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Immersions in a symplectic manifold

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Abstract. In this paper we give a homotopy classification of symplectic isometric immersions following Gromov's h-principle theorem.

Keywords. Symplectic immersions; h-principle.

1. Introduction

Let (N, σ) be a smooth symplectic manifold and M a manifold with a closed C^∞ 2-form ω on it. A smooth immersion $f: (M, \omega) \to (N, \sigma)$ is called *symplectic isometric* (or simply *symplectic*) if f pulls back σ onto ω. The differential of f gives rise to a bundle monomorphism $F: TM \to TN$ such that $F^*\sigma = \omega$. Moreover, the underlying map $M \to N$ of F, which is f itself here, pulls back the de Rham cohomology class of σ onto that of ω. So a natural question would be if the existence of such a bundle map ensures the existence of a symplectic immersion. Gromov proves that when dim $M <$ dim N then the question has an affirmative answer. We prove here further that the space of symplectic immersions is the same up to homotopy as the space of those bundle maps. Let Symp (M, N) denote the space of symplectic immersions of M into N with C^∞ compact-open topology and $\mathrm{Symp}_0\,(TM, TN)$ denote the associated space of bundle monomorphisms $F: TM \to TN$ with C^0 compact-open topology; i.e. F satisfies $F^*\sigma = \omega$ and its underlying (continuous) map f satisfies the cohomology condition $f^*[\sigma] = [\omega]$, where $[\sigma]$ and $[\omega]$ denote respectively the de Rham cohomology classes of σ and ω.

The main theorem may now be stated as follows.

Theorem 1.1. ([1], p. 334–335) *If dim $M <$ dim N then the differential map d: $\mathrm{Symp}(M, N) \to \mathrm{Symp}_0(TM, TN)$ is a weak homotopy equivalence. In fact, symplectic immersions satisfy the C^0-dense parametric h-principle in the space of continuous maps f: $M \to N$ which pull back the cohomology class of σ onto that of ω.*

It is interesting to note that when dim $N = 2$dim M, taking ω equal to zero we obtain the following result of Lees [2].

COROLLARY 1.2

The Lagrangian immersions satisfy C^0-dense parametric h-principle.

In the next theorem we state the relative version of the above h-principle.

Theorem 1.3. *Let $F: T(\mathrm{Op}\,A) \to TN$ be a bundle monomorphism such that $F^*\sigma = \omega$, where A is a compact set in M, and let the underlying map f be a symplectic immersion on a neighbourhood of a compact set $B \subset A$. If the relative cohomology class $[f^*\sigma - \omega]$ vanishes in $H^2(A, B)$ then F can be homotoped to a symplectic immersion such that the homotopy remains constant in a neighbourhood of B.*

It should be remarked that Gromov studied in [1, § 3.4.2] a more general problem, namely the h-principle of σ-regular isometric immersions for an arbitrary closed form σ. The general theorem arises from the h-principle of some auxiliary sheaf which comes as the solution sheaf of an infinitesimally invertible differential operator, and Gromov proves this by using sophisticated machinery like the Nash–Moser implicit function theorem along with his sheaf theoretical techniques. The maps into a symplectic manifold (N, σ) automatically satisfy the σ-regularity condition because of the non-degeneracy of the 2-form σ. Moreover, we may avoid the generalized implicit function theorem. Instead, we use Moser's Theorem on stability of symplectic forms, which says that if two symplectic forms on a compact manifold are homotopic within the same cohomology class then they are isotopic. Throughout this paper we shall extensively use different consequences of this stability theorem.

The proof of the theorems is based on sheaf theoretic techniques. The sheaf of symplectic isometric immersions arises as the solution space of a closed partial differential relation [1, p. 2]. To apply the above-mentioned technique one starts with a topological solution of the difeferential relation and embeds the manifold (M, ω) in a symplectic manifold (M', ω') such that ω' restricts to ω on M, so that the restrictions of symplectic isometric immersions $(M', \omega') \to (N, \sigma)$ to M are isometric with respect to ω. We refer to the sheaf of symplectic immersions $M' \to N$ as the extension sheaf. An important requirement for the applicability of the sheaf theoretic techniques is the existence of a microflexible extension (see § 2). However, as we shall see in Example 3.3, this is not the case here. Noting that the symplectic immersion $(M', \omega') \to (N, \sigma)$ are in 1–1 correspondence with the Lagrangian sections of the product symplectic manifold $(M' \times N, \sigma - \omega')$ and that with the cohomology condition on the topological solution we can reduce the problem to the case where $\sigma - \omega'$ is exact, we pass on to the auxiliary sheaf of exact Lagrangian sections of $M' \times N \to M'$. This sheaf has the same homotopy type as the sheaf of Lagrangian section of $M' \times N$ and, moreover, it is microflexible. Another key point is to note that Hamiltonian diffeotopies on (M', ω') act on the auxiliary sheaf and sharply move M in M' [1, p. 82]. Thus we obtain h-principle for the auxiliary sheaf and hence the h-principle for symplectic immersions. Following the proof we also obtain

COROLLARY 1.4

If the symplectic form on N is exact, namely $\sigma = \mathrm{d}\tau$, and if $\dim N = 2\dim M$, then the space of τ-exact Lagrangian immersions satisfy (everywhere C^0-dense) h-principle.

For any undefined term we refer to [1].

2. Brief review of the sheaf theoretic results

We now briefly describe the sheaf theoretic techniques to prove parametric h-principle. Let Φ denote the sheaf of solutions of some rth order partial differential relation $\mathscr{R} \subset J^r(M, N)$ defined for C^r-maps $M \to N$, and Ψ the sheaf of sections of the r-jet bundle

$J^r(M, N) \to M$ with images in $\mathscr{R}$. The natural topologies on $\Phi(U)$ and $\Psi(U)$ are respectively the C^r and C^0 compact open topologies.

