Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 2, June 1998, pp. 109–120.

Degree of approximation of functions associated with Hardy–Littlewood series in the generalized Hölder metric

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MS received 4 December 1996; revised 16 December 1997

Abstract. The paper studies the degree of approximation of functions associated with Hardy Littlewood series in the generalized Hölder metric.

Keywords. Banach space; Hölder metric; generalized Hölder metric, infinite matrix; Hardy Littlewood series.

1. Definition

Let f be a periodic function of period 2π and let $f \in L_p\,[0, 2\pi]$ for $p \geqslant 1$. Then the Fourier series of f at $t = x$ is given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x), \tag{1.1}$$

and conjugate series by

$$\sum (a_n \sin nx - b_n \cos nx) = \sum_{n=1}^{\infty} B_n(x). \tag{1.2}$$

Let us write

$$\Phi_x(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}, \tag{1.3}$$

$$\chi_x(u) = \int_u^{\pi} \Phi_x(w) \frac{1}{2} \cot w/2 \, \mathrm{d}w. \tag{1.4}$$

Let $s_n(x)$ and $s_n^*(x)$ respectively denote the partial sum and modified partial sum of (1.1), i.e.

$$s_n(x) = \sum_{k=0}^{n} A_k(x), \quad s_n^*(x) = \sum_{k=0}^{n-1} A_k(x) + \frac{1}{2}A_n(x).$$

It is known ([9], p. 50) that

$$s_n^*(x) - f(x) = \frac{2}{\pi} \int_0^{\pi} \frac{\Phi_x(u) \sin nu}{2 \tan u/2} \, \mathrm{d}u. \tag{1.5}$$

The space $L_p\,[0, 2\pi]$ where $p = \infty$ includes the space $C_{2\pi}$ of all continuous functions defined over $[0, 2\pi]$.

We write

$$\|f\|_c = \sup_{x\in[0,2\pi]} |f(x)|$$

$$= \|f\|_p \quad (p=\infty)$$

and for $p \geqslant 1$,

$$\|f\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p \mathrm{d}x\right)^{1/p} \quad (p \geqslant 1).$$

We write

$$\omega(\delta) = \omega(\delta, f) = \sup_{0\leqslant h\leqslant\delta} \|f(\cdot + h) - f(\cdot)\|_c, \tag{1.6}$$

when the norm has been taken with respect to x throughout the paper

$$\omega_p(\delta) = \omega_p(\delta, f) = \sup_{0\leqslant h\leqslant\delta} \|f(\cdot + h) - f(\cdot)\|_p, \tag{1.7}$$

$$\omega_p^{(2)}(\delta) = \omega_p^{(2)}(\delta, f) = \sup_{0\leqslant h\leqslant\delta} \|f(\cdot + h) + f(\cdot - h) - 2f(\cdot)\|_p, \tag{1.8}$$

which are respectively called modulus of continuity, integral modulus of continuity and integral modulus of smoothness (see [9] p. 42). In the case $0 < \beta \leqslant 1$ and $\omega(\delta, f) = O(\delta^\beta)$ we write $f \in lip\,\beta$, and if $\omega_p(\delta, f) = O(\delta^\beta)$, we write $f \in lip\,(\beta, p)$. The case $\beta > 1$ is of no interest as in this case f turns out to be constant. The class $lip\,(\beta, p)$ with $p = \infty$ will be taken as $lip\,\beta$.

Let

$$H_\alpha = \{f \in C_{2\pi}: |f(x) - f(y)| \leqslant K|x - y|^\alpha\},$$

where k is a positive constant, not necessarily same at each occurrence. It is known [7] that H_α is a Banach space with the norm $\|\cdot\|_\alpha$ defined by

$$\|f\|_\alpha = \|f\|_c + \sup_{x\neq y} \Delta^\alpha f(x, y), \tag{1.9}$$

where

$$\Delta^\alpha f(x, y) = |f(x) - f(y)|\,|x - y|^{-\alpha}, \quad x \neq y$$

and

$$\Delta^0 f(x, y) = 0.$$

The metric induced by the norm (1.5) on H_α is called the Hölder metric. Since

$$\|f\|_\beta \leqslant (2\pi)^{\alpha-\beta} \|f\|_\alpha, \quad 0 \leqslant \beta < \alpha \leqslant 1$$

it follows that $H_\alpha \subseteq H_\beta \subseteq C_{2\pi}$; that is $\{H_\alpha, \|\cdot\|_\alpha\}$ is a family of Banach spaces which decrease as α increases. Hölder metric has been generalized in [2], as follows.

