Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 2, June 1998, pp. 95–108.

On Ramanujan asymptotic expansions and inequalities for hypergeometric functions

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MS received 26 June 1997; revised 8 December 1997

Abstract. In this paper we first discuss refinement of the Ramunujan asymptotic expansion for the classical hypergeometric functions $F(a,b;c;x)$, $c \leqslant a+b$, near the singularity $x=1$. Further, we obtain monotonous properties of the quotient of two hypergeometric functions and inequalities for certain combinations of them. Finally, we also solve an open problem of finding conditions on $a,b>0$ such that

$$2F(-a,b;a+b;r^2) < (2-r^2)F(a,b;a+b;r^2)$$

holds for all $r \in (0,1)$.

Keywords. Hypergeometric functions; gamma function; elliptic integrals.

1. Introduction and main results

The Gaussian hypergeometric series (function) is defined by

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{z^n}{n!},$$

where a,b,c are complex numbers with $c \neq 0,-1,-2,\ldots$, $(a,0)=1$ for $a \neq 0$ and $(a,n+1) = (a+n)(a,n-1) = a(a+1)\cdots(a+n)$ for $n=0,1,2,\ldots$. In the exceptional case $c=-p$, $p=0,1,2,\ldots$, the function $F(a,b;c;z)$ is defined if $a=-m$ or $b=-m$, where $m=0,1,2,\ldots$ and $m \leqslant p$. The series $F(a,b;c;z)$ converges for $|z|<1$, and $F(a,b;c;z)$ can be continued analytically into the complex plane cut at $[1,\infty)$ (see [16]). The function $F(a,b;c;z)$ has a unique role among the special functions, since it is related to many other classes of special functions such as Bessel, Chebyshev, Legendre, Gegenbauer and Jacobi polynomials. Recall that $F(a,b;c;z)$ is called *zero-balanced* when $c=a+b$. In the special cases $a=1/2$ or $-1/2$, $b=1/2$ and $c=1$, we have

$$\mathcal{K}(x) = \frac{\pi}{2} F\left(\frac{1}{2},\frac{1}{2};1;x^2\right), \quad \mathcal{E}(x) = \frac{\pi}{2} F\left(-\frac{1}{2},\frac{1}{2};1;x^2\right), \quad 0<x<1,$$

and these functions are known as *Legendre's complete elliptic integrals of the first* and *second kind*, respectively. Set $\mathcal{K}'(x) = \mathcal{K}(x')$ and $\mathcal{E}'(x) = \mathcal{E}(x')$, $x' = \sqrt{1-x^2}$.

The basic identities due to Landen ([12], # 163·01, 164·02) (see also [11], p. 12 and [16], p. 507)

$$\mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathcal{K}(r), \quad \mathcal{K}\left(\frac{1-r}{1+r}\right) = \left(\frac{1+r}{2}\right)\mathcal{K}'(r)$$

give

$$\mu(r) = 2\mu\left(\frac{2\sqrt{r}}{1+r}\right), \quad \mu(r) = \frac{\pi}{2}\frac{F(1/2,1/2;1;1-r^2)}{F(1/2,1/2;1;r^2)}. \tag{1.1}$$

Several inequalities dealing with $\mathscr{K}(r)$, $\mathscr{K}'(r)$, $\mathscr{E}(r)$, $\mathscr{E}'(r)$ and $\mu(r)$ have been derived in recent papers, and therefore it is natural to seek suitable restriction on the parameters a, b so that these inequalities are valid for the hypergeometric function $F(a,b;a+b;x^2)$. We observe that the generalized μ-function defined by

$$m(r) = \frac{F(a,b;a+b;1-r^2)}{F(a,b;a+b;r^2)} \tag{1.2}$$

has brought new attention for obtaining additional applications of Ramanujan theory in the theory of modular equations [10]. Even though, in this paper, we are not going into the details of all such possible generalizations, we will point out some such results at the end of the paper.

Throughout the paper $B(a,b)$ denotes the Euler beta function

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{1.3}$$

provided the gamma function quotient is well defined. Further, we also use the notation

$$D(a,b,c) = \frac{B(c,a+b-c)}{B(a,b)}. \tag{1.4}$$

The motivation for the present study derives from the recent development on the Gauss–Ramanujan asymptotic formula reformulated in [1, 15]. If $c = a+b$, then as $x \to 1$ with $0 < x < 1$ we have the Ramanujan asymptotic formula ([9], p. 33–34) (see also [6], [13], Theorem 19)

$$F(a,b;a+b;x) = \frac{1}{B(a,b)}[R - \log(1-x) + O((1-x)\log(1-x))], \tag{1.5}$$

where

$$R := R(a,b) = -\psi(a) - \psi(b) - 2\gamma, \ \psi(a) = \Gamma'(a)/\Gamma(a), \tag{1.6}$$

and γ denotes the Euler–Mascheroni constant defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n} k^{-1} - \log n\right) = 0{\cdot}57721566\ldots.$$

If $c < a+b$, then as $x \to 1$ with $0 < x < 1$ the asymptotic formula (see [16, p. 299]) is given by

$$F(a,b;c;x) \sim D(a,b,c)(1-x)^{c-a-b},$$

where $D = D(a,b,c)$ is defined by (1.4). Refined versions of the Ramanujan asymptotic relation (1.5) were obtained in [1, 15] ($c = a+b$) and a generalization to the case $c < a+b$ was given in [15]. We summarize these as follows.