DEFINITION 2.1

The solution sheaf Φ and the relation $\mathscr{R}$ are said to satisfy parametric h-principle if the r-jet map $j^r:\Phi \to \Psi$ is a weak homotopy equivalence.

Before proceeding further we state some general definitions and results on topological sheaves.

DEFINITION 2.2

let $\mathscr{F}$ be a topological sheaf over M and A be a compact set in M. The symbol $\mathscr{F}(A)$ will denote the space of maps which are defined over some neighbourhood of A in M; in fact it is the direct limit of the spaces $\mathscr{F}(U)$ where U runs over all the open sets containing A. A map $f: P \to \mathscr{F}(A)$ on a polyhedron P is called continuous if there exists an open set $U \supset A$ such that each f_p is defined over U and the resulting map $P \to \mathscr{F}(U)$ is continuous with respect to the given topology on $\mathscr{F}(U)$.

DEFINITION 2.3

A topological sheaf $\mathscr{F}$ over M is flexible if the restriction maps $\mathscr{F}(A) \to \mathscr{F}(B)$ are Serre fibrations for every pair of compact sets (A, B), $A \supset B$. The restriction map $\mathscr{F}(A) \to \mathscr{F}(B)$ is called a microfibration if given a continuous map f'_0: $P \to \mathscr{F}(A)$ on a polyhedron P and a homotopy f_t, $0 \leqslant t \leqslant 1$, of $f'_0|_B$ there exists an $\varepsilon > 0$ and a homotopy f'_t of f'_0 such that $f'_t|\mathrm{Op}B = f_t$ for $0 \leqslant t \leqslant \varepsilon$. If for any pair of compact sets the restriction morphism is a microfibration then the sheaf $\mathscr{F}$ is called microflexible.

DEFINITION 2.4

A sheaf homomorphism f: $\mathscr{F} \to \mathscr{G}$ is a local weak homotopy equivalence if for each $x \in M$ the homomorphism $f(x)$:$\mathscr{F}(x) \to \mathscr{G}(x)$ is a weak homotopy equivalence.

Theorem 2.5. (sheaf homomorphism theorem [1, p. 77]) *Let $\mathscr{F}$ and $\mathscr{G}$ be two topological sheaves defined on a manifold M and let f: $\mathscr{F} \to \mathscr{G}$ be a morphism. If both sheaves are flexible and if f is a local weak homotopy equivalence then f is a weak homotopy equivalence.*

So to prove parametric h-principle for a relation $\mathscr{R}$ it suffices to show that the sheaves Φ and Ψ (as defined above) are flexible and the r-jet map j^r: $\Phi \to \Psi$ is a local weak homotopy equivalence. For any partial differential relation $\mathscr{R}$ the sheaf Ψ is always flexible [1, p. 40]. But to prove flexibility of Φ we need to impose certain extensibility conditions on $\mathscr{R}$.

Let M be embedded in a higher dimensional manifold M' and let $\mathscr{R}'$ be a relation on M'. We denote the corresponding sheaf of solutions by Φ'.

DEFINITION 2.6

Φ' is said to be an *extension* of Φ if the inclusion of M in M' induces a restriction homomorphism α:$\Phi'|_M \to \Phi$; moreover, $\alpha(x)$ is a surjection for each $x \in M$.

This means that if we restrict a solution of $\mathscr{R}'$ to M we obtain a solution of $\mathscr{R}$ and moreover every local solution of $\mathscr{R}$ can be lifted to a local solution of $\mathscr{R}'$.

Now, for a pair of compact subsets (A, B) in M we define the space $\Gamma(A, B)$ of compatible pairs of solutions inside $\Phi'(B)\Phi(A)$. This set consists of all pairs (f', f) such that $\alpha(f') = f|\mathrm{Op}B$.

DEFINITION 2.7

The extension Φ' will be called a microextension if the obvious map $\gamma: \Phi'(A) \to \Gamma(A, B)$ is a microfibration.

Now we explain the concept of diffeotopy sharply moving a submanifold in M'. It is worth recalling that the idea contained in this definition is a key point in the Smale–Hirsch immersion Theorem.

DEFINITION 2.8

We fix a metric d on M. An open set in M will be called 'small' if it is contained in a ball of small radius. A class of diffeotopies $\mathscr{D}$ on M' is said to sharply move M in M' if given any hypersurface S lying in a small open set of M and given any positive number ε we can obtain a diffeotopy $\{\delta_t\}$ in $\mathscr{D}$ which satisfies the following conditions:

(a) δ_0 is the identity map,
(b) each δ_t is identity outside an ε-neighbourhood of S in M,
(c) $d(\delta_1(S), M) > r$ for some $r > 0$.

Gromov gives the following sufficient condition for flexibility of Φ in his main Lemma [1, p. 82] and microextension Theorem [1,. 85].