For $0 < \alpha \leqslant 1$,

$$H(\alpha, p) = \{f \in L_p, \quad 0 < p \leqslant \infty: \|f(\cdot + h) - f(\cdot)\|_p \leqslant K|h|^\alpha\}$$

and define for $f \in H(\alpha, \rho)$

$$\|f\|_{(\alpha,p)} = \|f\|_p + \sup_{h\neq 0} \frac{\|f(\cdot + h) - f(\cdot)\|_p}{|h|^\alpha}$$

$$\|f\|_{(0,p)} = \|f\|_p. \tag{1.10}$$

It can be easily verified that (1.10) is a norm for $p \geqslant 1$ and that $H(\alpha, p)$ is a Banach space. Note that $H(\alpha, \infty)$ is the familiar H_α space introduced earlier by Prössdorff [7].

Let $A = (a_{n,k})$ be an infinite matrix that satisfies the following conditions:

$$\|A\| = \sup_n \sum_{k=0}^{\infty} |a_{n,k}| < \infty \tag{1.11}$$

and

$$\sum_{k=0}^{\infty} a_{n,k} = 1 \quad \text{for each} \quad n = 0, 1, 2, \ldots. \tag{1.12}$$

We write $A \in \mathscr{T}$ if conditions (1.11) and (1.12) hold. Let (μ_n) be a positive non-decreasing sequence such that

$$\sum_{k=\mu_n}^{\infty} (k+1)|a_{n,k}| = O(\mu_n). \tag{1.13}$$

Also we need the following additional notations:

$$\Psi(n) = \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+1}|, \tag{1.14}$$

$$D_n(u) = \frac{\sin(n+1/2)u}{2\sin u/2}, \tag{1.15}$$

$$F_x(u) = \chi_x(u) - \chi_x(\pi/\mu_n), \tag{1.16}$$

$$G(u) = F_x(u) - F_{x+y}(u), \tag{1.17}$$

$$K_n(u) = \sum_{k=0}^{\infty} a_{n,k} D_k(u). \tag{1.18}$$

2. Introduction

By writing Hardy–Littlewood series or in short HL-series we mean the series

$$\sum_{n=1}^{\infty} \frac{s_n(x) - f(x)}{n}. \tag{2.1}$$

We take this opportunity to acknowledge the fact that this nomenclature for the series (2.1) was first given by Mohanty (see [6]).

Hardy and Littlewood [5] have shown that (2.1) is summable $(C, 1)$ to the value

$$\frac{1}{\pi}\int_{0+}^{\pi} \left\{\left(\frac{\pi - u}{2}\right)\cot u/2 - \log(2\sin u/2)\right\}\Phi_x(u)\,du$$

whenever the integral

$$\int_{0+}^{\pi} \Phi_x(u)\frac{1}{2}\cot u/2\,du \tag{2.2}$$

exists, Further ([5], see also [9], p. 122), if

$$\int_0^t |\Phi_x(u)|\,du = o(t) \quad \text{as} \quad t \to o+, \tag{2.3}$$

then (2.1) converges if and only if (2.2) exists. The interest of the HL-series lies in its relationship to the integral (2.2), and these relations are very similar to those between the conjugate series $\sum B_n$ and the integral

$$\int_{0+}^{\pi} \frac{\psi_x(u)}{u} du, \tag{2.4}$$

where

$$\psi_x(u) = \frac{1}{2}\{f(x+u) - f(x-u)\}.$$

It is known [9] that if $f \in L$ then (2.4) exists almost everywhere.

On the other hand there exists a continuous function f for which the integral (2.2) diverges for almost all x [5].