1.7. Theorem. [1, 15] *For $a, b, c > 0$, let $B = B(a,b)$, $R = R(a,b)$ and $D = D(a,b,c)$ be as above. Then the following statements are true:*

(i) *The function* $f_1(x) \equiv (1 - F(a,b;c;x))/\log(1-x)$ *is strictly increasing from* $(0,1)$ *onto* $(ab/(a+b), 1/B)$.
(ii) *The function* $f_2(x) \equiv BF(a,b;a+b;x) + \log(1-x)$ *is strictly decreasing from* $(0,1)$ *onto* (R, B).
(iii) *The function* $f_3(x) \equiv BF(a,b;a+b;x) + (1/x)\log(1-x)$ *is increasing from* $(0,1)$ *onto* $(B-1, R)$ *if* $a \in (0,\infty)$ *and* $b \in (0, 1/a]$.
(iv) *The function* $f_3(x)$ *is decreasing from* $(0,1)$ *onto* $(R, B-1)$ *if* $a \in (1/3, \infty)$ *and* $b \geqslant (1+a)/(3a-1)$.
(v) *The function* $f_4(x) \equiv xF(a,b;a+b;x)/\log(1/(1-x))$ *is decreasing from* $(0,1)$ *onto* $(1/B, 1)$ *if* $a \in (0,\infty)$ *and* $b \in (0,1/a]$.
(vi) *The function* $f_4(x)$ *is increasing from* $(0,1)$ *to the range* $(1, 1/B)$ *if* $a \in (1/2, \infty)$ *and* $b \geqslant a/(2a-1)$.
(vii) *For* $a > b$ *and* $c < a+b$, *we have*

(1) *The function* $g(x) \equiv F(a,b;c;x)/(1-x)^{c-a-b}$, $x \in (0,1)$, *is strictly increasing with range* $(1, D)$ *if* $c > a$ *or* $c < b$.
(2) *The function* $g(x)$ *is strictly decreasing from* $(0,1)$ *onto* $(D,1)$ *if* $b < c < a$.

We observe that for $a \in (0,\infty)$ and $b \in (0,1/a]$, Theorem 1.7 gives the following precise form of the Ramanujan approximation:

$$R(a,b) < B(a,b)F(a,b;a+b;x) + \log(1-x)$$
$$< R(a,b) + \frac{1-x}{x}\log\frac{1}{1-x} \quad \text{for} \quad x \in (0,1).$$

For the proof of Theorem 1.7, an important fact used in [1, 15] was that all the coefficients of $F(a,b;a+b;x)$ should be positive. If, in the series $F(a,b;a+b;x)$, we choose a to be any complex quantity such that $\mathrm{Re}\,a > 0$ and b is the complex conjugate of a, then all the Maclaurin coefficients of the series $F(a,b;a+b;x)$ remain positive for $x \in (0,1)$. Since this case has not been handled in [1, 15], we first state our results, which are not covered in [1, 15], for this case.

1.8. Theorem. *Let a be a complex quantity such that* $\mathrm{Re}\,a > 0$ *and* $c > 0$.

(i) *The function* $g_1(x) \equiv F(a,\bar{a};c;x)/(1-x)^{c-2\mathrm{Re}\,a}$ *is increasing from* $(0,1)$ *with range* $(1, D(a,\bar{a},c))$ if $0 < c < 2\mathrm{Re}\,a$. *Increasing may be replaced by strictly increasing whenever* $c \neq a$.
(ii) *The function* $g_2(x) \equiv (F(a,\bar{a};c;x) - 1)/((1-x)^{c-2\mathrm{Re}\,a} - 1)$ *is increasing from* $(0,1)$ *onto*

$$\left(\frac{|a|^2}{c(2\mathrm{Re}\,a - c)}, D(a,\bar{a},c)\right)$$

if $0 < c < 2\mathrm{Re}\,a$. *Increasing may be replaced by strictly increasing if we also have* $c \neq a$.

1.9. Theorem. *Let* $B(a,b)$ *and* $R(a,b)$ *be defined by* (1.3) *and* (1.6), *respectively. Let* $a,b,c > 0$, *or* $b = \bar{a}$ *with a as a non-zero complex number and* $c > 0$. *Then we have*:

(i) *If* $c \geqslant \max\{0, a+b+ab\}$, *the function* $f(x) \equiv (1 - F(a,b;c;x))/\log(1-x)$ *is decreasing from* $(0,1)$ *onto* $(0, ab/c)$.