Theorem 2.9. *If Φ admits a microextension Φ' which is microflexible and if there exists a class of acting diffeotopy on Φ' which sharply moves M in M' then Φ is a flexible sheaf.*

3. Construction of an extension sheaf

Let (M, ω) and (N, σ) be as in § 1. Then the symplectic immersions $(M, \omega) \to (N, \sigma)$ correspond to the partial differential relation $\mathscr{R} \subset J^{(1)}(M, N)$ consisting of 1-jets $j^1_x f$, $x \in M$, of germs of local immersions f such that $f^*\sigma = \omega$ at x. Let Ψ denote the sheaf of bundle monomorphisms $F: TM \to TN$ which pull back the form σ onto ω. This may be identified with the sheaf of sections of the relation $\mathscr{R}$. To obtain an extension of $\mathscr{R}$, we first embed (M, ω) isometrically into a symplectic manifold (M', ω'). We start with an $F: TM \to TN$ in $\Psi(M)$ whose underlying map is $f: M \to N$ and consider the bundle f^*TN/TM over M. Without loss of generality we may assume that F is smooth. Observe that the total space of the bundle, which we denote by X, has the same dimension as N. We extend F to a bundle morphism $F': TX|M \to TN$ such that F' maps fibres of $TX|M$ isomorphically onto the fibres of TN. Since the form $F'^*\sigma$ restricts to the closed form ω on M, it extends to a closed form ω' on some neighbourhood M' of M in X. M' may be taken to be a tubular neighbourhood of M in X so that the inclusion i: $M \to M'$ is a homotopy equivalence. Since $F'^*\sigma$ is non-degenerate so is ω'. Thus, (M, ω) is isometrically embedded in the symplectic manifold (M', ω').

We shall denote the sheaf of symplectic isometric immersions of (M, ω) in (N, σ) by $\mathscr{S}$ and that of (M', ω') in (N, σ) by $\mathscr{S}'$. Let $\mathscr{R}'$ denote the space of 1-jets of germs of symplectic immersions of (M', ω') in (N, σ) and Ψ' the sheaf of section of $\mathscr{R}'$.

PROPOSITION 3.1

$\mathscr{S}'$ is an extension of $\mathscr{S}$.

Proof. It is easy to see that the isometric embedding of (M,ω) in (M',ω') induces a morphism $\alpha: \mathscr{S}'|_M \to \mathscr{S}$. To prove that $\alpha(x): \mathscr{S}'(x) \to \mathscr{S}(x)$ is onto we start with a local symplectic immersion f at a point $x \in M$. Let $\bar{f}$ be any extension of f to a local immersion in M'. Then, since the dimension of M' is the same as the dimension of N, the form $\bar{\omega} = \bar{f}^*\sigma$ is a symplectic form. Now the two linear symplectic forms $\bar{\omega}_x$ and ω'_x defined on T_xM' coincide on the subspace T_xM. Hence there exists a linear isomorphism l of T_xM' which pulls back $\bar{\omega}_x$ onto ω'_x and keeps T_xM pointwise fixed. We consider the germ of a local map f' whose 1-jet at x equals to $j^1_x\bar{f} \circ l$ so that $j^1_x f' \in \mathscr{R}'$. By construction the jet $j^1_x f'$ projects onto $j^1_x f \in \mathscr{R}$. Moreover we may assume without loss of generality that f' extends f. So we have the following:

- $f'^*\sigma = \omega'$ at x.
- f' equals f on $U \cap M$, where U is the domain of f. Hence, pullbacks of both the forms $f'^*\sigma$ and ω' are the same.

Therefore, by the relative Poincaré Lemma, we obtain a 1-form φ on a neighbourhood, say $\tilde{U}$, of x in U such that $d\varphi = f'^*\sigma - \omega'$ and $\varphi|\tilde{U} \cap M = 0$. Now, by applying Moser's theorem [3] we get a diffeomorphism δ on a neighbourhood, say U', of x in $\tilde{U}$, such that $\delta^*(f'^*\sigma) = \omega'$, $\delta|U' \cap M$ is identity, and $d\delta_x = id$. Then $f' \circ \delta$ is the required extension of f. ■

PROPOSITION 3.2

The 1-jet map $j^1: \mathscr{S} \to \Psi$ is a local weak homotopy equivalence.

Proof. The main point is to observe that an infinitesimal solution can be deformed to a local solution of the relation $\mathscr{R}$. To see this, we start with an infinitesimal solution f at x, so that $f^*\sigma = \omega$ at x. Proceeding as in the proof of the above lemma we may extend f to a map f' on a neighbourhood of x in M' such that $f'^*\sigma = \omega'$ at x. Set $f'^*\sigma = \omega''$. Since $\omega'' = \omega'$ at x, therefore ω'' is symplectic on a neighbourhood of x. Applying Moser's theorem we get a local isotopy δ_t at x such that $\delta_1^*\omega'' = \omega'$. Moreover, the homotopy keeps x and T_xM' pointwise fixed. Defining $\bar{f} = f' \circ \delta|_M$ on Op x we observe that $\bar{f}$ is a symplectic immersion and it is homotopic to f in the space of infinitesimal solutions of $\mathscr{R}$.

The remaining part of the proof is now a routine work in view of the above observation (and hence we omit it here). □

However, it can be seen from the following example that the extension sheaf $\mathscr{S}'$ is not microflexible.

Example 3.3. Consider the standard embedding of the closed unit disc in $\mathbb{R}^2$. If we deform it near the boundary by pushing it inside then it (the homotopy) cannot be extended symplectically on the whole of the disc.

This phenomenon may be explained as follows: If f_0 is a symplectic immersion over Op A and f_t a homotopy of f_0 such that $f_t|\text{Op}B$ is a symplectic immersion, then the relative cohomology class of $f_t^*\sigma - \omega$ in $H^2(A,B)$ determines the obstruction to extending $f_t|\text{Op}B$ to Op A as symplectic immersions. If the cohomology class $[f_t^*\sigma - \omega] = 0 \in H^2(A, B)$ then there exists a smooth family of 1-forms α_t which vanishes on OpB and $f_t^*\sigma - \omega = d\alpha_t$. Then Moser's stability Theorem applies and we can lift $f_t|\text{Op}B$ over A as symplectic immersion.

Since $\mathcal{S}'$ is not microflexible we cannot apply the sheaf theoretic techniques (described in § 2) on it. However, we shall see in the next section that there exists a topological sheaf on M' naturally associated to a subspace of the space of symplectic immersions which do satisfy microflexibility and has the same homotopy type as $\mathcal{S}'$.