At this stage, we remark that the above results on HL-series remain unaltered if we replace HL-series by

$$\frac{1}{2}c_0 + \sum_{n=1}^{\infty} \left\{\frac{s_n^*(x) - f(x)}{n}\right\}, \tag{2.5}$$

where

$$c_0 = \frac{2}{\pi}\int_0^{\pi} \Phi_x(u)\frac{u}{2}\cot u/2\, du$$

and the series (2.5) is summable $(C, 1)$ to the value (2.2) whenever this integral exists. Thus the convergence or summability problem of (2.5) is the same as that of (2.1), though their sums are different and hence we may term (2.5) as HL-series.

Prössdorff [7] studied the degree of approximation in the Hölder metric and proved the following theorem.

Theorem A [7]. *Let $f \in H_\alpha$ $(0 < \alpha \leqslant 1)$ and $0 \leqslant \beta < \alpha \leqslant 1$. Then*

$$\| \sigma_n(f) - f \|_{(\beta,\infty)} = O(1)\begin{cases} n^{\beta-\alpha}, & (0 < \alpha < 1) \\ n^{\beta-1}(1+\log n)^{1-\beta}, & \alpha = 1 \end{cases}$$

where $\sigma_n(f)$ is the Fejèr means of the Fourier series of f.

(Remark $\| \sigma_n(f) - f \|_{(\beta,\infty)}$ is in our notation). The case $\beta = 0$ of Theorem A is due to Alexits [1]. With regard to the approximation of functions in L_p norm, the following theorem is due to Quade [8].

Theorem B [8]. *Let $f \in lip(\alpha, p)$, $0 < \alpha \leqslant 1$. Then*

$$\| \sigma_n(t) - f \|_{(0,p)} = O(1)\begin{cases} n^{-\alpha} & (p > 1) \\ n^{-\alpha} & (p = 1, 0 < \alpha < 1) \\ (\log n)/n & (p = 1, \alpha = 1) \end{cases}.$$

(Remark $\| \sigma_n(f) - f \|_{(0,p)}$ is in our notation). In a recent paper [2], the degree of approximation in the generalized Hölder metric has been introduced and the following result has been obtained.

Theorem C [2]. *Let $s_n(x)$ be the nth partial sum of* (1.1). *Suppose that $A \in \mathscr{T}$ and let there exist a positive non-decreasing sequence (μ_n) such that* (1.13) *hold. Then for $p \geqslant 1$ and $f \in H(\alpha, p)$, $0 < \alpha \leqslant 1$, $0 \leqslant \beta < \alpha$,*

$$\left\| \sum_{k=0}^{\infty} a_{n,k} s_k - f \right\|_{(\beta,p)}$$

$$= O(1) \begin{cases} (1 + \log(\mu_n/\lambda_n))^{\beta/\alpha} \lambda_n^{\beta-\alpha} + \psi(n)\lambda_n^{1-\alpha+\beta}, & (0 < \alpha \leqslant 1) \\ \dfrac{(1 + \log(\mu_n/\lambda_n))^{\beta}}{\lambda_n^{1-\beta}} + \psi(n)\lambda_n^{\beta}(\log \lambda_n)^{1-\beta}, & (\alpha = 1) \end{cases}$$

where λ_n is any positive non-decreasing sequence such that $\lambda_n \leqslant \mu_n$.

3. Main results

The object of the present paper is to determine the degree of approximation of the series (2.5) by means of A-transform in the generalized Hölder metric. We prove the following theorem.

Theorem. *Suppose that $A \in \mathscr{T}$ and let there exist a positive non-decreasing sequence (μ_n) such that* (1.13) *hold. Let $M_n(x)$ be the A-transform of the series* (2.5). *Then for $p \geqslant 1$ and $f \in H(\alpha, p)$, $0 < \alpha \leqslant 1$, $0 \leqslant \beta < \alpha$*

$$\| M_n(x) - \chi_x(\pi/\mu_n) \|_{(\beta,p)}$$

$$= O(1) \begin{cases} (\log \mu_n)^{\beta} \left[\dfrac{\left(1 + \log\left(\dfrac{\mu_n}{\lambda_n}\right)\right)^{\beta}}{\lambda_n^{1-\beta}} + \lambda_n^{\beta}\psi(n)(\log \lambda_n)^{1-\beta} \right], & \alpha = 1 \\ (\log \mu_n)^{\beta/\alpha} \left[\left(1 + \log \dfrac{\mu_n}{\lambda_n}\right)^{\beta/\alpha} \lambda_n^{\beta-\alpha} + \psi(n)\lambda_n^{1-\alpha+\beta} \right], & 0 < \alpha < 1 \end{cases} \tag{3.1}$$

where $\psi(n)$ is defined in (1.14) *and λ_n is any positive non-decreasing sequence such that $\lambda_n \leqslant \mu_n$.*

Before we take up the proof, we will exhibit below the following.