(ii) *If* $a \in \mathbb{C}$ *and* $c = 2\mathrm{Re}\,a > 0$, *the function* $f(x) \equiv (1 - F(a,\bar{a};c;x))/\log(1-x)$ *is strictly increasing from* $(0,1)$ *onto* $(|a|^2/2\mathrm{Re}\,a, 1/B(a,\bar{a}))$, *and if* $0 < c < 2\mathrm{Re}\,a$, *it is strictly increasing from* $(0,1)$ *onto* $(|a|^2/2\mathrm{Re}\,a, \infty)$.

(iii) *For* $a \in \mathbb{C}$ *such that* $\mathrm{Re}\,a > 0$, *the function* $g(x) \equiv B(a,\bar{a})F(a,\bar{a};2\mathrm{Re}\,a;x) + \log(1-x)$ *is strictly increasing from* $(0,1)$ *onto* $(R(a,\bar{a}), B(a,\bar{a}))$.

1.10 Theorem. *Define* $F(x) = F(a,b;c;x)$. *Let* $a,b,c > 0$, *or* $b = \bar{a}$ *with* a *as a non-zero complex number and* $c > 0$. *Then we have the following*:

(i) *If* $c(a+b+2) \geqslant ab - 2$, *the function* $F''(x)/F'(x)$ *is increasing for* $x \in (0,1)$.

(ii) *If* $c(a+b+1) \geqslant ab$, *the function* $F'(x)/F(x)$ *is increasing for* $x \in (0,1)$. *Strict inequality in each of the above two cases on* c *implies that the corresponding function is strictly monotone for* $x \in (0,1)$.

(iii) *If* $\alpha, \beta > 0$ *and* $c \geqslant \max\{0, a+b-\beta, \alpha ab/\beta\}$, *the function* $(1-x)^\beta F^\alpha(x)$ *is strictly decreasing for* $x \in (0,1)$.

(iv) *If* $\alpha, \beta > 0$ *and* $c \leqslant \min\{0, a+b-\beta, \alpha ab/\beta\}$, *the function* $(1-x)^\beta F^\alpha(x)$ *is strictly increasing for* $x \in (0,1)$.

The special case $c = a + b$ of the following result has been used in [7] to solve a conjecture in [2], Problem 10, p. 80 and therefore, Theorem 1.11 will be a useful extension.

1.11. Theorem. *Define* $F(x) = F(a,b;c;x)$.

(1) *Suppose that* a *and* b *are related by any one of the following*:

(i) $a, b > 0$ *and* $c \geqslant a + b$,

(ii) $a, b \in (-1, 0)$ *and* $c > 0$,

(iii) $a \in \mathbb{C}\setminus\{0\}$ *and* $0 \neq c \geqslant \max\{0, 2\mathrm{Re}\,a\}$.

Then, for $K = \max\{(ab + (a+b-c))/(c+1), ab + 2(a+b-c)\}$, *the inequality*

$$x[(1-x)F''(x) - F'(x)] < K[F(x) - 1]$$

holds for $x \in (0,1)$.

(2) *Suppose that* a *and* b *are related by any one of the following*:

(i) $a, b > 0$ *and* $0 < c < a + b$,

(ii) $a \in \mathbb{C}\setminus\{0\}$ *and* $0 < c < 2\mathrm{Re}\,a$.

Then, for $K = \min\{(ab + (a+b-c))/(c+1), ab + 2(a+b-c)\}$, *the inequality*

$$x[(1-x)F''(x) - F'(x)] \geqslant K[F(x) - 1]$$

holds for $x \in (0,1)$.

The proofs of Theorems 1.8–1.11 will be given in § 3.

The authors [2, 4, 5] proved several functional inequalities involving the hypergeometric function $F(a,b;c;x)$ with a,b,c real. The aim of such functional inequalities derived in these papers was to obtain certain generalized inequalities which were modeled after various inequalities for combinations of $\mathscr{K}(r)$ and $\mathscr{E}(r)$. In this connection, we again consider the situation where a is a complex quantity, $b = \bar{a}$ and c is a real quantity such that $c \neq 0, -1, -2, \ldots$. The case $c = 2\mathrm{Re}\,a$ with $\mathrm{Re}\,a > 0$ is more interesting because this choice covers also the behaviour of $\mathscr{K}(r)$. Thus, it is natural to look for the extension in the neighbourhood of (1/2, 1/2, 1) which deals with the case $\mathscr{K}(r)$. Now we state our next result, which gives new functional inequalities.