4. Sheaf of exact Lagrangian sections

Throughout this section we assume that both σ and ω' are exact symplectic forms. Let p_1 and p_2 respectively denote the projections of $M' \times N$ onto the first and the second factor. The product form $p_2^*\sigma - p_1^*\omega'$ on $M' \times N$ is then an exact symplectic form. We denote it by $\sigma - \omega'$. Let τ be a 1-form such that $\sigma - \omega' = \mathrm{d}\tau$.

If $f: M' \to N$ is a symplectic isometric immersion then its graph map $g = (1, f)$: $M' \to M' \times N$ is a Lagrangian section of $(M' \times N,\ \sigma - \omega')$. Since $\sigma - \omega' = \mathrm{d}\tau$, the Lagrangian condition becomes equivalent to closeness of the form $g^*\tau$. It is easy to observe that the correspondence $f \mapsto g$ is bijective. We now construct the sheaf of exact Lagrangian sections as follows: This consists of pairs (g, φ), where $g: M' \to N$ is a section of the product bundle such that map $f = p_2 \circ g: M' \to N$ is an immersion, and φ is a function on M' satisfying $g^*\tau = \mathrm{d}\varphi$. (Such a g is called a τ-exact Lagrangian immersion.) We denote the sheaf of such pairs by $\mathcal{E}'$ and call it the sheaf of τ-exact Lagrangian sections. Observe that $\mathcal{S}'$ and $\mathcal{E}'$ are locally homotopically equivalent since the germ of a Lagrangian section at a point denotes a germ of an exact Lagrangian section; moreover the space of primitives φ for a τ-exact Lagrangian section g (meaning that φ satisfies the relation $g^*\tau = \mathrm{d}\varphi$) is isomorphic to $\mathbb{R}$. Consequently, the sheaf of sections corresponding to the relation, of which $\mathcal{E}'$ is the solution sheaf, has the same homotopy type as Ψ'. We now prove

PROPOSITION 4.1

The sheaf $\mathcal{E}'$ of τ-eact Lagrangian sections is microflexible.

Proof. Let (A, B) be a pair of compact sets in M'. Let g' be a τ-exact Lagrangian section over a A (meaning that it is defined on a neighbourhood of A) such that $g'^*\tau = \mathrm{d}\varphi'$ for a 0-form φ', and (g_t, φ_t) a homotopy of $(g', \varphi')|\text{Op}B$ in $\mathcal{E}'$.

We first prove the following simple lemma.

Lemma 4.2 *Let g_t be a homotopy of τ-exact Lagrangian sections. If g_0 is also a τ'-exact Lagrangian section for some 1-form τ' on $M' \times N$ satisfying $\sigma - \omega' = \mathrm{d}\tau'$, then every g_t is a τ'-exact Lagrangian section.*

Proof. Two such forms τ and τ' differ by a closed 1-form c on M'. So, we have the following relation

$$g_t^*\tau' = g_t^*\tau + g_t^*c$$

for every t. Then, by hypothesis, g_0^*c is an exact form. Since c is closed, g_t^*c is also exact. Consequently $g_t^*\tau'$ is exact for each t. ∎

Now by the standard theory of Lagrangian submanifolds [3], there exists a neighbourhood W of the Lagrangian submanifold $L' = \operatorname{Im} g'$ such that $(W, \mathrm{d}\tau)$ is symplectomorphic to a neighbourhood of the zero section $Z_{L'}$ in the cotangent bundle $(T^*L', \mathrm{d}\theta_{L'})$ with the standard symplectic form $\mathrm{d}\theta_{L'}$ on it. Under the above identification, a Lagrangian section of $M' \times N$ (inside W) which is a small deformation of g' corresponds to a C^0-small closed form on L'; moreover, a $\tau' = \delta^*\theta_{L'}$-exact Lagrangian corresponds to an exact form on L'. Clearly the sheaf of exact 1-forms is microflexible. Hence we can obtain lifts g'_t *of* g_t (for t small enough) which are τ'-exact Lagrangian sections. By the Lemma above they are also τ-exact. Moreover, for small t, the underlying maps will be immersions on Op A. Now, we can choose a homotopy φ'_t on Op A such that $g'^*_t \tau = \mathrm{d}\varphi'_t$. On Op B, we have $\mathrm{d}\varphi'_t = \mathrm{d}\varphi_t$. Hence $\varphi'_t - \varphi_t = c_t$, where c_t is a closed 0-form, that is a constant. So we may replace φ'_t by $\varphi'_t - c_t$. The homotopy $(g'_t, \varphi'_t - c_t)$ is the required lift. ■

We shall now describe a class of diffeotopies which would act on the sheaf $\mathscr{E}'$ and at the same time sharply move a submanifold of M' of positive codimension. Since ω' is symplectic we have an isomorphism $I_{\omega'}: \chi(M') \to \Lambda^1(M')$ from the space of vector fields $\chi(M')$ onto the space of 1-forms $\Lambda^1(M')$. A C^∞ diffeotopy δ_t of M' is called exact if δ_0 is identity and if $\delta'_t = \mathrm{d}\delta_t/\mathrm{d}t$ is a Hamiltonian vector field for each t. So we can write $\delta'_t \cdot \omega' (= I_{\omega'}(\delta'_t)) = \mathrm{d}\alpha_t$ for some smooth family of exact 1-forms $\mathrm{d}\alpha_t$ on M'. If α_t can be chosen to be identically zero on the maximal open subset where δ_t is constant, then such a diffeotopy is called a strictly exact diffeotopy.

PROPOSITION 4.3

The (strictly) exact diffeotopies of M' act on the sheaf $\mathscr{E}'$.