Fourier character of HL-series (2.5)

Let

$$\chi_x(u) = \int_u^{\pi} \Phi_x(w) \frac{1}{2} \cot w/2 \, dw. \tag{3.2}$$

It is known [3] that χ is even and Lebesgue integrable. Let

$$\chi_x(t) \sim \frac{1}{2} c_0 + \sum_{n=1}^{\infty} c_n \cos nt. \tag{3.3}$$

We have

$$c_0 = \frac{2}{\pi}\int_0^\pi \chi_x(t)\,\mathrm{d}t = \frac{2}{\pi}\int_0^\pi \left(\int_t^\pi \Phi_x(u)\cot u/2\,\mathrm{d}u\right)\mathrm{d}t$$

$$= \frac{2}{\pi}\int_0^\pi \Phi_x(u)\frac{1}{2}\cot u/2\,\mathrm{d}u\int_0^u \mathrm{d}t,$$

$$= \frac{2}{\pi}\int_0^\pi \Phi_x(u)\frac{u}{2}\cot u/2\,\mathrm{d}u \tag{3.4}$$

and for $n \geqslant 1$

$$c_n = \frac{2}{\pi}\int_0^\pi \chi_x(t)\cos nt\,\mathrm{d}t = \frac{2}{\pi}\int_0^\pi \cos nt\left(\int_t^\pi \Phi_x(u)\frac{1}{2}\cot u/2\,\mathrm{d}u\right)\mathrm{d}t$$

$$= \frac{2}{\pi}\int_0^\pi \Phi_x(u)\frac{1}{2}\cot u/2\,\mathrm{d}u\int_0^u \cos nt\,\mathrm{d}t$$

$$= \frac{2}{\pi n}\int_0^\pi \frac{\Phi_x(u)\sin nu\,\mathrm{d}u}{2\tan u/2}$$

$$= \frac{s_n^*(x) - f(x)}{n}. \tag{3.5}$$

Thus, we have the following.

PROPOSITION

The Hardy Littlewood series (2.5) *is Fourier series of even function* $\chi(u)$ *at* $u = 0$.

4. Lemmas

To prove the theorem we use the following Lemmas.

Lemma 1. *If* $\Phi_x(u) \in L_p$, $p \geqslant 1$ *then*

$$\chi(u) \in L_p, \quad p \geqslant 1. \tag{4.1}$$

Proof. The result is proved by Hardy [4] for $p = 1$. For $p > 1$, we choose q with $1/p + 1/q = 1$ and α such that $1/q < \alpha < 1$. We observe that

$$\alpha > \frac{1}{q} = 1 - \frac{1}{p},$$

$$\frac{p}{q} - p\alpha = p\left(1 - \frac{1}{p}\right) - p\alpha = p - p\alpha - 1.$$

Thus we have $0 < p - p\alpha < 1$. Now

$$|\chi_x(t)| \leqslant \int_t^\pi \left|\Phi_x(u)\frac{1}{2}\cot u/2\right|\mathrm{d}u$$

$$\leqslant \int_t^\pi \frac{|\Phi_x(u)|}{u^{1-\alpha}}\frac{\mathrm{d}u}{u^\alpha} \leqslant \left(\int_t^\pi \frac{|\Phi_x(u)|^p}{u^{p-p\alpha}}\mathrm{d}u\right)^{1/p}\left(\int_t^\pi \frac{\mathrm{d}u}{u^{q\alpha}}\right)^{1/q}$$

$$= \left(\int_t^\pi \frac{|\Phi_x(u)|^p}{u^{p-p\alpha}} \mathrm{d}u \right)^{1/p} \left[\frac{1}{(q\alpha - 1)} (t^{1-q\alpha} - \pi^{1-q\alpha}) \right]^{1/q}$$