1.12. *Lemma. Let a be a complex quantity such that* $\mathrm{Re}\,a > 0$ *and* $F(x) = F(a, \bar{a}; 2\mathrm{Re}\,a; x)$. *Then we have the following*:

(i) *If* $3(\mathrm{Re}\,a)^2 + 6\mathrm{Re}\,a + 2 \geqslant (\mathrm{Im}\,a)^2$ *then the function* $F''(x)/F'(x)$ *is strictly increasing for* $x \in (0, 1)$.
(ii) *If* $3(\mathrm{Re}\,a)^2 + 2\mathrm{Re}\,a \geqslant (\mathrm{Im}\,a)^2$ *then the function* $F'(x)/F(x)$ *is strictly increasing for* $x \in (0, 1)$.
(iii) *For* $\alpha, \beta > 0$ *such that* $\alpha|a|^2 < 2\beta\mathrm{Re}\,a$, *the function* $(1-x)^\beta F^\alpha(x)$ *is strictly decreasing for* $x \in (0, 1)$.
(iv) *If* $3(\mathrm{Re}\,a)^2 + 2\mathrm{Re}\,a \geqslant (\mathrm{Im}\,a)^2$ *then the inequality* $F''(x)F(1-x) \leqslant F''(1-x)F(x)$ *holds for* $x \in (0, 1/2]$.
(v) *If* $\mathrm{Re}\,a > 0$ *then the inequality* $x[(1-x)F''(x) - F'(x)] < |a|^2[F(x) - 1]$ *holds for* $x \in (0, 1)$.

Proof. The cases (i)–(iii) follow from Theorem 1.10 whereas (iv) and (v) follow from Theorem 1.11. □□

1.13. Theorem. *Let a be a complex quantity such that* $0 < \mathrm{Re}\,a \leqslant 1$ *and that satisfies the condition*

$$3(\mathrm{Re}\,a)^2 + 2\mathrm{Re}\,a \geqslant (\mathrm{Im}\,a)^2.$$

If

$$m(r) = \frac{F(a, \bar{a}; 2\mathrm{Re}\,a; 1-r^2)}{F(a, \bar{a}; 2\mathrm{Re}\,a; r^2)}, \tag{1.14}$$

then the function $m(\sqrt{1-e^{-t}})$ *is decreasing and convex for* $t \in (0, \infty)$. *In other words, the function* $1/m(r)$ *is increasing and convex for* $r \in (0, 1)$. *In particular, we have the following inequality*:

$$\frac{1}{m(r)} + \frac{1}{m(s)} \geqslant \frac{2}{m(\sqrt{rs})}, \tag{1.15}$$

or, equivalently,

$$m(r) + m(s) \geqslant 2m(\sqrt{1 - \sqrt{(1-r^2)(1-s^2)}}) \tag{1.16}$$

hold for all $r, s \in (0, 1)$.

Proof. In the proof of Lemma 2.4 in [7], if we use Lemmas 2.1, 1.12 and Theorems 1.10 and 1.11, then after some computation we can easily see that the function

$$G(t) = \frac{F(\mathrm{e}^{-t})}{F(1-\mathrm{e}^{-t})}, \quad \text{with} \quad F(x) = F(a, \bar{a}; 2\mathrm{Re}\,a; x)$$

is decreasing and convex for $t \geqslant 0$ whenever a is such that $0 < \mathrm{Re}\,a \leqslant 1$ and

$$3(\mathrm{Re}\,a)^2 + 2\mathrm{Re}\,a \geqslant (\mathrm{Im}\,a)^2.$$

The remaining part of the proof of the theorem follows easily by the same arguments which we have used for the proof of Theorem 1.9 in [7]. Thus we complete the proof. □□

2. Preliminary lemmas

Before establishing the main theorems, we need to prove some technical lemmas.

2.1. *Lemma.* [15] *Suppose that the power series* $f(z)=\Sigma_{n=0}^{\infty}a_n z^n$ *and* $g(z)=\Sigma_{n=0}^{\infty}b_n z^n$ *both converge for* $|z|<1$ *and that* $a_n \in \mathbb{R}$, $b_n>0$ *for all* $n\geqslant 0$. *Then* $f(x)/g(x)$ *is increasing(strictly) (decreasing (strictly)) for* $x\in(0,1)$ *if* a_n/b_n *is increasing (strictly) (decreasing (strictly)) for* $n\geqslant 0$.

A more general form of Lemma 2.1 has been presented in [15] and is one of the crucial facts in the proof of some of our main results.

2.2. *Lemma. For* $a\in\mathbb{C}\backslash\{0\}$ *and* $A>0$, *let* $Q(n)$ *be defined by*

$$Q(n)=\frac{|(a,n)|^2(n+A)}{(2\mathrm{Re}\,a,n)(1,n)},\quad n\geqslant 1.$$

(1) *If* A, a *are related by any one of the following conditions*:

(i) $A=|a|^2$ *and* $a\in\mathbb{C}\backslash\{1\}$,

(ii) $A<|a|^2$ *and* $a\in\mathbb{C}\backslash\{0\}$,

then the sequence $\{Q(n)\}$ *is strictly increasing to the limit* $1/B(a,\bar{a})$. ($Q(n)=1$ *if* $a=1$.)