Proof. Let δ_t be a strictly exact diffeotopy on M'. We define a diffeotopy $\bar{\delta}_t$ on $M' \times N$ by $\bar{\delta}_t(x, y) = (\delta_t(x), y)$, where $x \in M'$ and $y \in N$. It follows that $\bar{\delta}'_t \cdot (\sigma - \omega')$ is exact for each t. Let α_t be a smooth family of 0-forms on $M' \times N$ satisfying $\bar{\delta}'_t \cdot (\sigma - \omega') = \mathrm{d}\alpha_t$. Then,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\bar{\delta}^*_t \tau) = \mathscr{L}_{\bar{\delta}'_t} \tau &= \bar{\delta}^*_t(\mathrm{d}(\bar{\delta}'_t \cdot \tau) + \bar{\delta}'_t \cdot \mathrm{d}\tau) \\ &= \bar{\delta}^*_t(\mathrm{d}(\bar{\delta}'_t \cdot \tau) + \bar{\delta}'_t \cdot \sigma - \bar{\delta}'_t \cdot \omega') \\ &= \bar{\delta}^*_t(\mathrm{d}(\bar{\delta}'_t \cdot \tau) + \mathrm{d}\alpha_t) \\ &= \mathrm{d}\bar{\delta}^*_t((\bar{\delta}'_t \cdot \tau) + \alpha_t). \end{aligned}$$

If we define $\varphi_t = \int_0^t \bar{\delta}^*_t(\bar{\delta}'_t \cdot \tau + \alpha_t)\, \mathrm{d}t$ then $\bar{\delta}^*_t \tau = \tau + \mathrm{d}\varphi_t$. Now we are in a position to define the action. For $(g, \varphi) \in \mathscr{E}'$ and δ_t as above, we set

$$\delta^*_t(g, \varphi) = (\delta^*_t g, (\delta_t^{-1})^*(\varphi + g^* \varphi_t)),$$

where $\delta^*_t g = \bar{\delta}_t \circ g \circ \delta_t^{-1}$. ■

PROPOSITION 4.4

The strictly exact diffeotopies of the symplectic manifold (M', ω') sharply move M in M'.

Proof. (Gromov) To move a closed hypersurface S lying in a small open set U of M we start with a vector $\partial_0 \in T_{x_0}(M')$ transversal to U in M'. This ∂_0 extends to an exact field

$\partial = I_{\omega'}^{-1}(\mathrm{d}H)$ which is transversal to U, since U is chosen small. In order to make the corresponding exact isotopy δ_t sharply move S, we take the union $S_\varepsilon = \cup_t \delta_t(S) \in M'$ over $t \in [0, \varepsilon]$ and then multiply the Hamiltonian H by a properly chosen C^∞ function a on U which vanishes outside an arbitrarily small neighbourhood of Op S_ε and which equals r in a smaller neighbourhood of S_ε. The diffeotopy corresponding to the field $I_{\omega'}(\mathrm{d}(aH))$ can move S as sharply as we want; the required sharpness can be obtained with a proper choice of ε. ■

Now applying the Main lemma of Gromov [1, p. 82] we may conclude as follows.

PROPOSITION 4.5

The sheaf $\mathscr{E}'|_M$ is flexible.

It then follows from the sheaf homomorphism Theorem that $\mathscr{E}'|_M$ satisfies parametric h-principle.

Let $\mathscr{E}$ be the sheaf of pairs (g, φ) on M, where $g: M \to M' \times N$ is a section such that its composition with the projection map p_2 is an immersion and φ is a function on M satisfying the relation $g^*\tau = \mathrm{d}\varphi$. To descend h-principle from $\mathscr{E}'|_M$ to $\mathscr{E}$ we observe the following.

PROPOSITION 4.6

$\mathscr{E}'$ is a microextension of $\mathscr{E}$.

Proof. From Proposition 3.1 and the discussion preceding Proposition 4.1 it follows that $\mathscr{E}'$ is an extension of $\mathscr{E}$. To prove that $\mathscr{E}'$ is a microextension of $\mathscr{E}$ we consider a lifting problem

$$\begin{array}{ccc} P \times \{0\} & \xrightarrow{(G'_0, \psi'_0)} & \mathscr{E}'(A) \\ i \big\downarrow & & \big\downarrow \gamma \\ P \times I & \xrightarrow[(g', \varphi'), (g, \varphi)]{} & \Gamma(A, B) \end{array}$$

where $\alpha \circ (G'_0, \psi'_0) = (g_0, \varphi_0)$ and $(G'_0, \psi'_0)|\mathrm{Op}B = (g'_0, \varphi'_0)$ and where $\Gamma(A, B)$ is a subset of $\mathscr{E}'(B) \times \mathscr{E}(A)$ consisting of compatible solutions as defined in § 2. (To avoid too many symbols we assume P to be a point and denote $g(t)$ by g_t and so on.) We shall denote the underlying maps of G'_0, g'_t and g_t by F'_0, f'_t and f_t. Since they are immersions (which correspond to an open differential relation), we can obtain a lift of the corresponding microextension problem for immersions. Let us denote the lift by F_t, where $0 \leqslant t \leqslant \varepsilon$ for some positive number $\varepsilon \leqslant 1$. Now each F_t being an immersion between equidimensional spaces, pulls back σ onto a symplectic form on a neighbourhood of A. Let us set $F_t^*\sigma = \omega'_t$. We denote the corresponding graph map by G_t. Then we have the relation $F_t^*\sigma - \omega' = \mathrm{d}G_t^*\tau$. On the other hand we obtain a homotopy ψ'_t of ψ'_0 such that ψ'_t coincides with φ_t and φ'_t on the relevant spaces. The 1-form α_t defined by $\alpha_t = G_t^*\tau - \mathrm{d}\psi'_t$ satisfies the following.