$$\leqslant K \left(\int_t^\pi \frac{|\Phi_x(u)|^p \mathrm{d}u}{u^{p-p\alpha}} \right)^{1/p} t^{1/q-\alpha}$$

$$\Rightarrow |\chi_x(t)|^p \leqslant K \left(\int_t^\pi \frac{|\Phi_x(u)|^p \mathrm{d}u}{u^{p-p\alpha}} \right) t^{p/q - p\alpha}$$

$$= K \left(\int_t^\pi \frac{|\Phi_x(u)|^p \mathrm{d}u}{u^{p-p\alpha}} \right) t^{p - p\alpha - 1}.$$

Hence

$$\int_0^\pi |\chi_x(t)|^p \mathrm{d}t \leqslant K \int_0^\pi t^{p-p\alpha-1} \left(\int_t^\pi \frac{|\Phi_x(u)|^p}{u^{p-p\alpha}} \mathrm{d}u \right) \mathrm{d}t$$

$$= K \int_0^\pi \frac{|\Phi_x(u)|^p}{u^{p-p\alpha}} \mathrm{d}u \int_o^u t^{p-p\alpha-1} \mathrm{d}t$$

$$= K \int_0^\pi \frac{|\Phi_x(u)|^p}{u^{p-p\alpha}} \left(\frac{u^{p-p\alpha}}{p - p\alpha} \right) \mathrm{d}u$$

$$= \frac{K}{p - p\alpha} \int_0^\pi |\Phi_x(u)|^p \mathrm{d}u < \infty.$$

Thus it proves the lemma.

Lemma 2. *Let* $1 \leqslant p \leqslant \infty$. *Then*

(i) $$\| G(u) \|_p = O(1) \begin{cases} u^\alpha, & u > \pi/\mu_n \\ \mu_n^{-\alpha}, & u < \pi/\mu_n \end{cases} \tag{4.2}$$

(ii) $$\| G(u) \|_p = O(1) h^\alpha \begin{cases} \log \mu_n, & u > \pi/\mu_n \\ \log \frac{1}{u}, & u < \pi/\mu_n \end{cases}. \tag{4.3}$$

Proof (i). Since $f \in H(\alpha, p)$ then

$$\| \Phi(w) - \Phi_h(w) \|_p = O(|w|^\alpha).$$

From (1.6) we have

$$F_x(u) = \int_u^{\pi/\mu_n} \Phi_x(w) \frac{1}{2} \cot w/2 \, \mathrm{d}w.$$

Now from (1.17) and using generalized Minkowski's inequality for $p \geqslant 1$ we get

$$\| G(u) \|_p = \| F(u) - F_h(u) \|_p = \left(\int_0^\pi |F_x(u) - F_{x+h}(u)|^p \mathrm{d}x \right)^{1/p}$$

$$= \left\{ \int_0^\pi \mathrm{d}x \left| \int_u^\pi (\Phi_x(w) - \Phi_{x+h}(w)) \frac{1}{2} \cot w/2 \, \mathrm{d}w \right|^p \right\}^{1/p}$$

$$\leqslant \int_u^\pi \frac{\mathrm{d}w}{2\tan w/2}\, \| \Phi(w)-\Phi_h(w)\|_p$$

$$= O(1)\int_u^\pi w^{\alpha-1}\,\mathrm{d}w = O(1)\begin{cases} u^\alpha, & u>\pi/\mu_n \\ \mu_n^{-\alpha}, & u<\pi/\mu_n \end{cases}.$$

Proof (ii). Since $f \in H(\alpha, p)$ then

$$\| \Phi(w)-\Phi_h(w)\| = O(|h|^\alpha).$$

From (1.16), (1.17) and using generalized Minkowski's inequality for $p \geqslant 1$ as above we have

$$\| G(u)\|_p = \| F(u)-F_h(u)\|_p$$

$$\leqslant \int_u^\pi \frac{\mathrm{d}w}{2\tan w/2}\, \| \Phi(w)\Phi_{+h}(w)\|_p$$

$$= O(1)h^\alpha \begin{cases} \log \mu_n, & u>\pi/\mu_n \\ \log \dfrac{1}{u}, & u<\pi/\mu_n \end{cases}.$$