(2) *If* $A=2|a|^2/(2-|a-1|^2)$ *and* $|a-1|<\sqrt{2}$, *then the sequence* $\{Q(n)\}$ *is decreasing to the limit* $1/B(a,\bar{a})$.

Proof. Define $\phi(n)=n(|a|^2-A)+(A+1)|a|^2-2A\mathrm{Re}\,a$. From the definition of $Q(n)$ it is easy to verify by simple computation that

$$Q(n+1)>Q(n)\Leftrightarrow\phi(n)>0,\quad\text{for}\quad n\geqslant 1.$$

(i) If $A=|a|^2$ and $a\in\mathbb{C}\backslash\{1\}$ then for all $n\geqslant 1$ we have $\phi(n)=|a|^2|a-1|^2>0$, and therefore the sequence $\{Q(n)\}$ is strictly increasing for $n\geqslant 1$.

(ii) First we assume that $A<|a|^2$ and $|a-1|\geqslant\sqrt{2}$. Then the coefficient of n in the expression $\phi(n)$ is positive and therefore, for all $n\geqslant 1$, we have

$$\phi(n)\geqslant\phi(1)=2|a|^2-A(2-|a-1|^2)\geqslant 2|a|^2>0.$$

(1) Suppose that $A<|a|^2$ and $|a-1|<\sqrt{2}$. Then in this case $\phi(1)>0$ provided $A<2|a|^2/(2-|a-1|^2)$, which is clearly true because of the assumption $A<|a|^2$. From these two observations, it follows that the sequence $\{Q(n)\}$ is strictly increasing for $n\geqslant 1$.

(2) Suppose that $A=2|a|^2/(2-|a-1|^2)$ and $|a-1|<\sqrt{2}$. Then we note that the condition on A implies that $A>|a|^2$, and therefore the coefficient of n in the expression $\phi(n)$ is negative so that

$$\phi(n)\leqslant\phi(1)=2|a|^2-A(2-|a-1|^2)=0.$$

Since

$$Q(n+1)\leqslant Q(n)\Leftrightarrow\phi(n)\leqslant 0,$$

the conclusion in this case follows from the fact that $\phi(n)\leqslant 0$ for all $n\geqslant 1$.

To find the limit of the sequence, we rewrite $Q(n)$ as

$$Q(n)=\frac{|a|^2}{2\mathrm{Re}\,a}\frac{|(a+1,n-1)|^2}{(2\mathrm{Re}\,a+1,n-1)(1,n-1)}+A\frac{|(a,n)|^2}{(2\mathrm{Re}\,a,n)(1,n)}.$$

Now we recall the following well-known result that follows easily from the Stirling formula:

$$\lim_{n\to\infty}\frac{(a,n)(b,n)}{(c,n)(1,n)}=\begin{cases}\dfrac{\Gamma(c)}{\Gamma(a)\Gamma(b)} & \text{if } c+1=a+b\\ 0 & \text{if } c+1>a+b\\ \infty & \text{if } c+1<a+b\end{cases}. \tag{2.3}$$

Finally, from (2.3), we deduce that $\lim_{n\to\infty}Q(n)=1/B(a,\bar{a})$. □□

When $a=1/2$ in parts 1 (i) and 2 of Lemma 2.2, we find that for each $n\geqslant 1$,

$$\frac{1}{\pi(n+2/7)}<\left(\frac{(1/2,n)}{(1,n)}\right)^2<\frac{1}{\pi(n+1/4)},$$

which improves both sides of the well-known Wallis inequalities that appear in ([14], p. 192, 3.1.16):

$$\frac{1}{\sqrt{\pi(n+1/2)}}<\frac{(1/2,n)}{(1,n)}<\frac{1}{\sqrt{\pi n}}.$$

Thus, Lemma 2.2 generalizes and improves the Wallis inequality in terms of complex parameters introduced through hypergeometric functions. Another generalization of Wallis inequality has recently been obtained in [1, 15].

