes q^* H$ where V is a vector bundle on X, H is a vector bundle on T. Then $D(T, \mathcal{F})$ can be expressed in terms of V and H.

Lemma 2. $D(T, \mathcal{F}) \simeq (\det H)^{-\chi(V)}$.

Proof. By the projection formula, for $i = 0, 1$

$$R^i q_* (p^* V \otimes q^* H) = R^i q_* (p^* V) \otimes H,$$

hence, $D(T, \mathcal{F}) \simeq \det H)^{-\chi(V)}$.

DEFINITION 3.

A holomorphic map $\psi : \mathbb{P}^1 \to M$ is said to be of degree d if $D(\mathbb{P}^1, (id \times \psi)^* \ (E \otimes p^* V))$ is isomorphic to $\mathcal{O}(d)$, where V is a rank 2 degree $2g - 3$ bundle on X.

Remark. Note that E on $X \times M$ is rigidified implies that $D(M, E \otimes p^* V) \simeq \lambda(E_x)$. Now the fact that the determinant of cohomology commutes with base change implies that we have an isomorphism:

$$D(\mathbb{P}^1, (id_X \times \psi)^*(E \otimes p^*V)) \simeq \lambda(\psi^* E_x).$$

Lemma 4. *$\psi : \mathbb{P}^1 \to M$ is a holomorphic map of degree d if and only if* $\deg D(\mathbb{P}^1, (id \times \psi)^* E) = (g-1)d$.

Proof. Let $q : X \times \mathbb{P}^1 \to \mathbb{P}^1$ be the natural projection and let $\mathrm{d}(E)$ denote $D(\mathbb{P}^1, E)$. Let F and G be holomorphic vector bundles on $X \times \mathbb{P}^1$ of ranks r and n respectively. Then

$$(*) \cdot \mathrm{d}(F \otimes G) \cdot \mathrm{d}(F)^{-n} \cdot \mathrm{d}(G)^{-r} = \cdot \mathrm{d}(\lambda(F)\lambda(G)) \cdot \mathrm{d}(\lambda(F))^{-1} \cdot \mathrm{d}(\lambda(G))^{-1}$$

in $\mathrm{Pic}(\mathbb{P}^1)$. The proof of this can be deduced from the Grothendieck–Riemann–Roch theorem (see [B-R] Lemma 4.12). In the above formula, set $F = (id \times \psi)^* E$ and $G = p^* V$, where V is a rank 2 degree $2g-3$ bundle on X.

$\Rightarrow$: We first note that $\lambda(F) \simeq p^* \mathcal{O}(x_0) \otimes q^* \mathcal{O}(d)$ and that $\lambda(G) \simeq p^* L$, where L is a line bundle of degree $2g-3$. Substituting these in $(*)$ and using lemma 2, we easily see that $\deg \cdot \mathrm{d}(F) = \mathrm{d}(g-1)$.

$\Leftarrow$: Let $\mathrm{d}(F \otimes G)$ be $\mathcal{O}(r)$, for some $r \in \mathbb{Z}$. From the remark above, we have $\lambda(F) \simeq p^*(x_0) \otimes q^* \mathcal{O}(r)$. We can now use $(*)$ to evaluate r to be d.

We next need a result concerning the H–N filtration of vector bundles. We need to know that the H–N filtration 'globalizes' correctly in a family of vector bundles. For a proof of this lemma, see [L] proposition 11.1.2. This, in our case, can also be proved in an elementary fashion by slightly modifying the proof of Prop. 15.4 of [A–B].

Lemma 5. *Let $\mathcal{E}$ be a holomorphic bundle on $X \times C$ where C is a compact Riemann surface. Suppose $\mathcal{E}_x$ has the same H–N type for every $x \in X$. Then there is a holomorphic filtration of the bundle $\mathcal{E}$ by subbundles over $X \times C$ such that it induces the H–N filtration on each $\mathcal{E}_x$, $x \in X$.*

4. The jumping divisor

Let X be a compact Riemann surface (in this section alone we will assume it to be of arbitrary genus) and F be a rank 2 holomorphic bundle on $X \times \mathbb{P}^1$ such that $\det F_x$ is isomorphic to $\mathcal{O}(d)$ for all $x \in X$ and for some fixed $d > 0$.