5. Proof of the Theorem

Denoting the nth partial sum of (2.5) by $T_n(x)$ and using (3.6) we have

$$T_n(x) = \frac{1}{2}c_0 + \sum_{k=1}^{n} \frac{S_k^*(x)-f(x)}{k}$$

$$= \frac{1}{2}c_0 + \sum_{k=1}^{n} c_k$$

$$= \frac{2}{\pi}\int_0^\pi \chi_x(u) D_n(x)\,\mathrm{d}u.$$

We write

$$I_n(x) = M_n(x) - \chi_x(\pi/\mu_n) = \sum_{k=0}^{\infty} a_{n,k} T_k(x) - \chi_x(\pi/\mu_n). \tag{5.1}$$

Now

$$I_n(x) = \frac{2}{\pi}\sum_{k=0}^{\infty} \chi_x(u) \sum_{k=0}^{\infty} a_{n,k} D_k(u)\mathrm{d}u - \chi_x(\pi/\mu_n)\sum_{k=0}^{\infty} a_{n,k}\frac{2}{\pi}\int_0^\pi D_k(u)\,\mathrm{d}u$$

$$= \frac{2}{\pi}\int_0^\pi F_x(u) \sum_{k=0}^{\infty} a_{n,k} D_k(u)\mathrm{d}u$$

$$= \frac{2}{\pi}\int_0^\pi F_x(u) K_n(u)\mathrm{d}u, \tag{5.2}$$

where $F_x(u)$ and $K_n(u)$ are respectively defined in (1.16) and (1.18).

Note that the change of order of summation and integration is justified provided either side is convergent. We observe that by (1.12) the series for $K_n(u)$ is convergent (even absolutely) and

$$K_n(u) = O(u^{-1})$$

for all $0 < u \leqslant \pi$ and the integral (5.2) exists.

Now by generalized Minkowski's inequality for $p \geqslant 1$, we have

$$\| l_n(x) - l_n(x+y) \|_p \leqslant \frac{2}{\pi} \int_0^{\pi} \| F_x(u) - F_{x+y}(u) \|_p |K_n(u)| \, du$$

$$= \frac{2}{\pi} \left(\int_0^{\pi/\lambda_n} + \int_{\pi/\lambda_n}^{\pi} \right) \| G(u) \|_p \, |K_n(u)| \, du$$

$$= l_1 + l_2, \quad \text{say,} \tag{5.3}$$

$$l_1 = \frac{2}{\pi} \int_0^{\pi/\lambda_n} \| G(u) \|_p \, |K_n(u)| \, du$$

$$= \frac{2}{\pi} \left(\int_0^{\pi/\mu_n} + \int_{\pi/\mu_n}^{\pi/\lambda_n} \right) \| G(u) \|_p \, |K_n(u)| \, du$$

$$= l_{1,1} + l_{1,2} \quad \text{say.} \tag{5.4}$$

We first note that

$$K_n(u) = \frac{1}{2 \sin u/2} \left(\sum_{k=0}^{\mu_n} + \sum_{k=\mu_n+1}^{\infty} \right) a_{n,k} \sin\left(k + \frac{1}{2} \right) u$$

$$= O(u^{-1}) \left[\sum_{k=0}^{\mu_n} |a_{n,k}|(k+1)u + \sum_{k=\mu_n+1}^{\infty} (k+1)|a_{n,k}|u \right]$$

$$= O(\mu_n) + O(\mu_n) = O(\mu_n). \tag{5.5}$$

By Lemma 2(i) and (5.5) we get

$$l_{1,1} = \frac{2}{\pi} \int_0^{\pi/\mu_n} \| G(u) \|_p |K_n(u)| \, du$$

$$= O(1) \int_0^{\pi/\mu_n} \mu_n^{1-\alpha} \, du$$

$$= O(1) \, \mu_n^{-\alpha}. \tag{5.6}$$

By making use of the fact that

$$\left| \sum_{k=0}^{\infty} a_{n,k} \sin\left(k + \frac{1}{2} \right) u \right| \leqslant \| A \| < \infty \tag{5.7}$$