3. Proofs of main results

3.1. *Proof of Theorem* 1.8. The idea of the proof is exactly as in [15] and so we just sketch the proof. Consider the sequence $\{Q(n)\}$, where

$$Q(n)=\frac{|(a,n)|^2}{(c,n)(2\text{Re}\,a-c,n)}.$$

Using the ascending factorial notation $(a,n+1)=(a,n)(a+n)$, we may rewrite $Q(n)$ as

$$Q(n)=\left\{\frac{|a|^2}{2\text{Re}\,a}\frac{(a+1,n-1)(\bar{a}+1,n-1)}{(2\text{Re}\,a+1,n-1)(1,n-1)}\right\}$$
$$\times\frac{2\text{Re}\,a(2\text{Re}\,a+1,n-1)(1,n-1)}{c(2\text{Re}\,a-c)(c+1,n-1)(2\text{Re}\,a+1-c,n-1)}.$$

Using (2.3) we obtain the following:

$$\lim_{n\to\infty}Q(n)=\frac{1}{B(a,\bar{a})}\cdot B(c,2\text{Re}\,a-c)\equiv D(a,\bar{a},c)\quad\text{for}\quad c<2\text{Re}\,a.$$

Further, we easily get that

$$Q(n+1)\geqslant Q(n)\Leftrightarrow|c-a|^2\geqslant 0$$

and therefore $Q(n)$ is strictly increasing if and only if $|c-a|>0$. We note that $|c-a|=0$ if and only if a is a positive real number and $c=a$. Thus, the sequence $Q(n)$ is increasing to the limit $D(a,\bar{a},c)$, as $n\to\infty$. In particular, for each positive integer n,

$$\frac{|a|^2}{c(2\text{Re}\,a-c)}<Q(n)<D(a,\bar{a},c). \tag{3.2}$$

Next we consider

$$F(a,b;c;z)=\sum_{n\geqslant 0}\alpha_n z^n \quad\text{and}\quad (1-z)^{c-2\mathrm{Re}a}=\sum_{n\geqslant 0}\beta_n z^n$$

and from the above series expansions we find that $Q(n)=\alpha_n/\beta_n$. Further, in the series expansion,

$$F(a,\bar{a};c;x)-D(a,\bar{a},c)(1-x)^{c-2\mathrm{Re}a}$$
$$=\sum_{n=0}^{\infty}\left(\frac{|(a,n)|^2}{(c,n)(1,n)}-D(a,\bar{a},c)\frac{(2\mathrm{Re}a-c,n)}{(1,n)}\right)x^n$$

all coefficients are non-negative (negative) according as $Q(n)$ is increasing (strictly increasing). Therefore, the conclusion is an immediate consequence of the monotonous properties of $Q(n)$ and Lemma 2.1. Thus we obtain that if $0<c<2\mathrm{Re}a$, then the function $g_1(x)\equiv F(a,\bar{a};c;x)/(1-x)^{c-2\mathrm{Re}a}$ is increasing from $(0,1)$ onto the range $(1,D(a,\bar{a},c))$ and is strictly increasing if we also have $c\neq a$.

Part (ii) follows similarly. □□

3.3. *Proof of Theorem* 1.9. For $n\geqslant 1$, we define

$$\alpha_n=\frac{(a,n)(b,n)}{(c,n)(1,n)},\quad \beta_n=\frac{1}{n}\quad\text{and}\quad Q(n)=\frac{\alpha_n}{\beta_n}.$$

Then for $|z|<1$, we can write $F(a,b;c;z)-1=\Sigma_{n\geqslant 1}\alpha_n z^n$ and $-\log(1-z)=\Sigma_{n\geqslant 1}\beta_n z^n$. Using (2.3), we easily obtain

$$\lim_{n\to\infty}Q(n)=\begin{cases}\dfrac{1}{B(a,b)} & \text{if}\quad c=a+b\\ 0 & \text{if}\quad c>a+b\\ \infty & \text{if}\quad c<a+b\,.\end{cases}$$

Simple calculation yields that

$$Q(n+1)>Q(n)\Leftrightarrow n(a+b-c)+ab>0$$

and

$$Q(n+1)\leqslant Q(n)\Leftrightarrow n(a+b-c)+ab\leqslant 0.$$

From the above observations, it can be easily seen that the conclusion for each case follows from the method of proof of Theorem 1.8, from the respective conditions on a,b,c and from Lemma 2.1. Therefore we omit the details. □□

3.4. *Proof of Theorem* 1.10. Let $F(x)=F(a,b;c;x)$. From the definition of the hypergeometric series, we easily obtain the derivative formula for $F(x)$:

$$F'(x)=\frac{ab}{c}F(a+1,b+1;c+1;x) \tag{3.5}$$

and

$$F''(x)=\frac{(a,2)(b,2)}{(c,2)}F(a+2,b+2;c+2;x).$$

For convenience, we let $F(x)=\Sigma_{n=0}^{\infty}A_n x^n$, $F'(x)=\Sigma_{n=0}^{\infty}B_n x^n$ and $F''(x)=\Sigma_{n=0}^{\infty}C_n x^n$. Therefore, we can write

$$A_n=\frac{(a,n)(b,n)}{(c,n)(1,n)},\quad B_n=\frac{(a,n+1)(b,n+1)}{(c,n+1)(1,n)},\quad C_n=\frac{(a,n+2)(b,n+2)}{(c,n+2)(1,n)}$$

so that by a simple calculation we have

$$\frac{C_n}{B_n}=\frac{(a+n+1)(b+n+1)}{(c+n+1)}\quad\text{and}\quad\frac{B_n}{A_n}=n+a+b-c+\frac{(c-a)(c-b)}{n+c}. \tag{3.6}$$