A theorem of Grothendieck implies that $F_x \simeq \mathcal{O}(l_x) \oplus \mathcal{O}(-l_x + d)$ where we assume $l_x \geq -l_x + d$. Since $d > 0, l_x > 0$. Let $l = \min\{l_x / x \in X\}$. Then $h^0(\mathbb{P}^1, F_x \otimes \mathcal{O}(-l-1)) = 0$ at some $x \in X$. By the theorem of semicontinuity, the set U of all points $x' \in X$ such that $h^0(\mathbb{P}^1, F_{x'} \otimes \mathcal{O}(-l-1)) = 0$, is open. This in turn implies that for all $y \in U$, $F_y \otimes \mathcal{O}(-l-1) \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-2l + d - 1)$. Hence U is the largest open set such that for $x \in U$, $F_x \simeq \mathcal{O}(l) \oplus \mathcal{O}(-l + d)$.

Let the points in $X - U$ be $x_1, x_2, \ldots, x_n$. Clearly $F_{x_i} \simeq \mathcal{O}(l_{x_i}) \oplus \mathcal{O}(-l_{x_i} + d)$ where $l_{x_i} > l$.

The local moduli of the bundle F_{x_i} is the germ of the vector space $H^1(\mathbb{P}^1 \ \mathrm{End}\ F_{x_i})$ at the origin, and the differential of the classifying map from analytic open sets V_{x_i} of x_i, to the local moduli of F_{x_i} is the Kodaira–Spencer map at x_i for the family F, when thought of as a family of vector bundles on $\mathbb{P}^1$ parametrised by X. Fixing a basis for the vector space $H^1(\mathbb{P}^1, \mathrm{End}\ F_{x_i})$, the classifying map is given by a coordinate-wise power series. Let $m_{i,1} \cdots m_{i,r}$ be non-negative integers such that $m_{i,j}$ is the first non-zero coefficient in the jth coordinate of this power series. Define $m_i \doteq \min\{m_{i,1}, m_{i,2} \cdots m_{i,r}\}$.

DEFINITION 6.

The jumping divisor of F on X is defined as the divisor $D = \sum_{i=1}^{n} m_i x_i$.

Lemma 7. *There exists a bundle F' on $X \times \mathbb{P}^1$ along with a map $F' \to F$ which is an isomorphism outside the divisor D, and such that jumping divisor of F' on X is the zero divisor.*

Proof. Let $D = \sum_{i=1}^{r} m_i x_i$ be the jumping divisor of F. That is, $F_{x_i} \simeq \mathcal{O}(l_i^0) \oplus \mathcal{O}(-l_i^0 + d)$ and $l_i^0 > 1$.

Consider the vector space $\mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{O}(l_i^0), \mathcal{O}(-l_i^0 + d))$ and the universal extension

$$0 \to p^*\mathcal{O}(-l_i^0 + d) \to \mathcal{V}_i \to p^*\mathcal{O}(l_i^0) \to 0 \tag{1}$$

on $\mathrm{Ext}^1 \times \mathbb{P}^1$. It is not hard to see that $\mathcal{V}_i$ is a locally complete family of deformations of F_{x_i}. (In fact, the Kodaira Spencer map for $\mathcal{V}_i$ at the origin of Ext^1 is the natural identification of Ext^1 with $H^1(\mathbb{P}^1, \mathrm{End}\, F_{x_i})$). Hence there are analytic open sets U_i of x_i and local classifying maps ψ_i from U_i to Ext^1, such that $(id \times \psi_i)^*\mathcal{V}_i \simeq F|_{U_i \times \mathbb{P}^1}$. Denote by D_i the scheme Spec $\mathcal{O}_{X,x_i}/\mathcal{M}_{x_i}^{m_i}$ and by τ_i the torsion sheaf $\mathcal{O}_{x_i}/\mathcal{M}_{x_i}^{m_i}$ on X. Since V_i is split over $\{0\} \times \mathbb{P}^1, \psi_i^*(1)$ is split on $D_i \times \mathbb{P}^1$. Let $\varrho_i : F|_{D_i \times \mathbb{P}^1} \to q^*\mathcal{O}(-l_i^0 + d) \to 0$ be one such splitting. Let τ denote the torsion sheaf

$$p^*\tau_1 \otimes q^*\mathcal{O}(-l_{x_1}^0 + d) \oplus p^*\tau_2 \otimes q^*\mathcal{O}(-l_{x_2}^0 + d) \oplus \cdots \oplus p^*\tau_r \otimes q^*\mathcal{O}(-l_{x_r}^0 + d).$$