and by Lemma 2 (i) we have

$$l_{1\,2} = \frac{2}{\pi} \int_{\pi/\mu_n}^{\pi/\lambda_n} \| G(u) \|_p \, |K_n(u)| \, du$$

$$= O(1) \int_{\pi/\mu_n}^{\pi/\lambda_n} u^{\alpha} u^{-1} \, du$$

$$= O(1) \left(\frac{1}{\lambda_n^{\alpha}} \right). \tag{5.8}$$

Combining (5.6) and (5.8) we obtain

$$l_1 = O(1)\mu_n^{-\alpha} + O(1)\frac{1}{\lambda_n^{\alpha}}$$

$$= O(1)\left(\frac{1}{\lambda_n^{\alpha}}\right). \tag{5.9}$$

By Abel's transformation

$$\sum_{k=0}^{\infty} a_{n,k}\sin\left(k+\frac{1}{2}\right)u = O(u^{-1})\sum_{k=0}^{\infty}|a_{n,k}-a_{n,k+1}|. \tag{5.10}$$

Now by (5.10), (1.14) and Lemma 2(i) we get

$$l_2 = \frac{2}{\pi}\int_{\pi/\lambda_n}^{\pi} \|G(u)\|_p |K_n(u)|\,\mathrm{d}u$$

$$= O(1)\int_{\pi/\lambda_n}^{\pi} u^{\alpha-2}\psi(n)\mathrm{d}u$$

$$= O(1)\psi(n)\begin{cases}\log\lambda_n, & \alpha = 1\\ \lambda_n^{1-\alpha}, & 0<\alpha<1\end{cases}. \tag{5.11}$$

Again by Lemma 2(ii) and (5.5), we get

$$l_{1,1} = O(1)|y|^{\alpha}\int_0^{\pi/\mu_n}\log\frac{1}{u}|k_n(u)|\,\mathrm{d}u$$

$$= O(1)|y|^{\alpha}\mu_n\int_0^{\pi/\mu_n}\log\frac{1}{u}\mathrm{d}u$$

$$= O(1)|y|^{\alpha}\log\mu_n. \tag{5.12}$$

By Lemma 2(ii) and (5.7) we obtain

$$l_{1,2} = O(1)|y|^{\alpha}\log\mu_n\int_{\pi/\mu_n}^{\pi/\lambda_n}|K_n(u)|\,\mathrm{d}u$$

$$= O(1)|y|^{\alpha}\log\mu_n\int_{\pi/\mu_n}^{\pi/\lambda_n}u^{-1}\,\mathrm{d}u$$

$$= O(1)|y|^{\alpha}\log\mu_n\log\left(\frac{\mu_n}{\lambda_n}\right). \tag{5.13}$$

Combining (5.12) and (5.13), we obtain

$$l_1 = O(1)|y|^{\alpha}\log\mu_n + O(1)|y|^{\alpha}\log\mu_n\log\left(\frac{\mu_n}{\lambda_n}\right)$$

$$= O(1)|y|^{\alpha}(\log\mu_n)\left(1+\log\frac{\mu_n}{\lambda_n}\right). \tag{5.14}$$

Again by Lemma 2(ii), (1.14) and (5.10) we get

$$l_2 = O(1)|y|^\alpha \int_{\pi/\lambda_n}^{\pi} \log\mu_n |K_n(u)|\,du$$

$$= O(1)|y|^\alpha \log\mu_n \int_{\pi/\lambda_n}^{\pi} \frac{\psi(n)}{u^2}\,du$$

$$= O(1)|y|^\alpha \log\mu_n\,\psi(n)\lambda_n. \tag{5.15}$$

Combining (5.9) and (5.14) we obtain

$$l_1 = l_1^{\beta/\alpha} l_1^{1-\beta/\alpha}$$

$$= O(1)|y|^\beta (\log\mu_n)^{\beta/\alpha}\left(1+\log\frac{\mu_n}{\lambda_n}\right)^{\beta/\alpha}\frac{1}{\lambda_n^{\alpha-\beta}}. \tag{5.16}$$