(i) Now we assume that either a,b,c are all positive real numbers, or a is a non-zero complex number such that $b=\bar{a}$ and $c>0$. Therefore by a simple computation we can easily find that

$$\frac{C_n}{B_n}\leqslant\frac{C_{n+1}}{B_{n+1}}\Leftrightarrow\phi(n)\leqslant 0, \tag{3.7}$$

where

$$\phi(n)=n^2+(2c+3)n+c(a+b+3)+2-ab. \tag{3.8}$$

We remark that the inequality (3.7) continues to hold if we replace both inequalities in (3.7) by strict inequalities, respectively. Since $c>0$, the function $\phi(n)$ is increasing for $n\geqslant 0$, and therefore, it follows that

$$\phi(n)\geqslant\phi(0)=c(a+b+3)+2-ab.$$

This observation shows that, if a,b,c are related by the condition $c(a+b+3)+2-ab\geqslant 0$ then the sequence $\{C_n/B_n\}$ is increasing for all $n\geqslant 0$, and hence, by Lemma 2.1, it follows that the function $F''(x)/F'(x)$ is increasing for $x\in(0,1)$.

(ii) Again, by Lemma 2.1, it suffices to show that the ratio of the coefficients B_n/A_n is strictly decreasing for $n\geqslant 0$. By a simple computation, we note that the inequality

$$\frac{B_n}{A_n}<\frac{B_{n+1}}{A_{n+1}}$$

is equivalent to

$$(n+c)(n+c+1)>(c-a)(c-b).$$

Since $c>0$, the last inequality holds for all $n\geqslant 0$ if it holds for $n=0$. Putting $n=0$ in the last inequality we have $c(c+1)>(c-a)(c-b)$, which is equivalent to $c(1+a+b)>ab$. Therefore the conclusion follows from Lemma 2.1 if a,b,c are related by $c(1+a+b)>ab$ and we complete the proof.

(iii) Consider the function $f(x)=(1-x)^{\beta}(F(a,b;c;x))^{\alpha}$. Then using the derivative formula for $F(a,b;c;x)$, we have

$$f'(x)=(1-x)^{\beta-1}(F(a,b;c;x))^{\alpha-1}\times\left[-\beta F(a,b;c;x)+\frac{\alpha ab}{c}(1-x)F(a+1,b+1;c+1;x)\right].$$

Using the series expansion for the square bracketed term of the above expression, we can write

$$f'(x) = (1-x)^{\beta-1}(F(a,b;c;x))^{\alpha-1}$$
$$\times \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n+1)(1,n)} [\alpha ab - \beta c + (a+b-c-\beta)n]x^n.$$

The conditions on c, i.e., $\alpha, \beta, a, b, c > 0$ and $\alpha ab < \beta c$, imply that

$$\alpha ab - \beta c + (a+b-c+\beta)n \leqslant \alpha ab - \beta c \leqslant 0$$

for all $n \geqslant 0$. This observation shows that $f'(x) < 0$ for $x \in (0,1)$ and therefore, the function f is decreasing for $x \in (0,1)$.
(iv) Follows from part (iii) and from the fact that the given condition on c implies that

$$\alpha ab - \beta c + (a+b-c-\beta)n \geqslant 0$$

for all $n \geqslant 0$. □□

3.9. *Proof of Theorem* 1.11. (i) Define $F(x) = F(a,b;c;x)$, where $a,b,c > 0$ and $c \geqslant a+b$. Then from the series expansions for $F'(x)$ and $F''(x)$, we easily compute that

$$(1-x)F''(x) - F'(x)$$
$$= \sum_{n=0}^{\infty} \frac{(a,n+1)(b,n+1)}{(c,n+2)(1,n)} [n(a+b-c) + a + b + ab - c]x^n,$$

so that using the series expansion for $F(x)$ and simplification, we find that the inequality

$$x[(1-x)F''(x) - F'(x)] < K[F(x) - 1] \tag{3.10}$$

is equivalent to the inequality

$$\sum_{n=1}^{\infty} \frac{(a,n)(b,n)}{(c,n+1)(1,n)} \{\phi(n)\} x^n > 0,$$

where

$$\phi(n) = n^2(c-a-b) + n(K-ab) + Kc,$$

and K is defined in Theorem 1.11. We divide the proof into two parts.

Let $c = a+b$. Then in this case $K = ab$ so that for all $n \geqslant 1$ we have $\phi(n) = cab = (a+b)ab > 0$ and therefore (3.10) holds since $a, b > 0$.