Let ϕ be the surjection

$$F \to \tau \to 0$$

of $\mathcal{O}_{X \times \mathbb{P}^1}$ modules obtained by extending $(\varrho_1, \varrho_2, \ldots, \varrho_r)$ by zero outside the divisor $D \times \mathbb{P}^1$. Set $F_1 := \ker \phi$. Then it can be checked that F_1 is a locally free sheaf on $X \times \mathbb{P}^1$. Restricting the short exact sequence

$$0 \to F_1 \to F \to \tau \to 0$$

to $\{x_i\} \times \mathbb{P}^1$, we have following exact sequence

$$\begin{array}{ccccccccc} 0 \to \mathcal{O}(-l_i^0 + d) \to & F_{1_{x_i}} & & \to & & \mathcal{O}(l_i^0) \oplus \mathcal{O}(-l_i^0 + d) \to \mathcal{O}(-l_i^0 + d) \to 0 \\ & & \searrow & & \nearrow & & \\ & & & \mathcal{O}(l_i^0) & & & \\ & & \nearrow & & \searrow & & \\ & 0 & & & & 0 & \end{array}$$

Claim: $F_{1_{x_i}}$ is *non-split* extension of $\mathcal{O}(l_i^0)$ by $\mathcal{O}(-l_i^0 + d)$. (See [RF] pages 132–134).

Write $F_{1_{x_i}} \simeq \mathcal{O}(l_i^1) \oplus \mathcal{O}(-l_i^1 + d)$

We have non-split exact sequence of bundles

$$0 \to \mathcal{O}(-l_i^0 + d) \to \mathcal{O}(l_i^1) \oplus \mathcal{O}(-l_i^1 + d) \to \mathcal{O}(-l_i^0) \to 0$$

This forces $l_i^1 < l_i^0$.

The jumping divisor for F_1 be $D_1 = \sum_{i=1}^{r} m_i^1 x_i$. Then applying the above process successively, after a finite number of steps we get a bundle $F' := F_n$ such that the jumping divisor of F' is the zero divisor.

5. Degree one maps[1]

In this section we study degree 1 maps from $\mathbb{P}^1$ to M and identify the space of degree 1 maps as the $PGL(2)$ bundle over G, where G is a Grassmann bundle over $\mathrm{Pic}^0(X)$.

Let $L \in \mathrm{Pic}^0(X)$, we know that a non-split extension of $L^{-1}(x_0)$ by L is a stable bundle and that any two extensions which differ by a non-zero scalar are isomorphic. The family of non-split extensions of $L^{-1}(x_0)$ by L modulo non-zero scalars is parametrized by $\mathbb{P}(H^1(X, \mathrm{Hom}(L^{-1}(x_0), L)))$. For $L \in \mathrm{Pic}^0(X)$, we denote this projective space by $\mathbb{P}_{\mathcal{L}}$. By [R], the induced map from $\mathbb{P}_L$ to M induces an isomorphism from $\mathrm{Pic}(M)$ to $\mathrm{Pic}(\mathbb{P}_L)$. There is a universal family of extensions on $X \times \mathbb{P}_L$:

$$0 \to p^*L \otimes q^*\mathcal{O}(1) \to V \to p^*L^{-1}(x_0) \to 0. \qquad (\dagger)$$

It is also clear that V is the pull back of the rigidified Poincaré bundle.