(a) $\alpha_0 = 0$
(b) α_t vanishes on some open neighbourhood of A in M,
(c) α_t vanishes on an open neighbourhood of B in M'
(d) $F_t^*\sigma - \omega' = \mathrm{d}\alpha_t$.

Consider the vector fields $X_t = I_{\omega_t'}^{-1}(\mathrm{d}\alpha_t/\mathrm{d}t)$. The vector fields vanish on $\mathrm{Op}_M A$ as well as on $\mathrm{Op}_{M'} B$. Hence it can be integrated on a neighbourhood of A in M' to obtain a family of diffeomorphisms $\{\delta_t; 0 \leqslant t \leqslant \tilde{\varepsilon}\}$ for some $\tilde{\varepsilon} \leqslant \varepsilon$ such that

(e) δ_0 is identity on $\mathrm{Op}_{M'} A$,
(f) $\delta_t | \mathrm{Op}_M A = id$,
(g) $\delta_t | \mathrm{Op}_{M'} B = id$,
(h) $\delta_t^* \omega_t' = \omega'$.

The required partial lift of the original lifting homotopy problem can now be given by the graph map of $F_t' = F_t \circ \delta_t$. In fact, since F_t' is a symplectic immersion $G_t'^*\tau$ is closed. On the other hand, $i: M \to M'$ induces an isomorphism $i^*: H^2_{deR}(M') \to H^2_{deR}(M)$, and we know from our initial data that $i^* G_t'^* \tau$ is exact. Hence, $G_t'^*\tau$ is also exact. It is now a trivial matter to fix ψ_t'. □

With the microextension Theorem of Gromov [1, p. 85] we now observe from Proposition 4.5 and 4.6 that the sheaf $\mathscr{E}$ is flexible. We have already proved the local h-principle in Proposition 3.2. So again appealing to the sheaf homomorphism Theorem we may conclude that $\mathscr{E}$ satisfies parametric h-principle.

Corollary 1.4 is now an easy consequence of the above h-principle. If the symplectic form on N is exact, namely $\sigma = \mathrm{d}\tau$, and if $\dim N = 2 \dim M$, then for $\omega = 0$ the space $\mathscr{E}(M)$ corresponds to the space of τ-exact Lagrangian immersions. This proves that the space of τ-exact Lagrangian immersions satisfy C^0-dense h-principle.

We now prove as follows.

PROPOSITION 4.7

$\mathscr{E}(M)$ has the same homotopy type as the space $\mathscr{S}(M)$ of symplectic isometric immersions.
To prove this we need the following.

Lemma 4.8 *Let (M,ω) and (N,σ) be two symplectic manifolds with exact symplectic forms and let M be compact. Let g be a section of $M \times N \to M$ such that the underlying map $f = p_2 \circ g: M \to N$ is a symplectic immersion. Then f can be homotoped to a symplectic immersion f_1 such that $g_1 = (1, f_1)$ is a τ-exact Lagrangian immersion. In fact, $(p_2)_*: \mathscr{E}(M) \to \mathscr{S}(M)$ induces surjective maps at each homotopy level.*

Proof. Since both σ and τ are exact, there exists a 1-form τ on $M \times N$ such that $p_2^*\sigma - p_1^*\omega$ equals $\mathrm{d}\tau$. Therefore, if f is a symplectic immersion, then $g^*\tau$ is closed. We denote the form $g^*\tau$ by c. Since ω is a symplectic form there exists a unique vector field ∂ defined by $I_\omega^{-1}(-c)$. Now, M being compact, the vector field ∂ can be integrated on M to obtain an isotopy δ_t so that $\delta_0 = id$ and $(\mathrm{d}/\mathrm{d}t)\delta_t = \partial$.

Define $f_t = f \circ \delta_t$ for $t \in I$. We first prove that each $f_t: M \to N$ is a symplectic immersion. This will imply that $g_t = (1, f_t)$ is a Lagrangian section of $M \times N$. To prove this we observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta_t^*\omega = \delta_t^*\{\partial \cdot d\omega + \mathrm{d}(\partial \cdot \omega)\} = \delta_t^*(0 + \mathrm{d}(-c)) = 0.$$

Therefore, $\delta_t^*\omega = \delta_0^*\omega = \omega$. And hence, each f_t is a symplectic immersion with respect to σ and ω.

Next we prove that the 1-form c is invariant under each δ_t, that is, $\delta_t^* c = c$ for every $t \in I$. In fact, differentiating $\delta_t^* c$ with respect to t we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta_t^* c = \delta_t^*(\partial \cdot \mathrm{d}c + \mathrm{d}(\partial \cdot c)).$$

Since the form c is closed and satisfies the relation $c = -\partial \cdot \omega$, the right hand side is equal to zero. Thus, $\delta_t^* c = \delta_0^* c = c$.

Finally we will show that $g_1^*(\tau)$ is exact, that is, g_1 is a τ-exact Lagrangian immersion. Let $\bar{\delta}_t$ denote the map $M \to M \times M$ defined by $\bar{\delta}_t(x) = (x, \delta_t(x))$. Then $g_t = (1 \times f) \circ \bar{\delta}_t$.