Again combining (5.11) and (5.15) we get

$$l_2 = l_2^{\beta/\alpha} l_2^{1-\beta/\alpha}$$

$$= \begin{cases} |y|^\beta (\log\mu_n\psi(n)\lambda_n)^{\beta/\alpha}(\psi(n)\log\lambda_n)^{1-\beta/\alpha}, & \alpha=1\\ |y|^\beta (\log\mu_n\psi(n)\lambda_n)^{\beta/\alpha}(\psi(n)\lambda_n^{1-\alpha})^{1-\beta/\alpha}, & 0<\alpha<1\end{cases}$$

$$= O(1)|y|^\beta\psi(n)\begin{cases}\lambda_n^\beta(\log\mu_n)^\beta(\log\lambda_n)^{1-\beta}, & \alpha=1\\ \lambda_n^{1-\alpha+\beta}(\log\mu_n)^{\beta/\alpha}, & 0<\alpha<1\end{cases}. \tag{5.17}$$

Hence

$$\sup_{y\neq 0}\frac{\| l_n(x+y)-l_n(x)\|_p}{|y|^\beta}$$

$$= O(1)(\log\mu_n)^{\beta/\alpha}(1+\log(\mu_n/\lambda_n))^{\beta/\alpha}\frac{1}{\lambda_n^{\alpha-\beta}}$$

$$+ O(1)\psi(n)\begin{cases}\lambda_n^\beta(\log\mu_n)^\beta(\log\lambda_n)^{1-\beta}, & \alpha=1\\ \lambda_n^{1-\alpha+\beta}(\log\mu_n)^{\beta/\alpha}, & 0<\alpha<1.\end{cases} \tag{5.18}$$

It follows from the analysis of the proof of (5.9) and (5.11) that

$$\| l_n(x)\|_p = O(1)\left(\frac{1}{\lambda_n^\alpha}\right) + O(1)\psi(n)\begin{cases}\log\lambda_n, & \alpha=1\\ \lambda_n^{1-\alpha}, & 0<\alpha<1.\end{cases} \tag{5.19}$$

Now we combine (5.18) and (5.19) to obtain the degree of approximation for $\| l_n(x)\|_{(\beta,p)}$ as

$$\| l_n(x)\|_{(\beta,p)} = O(1)\left[\frac{1}{\lambda_n^\alpha} + \psi(n)\begin{cases}\log\lambda_n, & \alpha=1\\ \lambda_n^{1-\alpha}, & 0<\alpha<1\end{cases}\right.$$

$$+ (\log\mu_n)^{\beta/\alpha}(1+\log\mu_n/\lambda_n)^{\beta/\alpha}\frac{1}{\lambda_n^{\alpha-\beta}}$$

$$\left. + \psi(n)\begin{cases}\lambda_n^\beta(\log\mu_n)^\beta(\log\lambda_n)^{1-\beta}, & \alpha=1\\ \lambda_n^{1-\alpha+\beta}(\log\mu_n)^{\beta/\alpha}, & 0<\alpha<1\end{cases}\right].$$

Hence the result follows.

Acknowledgement

The authors are grateful to the referee for useful suggestions.

References

[1] Alexitis G, *Convergence problem on orthogonal series* (New York: Pergamon Press) (1961)
[2] Das G, Ghosh T and Ray B K, Degree of approximation of functions by their Fourier series in the generalized Hölder metric. *Proc. Indian Acad. Sci.* **106** (1996) 139–153
[3] Hardy G H, Notes on some points in the integral calculus (LXVI): The arithmetic mean of Fourier constant, *Messenger Math.* **58** (1928) 50–52
[4] Hardy G H, *Divergent Series* (Oxford: Clarendon Press) (1949)
[5] Hardy G H and Littlewood J E, The allied series of Fourier series, *Proc. London Math. Sci.* **24** (1926) 211–246
[6] Mohanty R, On the absolute convergence of Hardy–Littlewood series. *J. Orissa Math. Soc.* **12–15** (1993–1996) 237–240
[7] Prössdorff S, Zur Konvergenz der Fourier reihen Hölder stetiger Funktionen, *Math. Nachr.* **69** (1975) 7–14
[8] Quade E S, Trigonometric approximation in the mean, *Duke Math. J.* **3** (1937) 529–543
[9] Zygmund A, *Trigonometric Series* (Cambridge University Press, New York) (1968), vol. I