Let $\mathbb{P}^1$ to $\mathbb{P}_L$ be a linear embedding. Restricting the exact sequence $(\dagger)$ to $\mathbb{P}^1$, we have a rank 2 family on $X \times \mathbb{P}^1$. Therefore it induces a map from $\mathbb{P}^1$ to M. By a simple calculation of the determinant of cohomology of V, we can conclude that this map from $\mathbb{P}^1$ to M induced by the family V is of degree 1. We shall prove that every degree 1 map is of this form:

Theorem 8. *Let $\psi : \mathbb{P}^1 \to M$ be a degree one map. Then it factors as $\mathbb{P}^1 \to \mathbb{P}_L \to M$ for some $L \in \mathrm{Pic}^0(X)$ where $\mathbb{P}^1$ to $\mathbb{P}_L$ is a linear embedding.*

Proof. On $X \times \mathbb{P}^1$ set $F = (id \times \psi)^*E$.

Claim: The jumping divisor of F is the zero divisor.

Suppose that the claim is true. Then $F_p \simeq \mathcal{O}(l) \oplus \mathcal{O}(-l+1)$ and $l > 0$.

By lemma 5, there is a line sub-bundle $\mathcal{L}$ of F. Since $\mathcal{L}_x \simeq \mathcal{O}(l)$ for all $x \in X$, $\mathcal{L} \simeq p^*L \otimes q^*\mathcal{O}(l)$ where L is a line bundle on X of degree, say d. Hence there is an exact sequence on $X \times \mathbb{P}^1$:

$$0 \to p^*L \otimes q^*\mathcal{O}(l) \to F \to p^*L^{-1}(x_0) \otimes q^*\mathcal{O}(-l+1) \to 0.$$

If we take the determinant of cohomology of F, then

$$D(\mathbb{P}^1, F) \simeq D(\mathbb{P}^1, p^*L \otimes q^*\mathcal{O}(l)) \otimes D(\mathbb{P}^1, p^*L^{-1}(x_0) \otimes q^*\mathcal{O}(-l+1))$$

Equating the degrees of the line bundles in the above isomorphism, we have

$$g - 1 = l(g - d - 1) + (l - 1)(2 - g - d).$$

This implies $-2dl + d + l = 1$. Since l is a positive integer and d is an integer, we have $d = 0$ and $l = 1$.

It only remains to prove the claim. Let the jumping divisor of F be $\sum_{i=1}^{n} m_i x_i$ such that

$$F_{x_i} \simeq \mathcal{O}(l_i) \oplus (-l_i + 1).$$

Applying lemma 7, we have to locally free sheaf F' on $X \times \mathbb{P}^1$ with the property that $F'_x \simeq \mathcal{O}(l) \oplus \mathcal{O}(-l+1)$ for all $x \in X$. Then, from lemma 5 we have:

$$0 \to p^*L_1 \otimes q^*\mathcal{O}(l) \to F' \to p^*L_2 \otimes q^*\mathcal{O}(-l+1) \to 0$$

[1] We came across the preprint [VM] after this work was done. Although the results of this section are also found in § 3 of [VM], proof techniques are quite different.

If the degree of L_1 is d, then the degree of L_2 is of degree $1-d-\sum_j\sum_i m_i^j$ since the degree of F'_p is $1-\sum_j\sum_i m_i^j$.

Therefore $D(\mathbb{P}^1,F')\simeq \mathcal{O}(l)^{-\chi(L_1)}\otimes\mathcal{O}(-l+1)^{-\chi(L_2)}$.

Hence the degree of the bundle $D(\mathbb{P}^1,F')$ is equal to

$$l+d+g-2-2dl-\sum_j\sum_i m_i^j l+\sum_j\sum_i m_i^j. \tag{1}$$

On the other hand, we can compute the degree of the determinantal bundle of F' by going back to the definition of F' (cf. lemma 7). It is

$$g-1-\sum_i\sum_i m_i^j(l_i^j-1). \tag{2}$$

Equating (1) and (2) we have:

$$l+d-2dl+\sum_j\sum_i m_i^j(l_i^j-l)-1=0.$$

But since $l_i^j>l$, the above equation can be written as

$$l-2dl+d-1<0.$$

Hence $d>1-l/(1-2l)$, as $l>0$.

Since F'_p is semi-stable of degree $1-\sum_j\sum_i m_i^j$ and L_1 is a sub-bundle of degree $d\geq 1$, this is a contradiction to the semi-stability of F'_p. This proves the claim.