If we set $\tau' = (1 \times f)^*\tau$ then

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t^*\tau = \frac{\mathrm{d}}{\mathrm{d}t}\bar{\delta}_t^*\tau' = \bar{\delta}_t^*\{\bar{\partial} \cdot \mathrm{d}\tau' + \mathrm{d}(\bar{\partial} \cdot \tau')\}, \tag{1}$$

where $\bar{\partial} = (\mathrm{d}/\mathrm{d}t)\bar{\delta}_t = (0, \partial)$. Now, it is easy to see that $(1 \times f)^*\mathrm{d}\tau = p_2^*\omega - p_1^*\omega$. Hence it follows that,

$$\begin{aligned}\bar{\delta}_t^*(\bar{\partial} \cdot \mathrm{d}\tau')(V) &= \mathrm{d}\tau'((0, \partial), (V, \mathrm{d}\delta_t(V)))\\ &= \omega(\partial, \mathrm{d}\delta_t(V)) - \omega(0, V)\\ &= \delta_t^*(\partial \cdot \omega)(V)\\ &= -\delta_t^*(c)(V) = -c(V).\end{aligned}$$

Equation (1) then reduces to the following:

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t^*\tau = -c + \bar{\delta}_t^*\mathrm{d}(\bar{\partial} \cdot \tau') = -c + \mathrm{d}\alpha_t,$$

where $\alpha_t = \bar{\delta}_t^*(\bar{\partial} \cdot \tau')$.

Integrating the above relation with respect to t we get $g_t^*\tau - c = -tc + \mathrm{d}\beta_t$, where $\beta_t = \int_0^t \alpha_t$. Substituting $t = 1$ we see that $g_1^*\tau$ is exact. Thus we prove that p_2 induces a surjective map $\pi_0(\mathscr{E}(M)) \to \pi_0(\mathscr{S}(M))$. The above arguments can easily be extended for a family of maps parametrized by a compact polyhedron to complete the proof of the lemma. □

Remark. If f in the above lemma is such that the 1-form c equals $\mathrm{d}\varphi$ on a neighbourhood of a compact set A, then taking an extension $\varphi': M \to \mathbb{R}$ of $\varphi | \mathrm{Op}A$ and replacing c by $c - \mathrm{d}\varphi'$ in the proof, we can homotope f to an f_1 as above, keeping f fixed on Op A. Also, starting with a family of symplectic immersions f_s parametrized by a compact polyhedron P, we can deform it to a desired one, so that if $g_s = (1, f_s)$ is an exact Lagrangian for s in the boundary of P then f_s remains fixed under the deformation.

Proof of the Proposition. Passing to the extension sheaf we consider the following sequence of maps: $\mathscr{E}'|_M \xrightarrow{(p_2)_*} \mathscr{S}'|_M \xrightarrow{J^1} \Psi'|_M$. The C^0-dense parametric h-principle for $\mathscr{E}'$ says that the composition is a weak homotopy equivalence. Hence p_2 induces injective maps in the homotopy level.

On the other hand, noting the fact that the closed 2-form ω' on M' is symplectic it follows from the lemma above that p_2 induces surjective maps between homotopy groups of the spaces $\mathscr{E}'(A)$ and $\mathscr{S}'(A)$ for every compact subset A of M. Thus, $(p_2)_*:\pi_i(\mathscr{E}(A))\to\pi_i(\mathscr{S}(A))$ is an isomorphism for every compact set $A\subset M$. It also follows from the above remark that, for a pair of compact sets (A, B) in M, the fibres of the restriction maps $\mathscr{E}'(A)\to\mathscr{E}'(B)$ and $\mathscr{S}'(A)\to\mathscr{S}'(B)$ are of the same weak homotopy type.

Since M can be covered by an increasing sequence of compact sets in M, we can conclude that $(p_2)_*:\mathscr{E}'|_M\to\mathscr{S}'|_M$ is a weak homotopy equivalence.

Now, proceeding as in Proposition 4.6 we may observe that both restriction maps $\mathscr{S}'(M)\to\mathscr{S}(M)$ and $\mathscr{E}'(M)\to\mathscr{E}(M)$ are fibrations. Moreover, for any $g\in\mathscr{E}$, the fibres in $\mathscr{E}'(M)$ and $\mathscr{S}'(M)$ over g and $p_2\circ g$ respectively are homotopically equivalent. Since we have proved above that $\mathscr{E}'(M)$ and $\mathscr{S}'(M)$ are of the same weak homotopy type, we conclude through homotopy exact sequence of fibrations that $\mathscr{E}(M)$ and $\mathscr{S}(M)$ are also of the same weak homotopy type. This completes the proof of the proposition. □

This leads us to the following intermediate theorem.

Theorem 4.9. *If the differential forms σ and ω are exact then the space of symplectic immersions of M into N satisfies C^0 dense parametric h-principle.*

As an application to Theorem 4.8 we may look at the question of isometric immersibility of a symplectic manifold in the Euclidean symplectic manifold $\mathbb{R}^{2n}$ with its canonical symplectic structure $\omega_0=\Sigma_{i=1}^n \mathrm{d}x_i\wedge \mathrm{d}y_i$. One simple observation is that a symplectic manifold with a non-exact symplectic form cannot admit such an immersion in $\mathbf{R}^{2n}$. So an appropriate question would be if an arbitrary manifold M with an exact symplectic form ω on it admits a symplectic immersion in $\mathbb{R}^{2n}$ for some values of n.

Example 4.10. If $(M, \mathrm{d}\tau)$ is a symplectic manifold of dimension $2m$ then it admits a symplectic immersion in $(\mathbb{R}^{2n},\omega_0)$ for $n\geqslant 2m$.

To prove the existence of a symplectic immersion, it is enough to produce a section of the complex Stiefel bundle over M with fibre $\mathscr{F}_m(\mathbb{C}^n)$, which has the same homotopy type as the space of symplectic isometric bundle maps $TM\to T\mathbb{R}^{2n}$. In fact, the space of all symplectic linear maps $\mathbb{R}^{2m}\to\mathbb{R}^{2n}$ between the euclidean symplectic spaces has the homotopy type of the space of complex linear isometries $\mathbb{C}^m\to\mathbb{C}^n$, that is, the space $\mathscr{F}_m(\mathbb{C}^n)$ of all complex m-frames in $\mathbb{C}^n$. The obstruction to the existence of such a section lies in the cohomology group $H^{2m}(M,\pi_{2m-1}(\mathscr{F}_m(\mathbb{C}^n)))$. From the standard homotopy theory it follows that $\mathscr{F}_m(\mathbb{C}^n)$ is $2(n-m)$ connected. Thus, M admits a symplectic immersion in $\mathbb{R}^{2n}$ if $2m-1\leqslant 2(n-m)$ i.e., if $2m\leqslant n$.