Let us summarise what we have done in terms of the moduli of degree one maps from $\mathbb{P}^1$ to M. First note that if $f:\mathbb{P}^1\to M$ is a degree 1 map and $\phi\in\mathrm{Aut}(\mathbb{P}^1)$, then $fo\phi$ is again a degree 1 map. Let $\mathcal{L}$ be the rigidified Poincaré bundle on $X\times\mathrm{Pic}^0(X)$ rigidified at x_0. Then $R^1q*(\mathcal{L}^2(-x_0))$ is a locally free sheaf on $\mathrm{Pic}^0(X)$ of rank g. Take the associated projective bundle and call it P. Let G be the Grassmann bundle of lines associated to the projective bundle P over $\mathrm{Pic}^0(X)$. Then G comes with a natural $\mathbb{P}^1$ bundle over it. Denote by S the total space of the natural principal $PGL(2)$ bundle associated to this. We have shown *set-theoretically* that this is the space of degree 1 maps. A simple calculation shows that the dimension of S is $3g-1$.

Let $\mathrm{Hom}_d(\mathbb{P}^1,M)$ denote the set of degree d morphisms from $\mathbb{P}^1$ to M. It can be seen to be an open subscheme of $\mathrm{Hil}\,b_P(\mathbb{P}^1\times M)$, where the polynomial P is the Hilbert polynomial of the graph of a degree d map with respect to the polarisation $\mathcal{O}(1)\otimes u$ on $\mathbb{P}^1\times M$ (see [K]).

The expected dimension of $\mathrm{Hom}_d(\mathbb{P}^1,M)$ at f equals $\deg(f^*T_M)+\dim M$ and it is attained if $H^1(\mathbb{P}^1,f^*T_M)=0$. See [K] for details. We shall now show that $\mathrm{Hom}_1(\mathbb{P}^1,M)$ is smooth of dimension $=-\deg(f^*K_M)+3g-3=3g-1$ (since one knows that $-K_M\simeq u^2$ on M (see [R]) and $f^*u\simeq\mathcal{O}_{\mathbb{P}^1}(1)$).

To compute $h^1(\mathbb{P}^1,f^*T_M)$, first of all note that

$$T_M\simeq R^1q*(adE)$$

where adE is the bundle of traceless endomorphisms of the rigidified Poincaré bundle on $X\times M$. So if $F=(1\times f)^*E$, $f^*T_M\simeq R^1q*(adF)$. So we want to compute $h^1(\mathbb{P}^1,R^1q_*adF)$. But since $R^iq_*adF=0\,\forall i\neq 1$, the Leray spectral sequence associated to the projection p degenerates at E_2 and we have

$$H^1(\mathbb{P}^1,R^1p_*adF)\simeq H^2(X\times\mathbb{P}^1,adF).$$

Consider now R^ip_*adF. Since $F_x \simeq \mathcal{O} \oplus \mathcal{O}(1) \forall x \in X$, we see easily that $R^ip_*adF = 0 \forall i \geq 1$. Hence by the Leray spectral sequence for the projection p, we have:

$$H^2(X \times \mathbb{P}^1, \ adF) \simeq H^2(X, q_*(adF)) = 0.$$

Since the family of degree 1 maps parametrised by the space S is injective, the classifying map from S to $\mathrm{Hom}_1(\mathbb{P}^1, M)$ is bijective. But as the dimension of S equals the dimension of $\mathrm{Hom}_1(\mathbb{P}^1, M)$, it is an isomorphism.

When the genus of X is 2, the work [N-R 1] shows that the space of lines in the moduli space can be identified with $\mathrm{Pic}^0(X)$ which is the $PGL(2)$–quotient of $\mathrm{Hom}_2(\mathbb{P}^1, M)$.

6. Degree two maps

In this section we shall study degree 2 maps from $\mathbb{P}^1$ to M and identify a component of the $\mathrm{Hom}_2(\mathbb{P}^1, M)$ as the variety $\mathcal{SU}(2, \mathcal{O}(x - x_0))x \in X$.

If $F = (1 \times f)^*E$ and $f \in \mathrm{Hom}_2(\mathbb{P}^1, M)$, then Case (1): $F_x \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$ at all but finitely many $x \in X$, Case (2): $F_x \simeq \mathcal{O}(l) \oplus \mathcal{O}(-l+2)$ (where $l > 1$) at all but finitely many $x \in X$.

Case 1.

Let V be a rank 2 stable bundle on X with determinant isomorphic to $\mathcal{O}(x - x_0)$. Consider the following tautological surjection of sheaves of modules on $X \times \mathbb{P}(V_x^*)$ for some $x \in X$:

$$p^*V^* \otimes q^*\mathcal{O}(-1) \rightarrow p^*k(x) \rightarrow 0$$

Let V_1^* denote the kernel of the above map. We write V_1 for the dual of V_1^*. Then V_1 is a locally free sheaf on $X \times \mathbb{P}^1$. It is easy to see that V_{1_t} $(\forall t \in \mathbb{P}^1)$ is stable of degree 1 and that $D(\mathbb{P}^1, V_1 \otimes p^*\dot{F}) \simeq \mathcal{O}(2)$, where F' is any rank 2 degree $2g - 3$ bundle on X.

Therefore, we have a degree 2 map $\psi : \mathbb{P}^1 \rightarrow M$ given by the family V. We shall prove in the following theorem that every degree 2 map which falls under case (1) is of this form.

Theorem 9. *Let $\psi : \mathbb{P}^1 \rightarrow M$ be a degree* 2 *map such that $(id \times \psi)^*E_{x \times \mathbb{P}^1}$ is generically isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Then there is a stable bundle V on X of rank* 2 *and determinant $\simeq \mathcal{O}(x - x_0)$ for some $x \in X$ such that*

$$0 \rightarrow p^*V \otimes q^*\mathcal{O}(1) \rightarrow (id \times \psi)^*E \rightarrow p^*k(x) \rightarrow 0$$

Proof.

Write F for $(id \times \psi)^*E$ and let the jumping divisor of F be D.

Claim: $D = (x)$ for some $x \in X$ and $F_x \simeq \mathcal{O}(2) \oplus \mathcal{O}$.

Suppose that the claim is true. Consider the following map of sheaves of $\mathcal{O}_{X \times \mathbb{P}^1}$ modules

$$F \rightarrow p^*k(x) \rightarrow 0.$$

Let F' denote the kernel. Then F' is locally free and $F'_x \simeq \mathcal{O}(1) \oplus \mathcal{O}(1) \forall x \in X$. So $F' \simeq p^*V \otimes q^*\mathcal{O}(1)$.

Clearly $\det(V) \simeq \mathcal{O}(x - x_0)$. It can be seen easily that V is semi-stable. We prove that V is in fact stable.

Suppose that V is semi-stable but not stable. There exists a line sub-bundle L of V of degree 0. Then $p^*L \otimes q^*\mathcal{O}(1)$ is a sub-bundle of F' and if the image sheaf of $p^*L \otimes q^*\mathcal{O}(1)$ in F is a sub-bundle then, we have an exact sequence of vector bundles on $X \times \mathbb{P}^1$:

$$0 \to p^*L \otimes q^*\mathcal{O}(1) \to F \to p^*L^{-1}(x_0) \otimes q^*\mathcal{O}(1) \to 0 \tag{1}$$

Restricting the exact sequence (1) to $\{x\} \times \mathbb{P}^1$, we have

$$0 \to \mathcal{O}(1) \to \mathcal{O}(2) \oplus \mathcal{O} \to \mathcal{O}(1) \to 0$$

which is impossible. That is, the sheaf inclusion of $p^*L \otimes \mathcal{O}(1)$ in F goes down in rank at some point $(x,p) \in X \times \mathbb{P}^1$. If L' denotes the sub-bundle of F_p generated by the image of L in F_p, then the degree of L' is strictly bigger than zero. This contradicts the stability of F_p. Hence V is *stable* of degree 0.

To prove the claim, let the jumping divisor of F be $D = \sum_{i=1}^n m_i x_i$ with $F_{x_i} \simeq \mathcal{O}(l_i) \oplus \mathcal{O}(-l_i+2)$. Notice that D is not the zero divisor. Applying lemma 7, we have as in the lemma, a bundle F' with $F'_x \simeq \mathcal{O}(1) \oplus \mathcal{O}(1) \forall x \in X$.