5. Proof of the main theorem

Let us now go back to the general case where $\sigma-\omega$ is not necessarily exact on $M\times N$. However, if $f:M\to N$ is a continuous map such that $f^*[\sigma]=[\omega]$ then f can be extended to a map $f':M'\to N$ such that $f'^*[\sigma]=[\omega']$. Then, in some neighbourhood W of graph f, there exists a 1-form τ such that $\sigma-\omega'=\mathrm{d}\tau$. We shall denote by $\mathscr{S}_W$ the sheaf of symplectic immersions $M\to N$ whose graphs lie in W. Then from Theorem 4.8 it follows that $\mathscr{S}_W$ satisfies parametric h-principle. We now come to the proof of Theorem 1.1.

Proof of Theorem 1.1 In view of the above discussion it remains only to prove that d induces injective maps at each homotopy level; namely, $d_*:\pi_i(\mathcal{S}(M))\to\pi_i$ $(\mathrm{Symp}_0(TM,TN))$ is injective for each integer i. Let f_0 and f_1 be two symplectic immersions on M such that their differentials $\mathrm{d}f_0$ and $\mathrm{d}f_1$ are homotopic in $\mathrm{Symp}_0(TM,TN)$; that is, there exists a homotopy $F_t:TM\to TN$ of symplectic bundle maps between $\mathrm{d}f_0$ and $\mathrm{d}f_1$ such the underlying maps $f_t:M\to N$ satisfy $f_t^*[\sigma]=[\omega]$. For each t we can choose a neighbourhood W_t of graph f_t on which $\sigma-\omega$ is exact. Let $\mathcal{S}_t$ denote the sheaf of those symplectic immersions whose graphs lie in W_t, and let Ψ_t be defined as in § 3 corresponding to $\mathcal{S}_t$. We can cover the set $\cup_t f_t(M)$ by finite by many such W_t's such that any two consecutive ones (ordered by the real number) intersect in a set which contains completely the graph of some f_t. Without any loss of generality we may assume that the neighbourhoods $\{W_0, W_1\}$ have this property. Let for some T, the graph of f_T lies in $W_0\cap W_1$. Then by h-principle of the sheaf $\mathcal{S}_{W_0\cap W_1}$ we obtain a symplectic immersion f, C^0-close to f_T, such that $\mathrm{d}f$ and F_T are homotopic within Ψ_{W_0}. In fact, the homotopy lies in Symp_0 (TM,TN) and the underlying maps of the homotopy have their graphs in $W_0\cap W_1$. Then applying parametric h-principle for $\mathcal{S}_0$ we conclude that f and f_0 are homotopic within the space $\mathcal{S}_0$. On the other hand f is homotopic to f_1 within the space $\mathcal{S}_1$. Joining these two homotopies we obtain a homotopy between f_0 and f_1 in the space of symplectic immersions. This proves that the differential d induces an isomorphism between the homotopy groups at the zero level.

Working with a family of such maps parametrized by spheres S^i, we can similarly prove the isomorphism between the higher homotopy groups of the relevant spaces which gives the desired h principle. □

We now prove the relative or the extension version of h-principle for symplectic immersions.

Proof of Theorem 1.3. Let F be as in the statement of Theorem 1.3. Therefore, by hypothesis, the underlying map f is symplectic over Op B and the de Rham cohomology class $[f^*\sigma-\omega]$ vanishes in $H^2(A,B)$. Hence we can find a 1-form φ on M which vanishes on a neighbourhood of B and satisfies the condition $f^*\sigma-\omega=\mathrm{d}\varphi$. Let $g=(1,f)$ denote the graph map of f and let Y be a tubular neighbourhood of Im g with a retraction $r:Y\to\mathrm{Im}\ g$ so that $r\circ g=g$. Therefore, the composition $p_1\circ r\circ g$ is the identity map on M. Hence we can find a 1-form τ on Y, namely $\tau=(p_1\circ r)^*\varphi$, such that $g^*\tau=\varphi$. Thus, the g-pullbacks of the closed forms $(\sigma-\omega)$ and $\mathrm{d}\tau$ are the same and so, by the relative Poincaré Lemma, $\sigma-\omega$ equals to the exact form $\mathrm{d}(\tau+\tau')$ on a neighbourhood W of Im g, where τ' vanishes on the graph of f. Moreover, $g^*(\tau+\tau')=\varphi$ vanishes on Op B so that $(g,\varphi)|\mathrm{Op}B$ is in $\mathcal{E}_W(B)$, where $\mathcal{E}_W$ denotes the sheaf of $\tau+\tau'$-exact Lagrangian sections with images in W. Now consider the following commutative diagram

$$\begin{array}{ccc}\mathcal{E}_W(A) & \longrightarrow & \Psi_W(A)\\ \downarrow & & \downarrow\\ \mathcal{E}_W(B) & \longrightarrow & \Psi_W(B)\end{array}$$

where the horizontal arrows are weak homotopy equivalences and the vertical ones are fibrations. Hence the fibres over $g|_B$ and $\mathrm{d}f|_{TB}$ are also weak homotopy equivalent. The theorem follows as F is in the fibre over $\mathrm{d}f|_{TB}$. □

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