Degree of F'_p is $1 - \sum_j \sum_i m_i^j$. Hence it can be seen that $F' \simeq p^*V \otimes q^*\mathcal{O}(1)$ for some vector bundle V on X of degree $1 - \sum_j \sum_i m_i^j$. From this we can compute the degree of $D(\mathbb{P}^1, F')$. It is equal to

$$2g - 3 + \sum_j \sum_i m_i^j \tag{2}$$

On the other hand, we can compute the degree of $D(\mathbb{P}^1, F')$ by going back to the definition of F' (cf. proof of lemma 7). It comes out to be

$$2g - 2 + \sum_j \sum_i m_i^j (2 - l_i^j) \tag{3}$$

Equating (2) and (3), we have

$$\sum_j \sum_i m_i^j (1 - l_i^j) + 1 = 0.$$

Since we know that $l_i^j \geq 2$ and $m_i^j \geq 1$, the above equality is true only when $m_i = 0$ for $i \geq 2$ and $l_1^0 = 2$.

Hence the theorem.

Let M' denote the inverse image of the copy of X in $\mathrm{Pic}^0(X)$ (defined by sending x to $\mathcal{O}(x - x_0)$ under the determinant map

$$\Lambda : \mathcal{SU}(2,0) \to \mathrm{Pic}^0(X)$$

where $\mathcal{SU}(2,0)$ stands for the moduli space of stable bundles on X of rank 2 and degree 0. Note that M' is smooth of dimension $3g - 2$. Let Q denote the natural $PGL(2)$-principal bundle on $X \times M'$ associated to the projective Poincaré bundle on $X \times M'$. Consider the map $(id \times \Lambda) : X \times M' \to X \times X$. Note that the natural projection of $(id \times \Lambda)^{-1}(\Delta)$ on M' is an isomorphism. Let S_1 stand for the pull-back of $Q|(id \times \Lambda^{-1}(\Delta))$ to M' under the inverse of this isomorphism.

On the other hand, if $f \in \mathrm{Hom}_2(\mathbb{P}^1, M)$ is as in case (1), then a spectral sequence argument (along the lines of discussion which follows the proof of theorem 8) shows that the scheme $\mathrm{Hom}_2(\mathbb{P}^1, M)$ is smooth of dimension $3g + 1$ at f. It we now note that S_1 is *injectively* parameterized, we get an injective map from S_1 to $\mathrm{Hom}_2(\mathbb{P}^1, M)$ whose image consists of points which come under case (1). This implies that such points form an irreducible component of the scheme $\mathrm{Hom}_2(\mathbb{P}^1, M)$ isomorphic to S_1. This is closely related to the work [N-R 2]. We shall consider this connection in a later work.

Case 2.

Theorem 10. *Let* $\psi : \mathbb{P}^1 \to M$ *be a degree* 2 *map such that* $(Id \times \psi)^* E$ *is generically* $\mathcal{O}(l) \oplus \mathcal{O}(-l+2)$ *for some* $l > 1$. *Then the bundle* $(1 \times \psi)^* E$ *can be expressed as*

$$0 \to p^* L \otimes q^* \mathcal{O}(2) \to (1 \times \psi)^* E \to p^* L^{-1}(x_0) \to 0$$

for some line bundle L *on* X *of degree* 0.

Proof. We denote by F the bundle $(id \times \psi)^* E$.

Arguing along the lines of the proofs of lemmas 8 and 9, we can conclude that the jumping divisor of F is the zero divisor.

Hence $F_x \simeq \mathcal{O}(l) \oplus \mathcal{O}(-l+2)$ for every $x \in X$. Applying the lemma 7 we have an exact sequence of vector bundles

$$0 \to L' \to F \to L'' \to 0.$$

We know that $L'_x \simeq \mathcal{O}(2) \forall x \in X$. Hence we can conclude that $L' \simeq p^* L \otimes q^* \mathcal{O}(2)$ for some line bundle L on X.

Similarly we can conclude that $L'' \simeq p^* L^{-1}(x_0)$.

Hence, rewriting the above exact sequence, we have:

$$0 \to p^* L \otimes q^* \mathcal{O}(2) \to F \to p^* L^{-1}(x_0) \to 0.$$

Now arguing again as in the proof of theorem 8, it can be shown that degree of L is 0. Hence the theorem.

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