Next, we assume $c > a+b$. Clearly for large n, $\phi(n) > 0$. Now, for all $n \geqslant 1$, the condition on c and K gives

$$\phi'(n) = 2n(c-a-b) + (K-ab) \geqslant 2(c-a-b) + (K-ab) \geqslant 0,$$

so that $\phi(n)$ is an increasing function of n. Therefore, using the condition on K, we deduce that

$$\phi(n) \geqslant \phi(1) = K(1+c) + c - a - b - ab \geqslant 0$$

for $n \geqslant 1$. This observation shows that the inequality (3.10) holds under the given condition on K.

The other parts may be checked similarly. □□

4. Concluding remarks

In this section we first state a few preliminary results in the form of a proposition which extends Theorem 1.7 in ([3], eq. (18)), Theorems 1.1(2) and 2.1(6) in [4].

4.1. PROPOSITION

(i) *For* $a,b>0$ *and* $x\in(0,1)$, *we have*

$$F(a,b;2b;x)<(1+x)^{2a}F\left(a,a-b+\frac{1}{2};b+\frac{1}{2};x^2\right). \tag{4.2}$$

(ii) *For* $a\in\mathbb{R}, b,c\in(0,\infty)$ *and* $x\in(0,1)$,

$$F(a,b;c;x)+F(-a,b;c;x)\geqslant 2.$$

(iii) *For* $a,b,c\in(0,\infty)$ *and* $x\in(0,1)$ *the function*

$$\frac{F(a,b;c;x)-F(-a,b;c;x)}{x}$$

is strictly increasing and convex on $(0,1)$ *and has the limit* $2ab/c$ *as* $x\to 0$.

Proof. (i) Recall the Gauss transformation

$$F\left(a,b;2b;\frac{4x}{(1+x)^2}\right)=(1+x)^{2a}F\left(a,a-b+\frac{1}{2};b+\frac{1}{2};x^2\right),\quad x\in(0,1), \tag{4.3}$$

where $2b\neq 0,-1,-2,\ldots$ (see [8], p. 111, eq. (5)) and also ([9], Entry 3 in ch. 11, p. 50). Proof of (4.2) follows from (4.3) since $x<4x/(1+x)^2$ and since the function $F(a,b;2b;x^2)$ is increasing on $[0,1)$.

(ii) Suppose that $a\in\mathbb{R}$ and $b,c\in(0,\infty)$. Now, we can write

$$F(a,b;c;x)+F(-a,b;c;x)-2=\sum_{n=2}^{\infty}\frac{(b,n)}{(c,n)(1,n)}[(a,n)+(-a,n)]x^n,\quad x\in(0,1). \tag{4.4}$$

Using the triangle inequality we see that

$$|(-a,n)|=|a||-a+1|\cdots|-a+n-1|\leqslant|a|(|a|+1)\cdots(|a|+n-1)=(|a|,n).$$

This observation shows that all the coefficients of the power series of the function (4.4) are positive and therefore the conclusion follows.

(iii) Suppose that $a,b,c\in(0,\infty)$. Now, we can write

$$\frac{F(a,b;c;x)-F(-a,b;c;x)}{x}=\frac{2ab}{c}+\sum_{n=2}^{\infty}\frac{(b,n)}{(c,n)(1,n)}[(a,n)-(-a,n)]x^{n-1}. \tag{4.5}$$

As in the proof of part (ii), the triangle inequality immediately gives $|(-a,n)|\leqslant(|a|,n)=(a,n)$ so that $(a,n)-(-a,n)>0$ for $a>0$ and all $n\geqslant 2$. Thus, all the coefficients of the power series of the function (4.5) are positive and the constant term is $2ab/c$ and the conclusion follows. □□

In ([5], Problem 9, p. 80), the authors state another problem which is based on the inequality

$$2\mathscr{E}(r) < (2-r^2)\mathscr{K}(r), \quad 0<r<1.$$

4.6. *Problem.* Is it true that for $a, b > 0$

$$2F(-a, b; a+b; r^2) < (2-r^2)F(a, b; a+b; r^2)?$$

Clearly, the above inequality is not true for a close to zero. The answer to Problem 4.6 will be divided into three different parts which are as follows.

4.7. Theorem. *Let $a \in (0,1]$ and $b>0$. Then a necessary and sufficient condition for*

$$2F(-a, b; a+b; r^2) < (2-r^2)F(a, b; a+b; r^2), \quad r \in (0,1), \tag{4.8}$$

is that $a \in (1/4, 1]$ and $b \in [a/(4a-1), \infty)$.

Proof. For convenience, we let $r^2 = t > 0$. Writing (4.8) as

$$(1-t/2)F(a, b; a+b; t) - F(-a, b; a+b; t) > 0$$

and then using the series expansion for $F(a, b; c; t)$, we easily see that inequality (4.8) is equivalent to

$$B_1 t + \sum_{n=2}^{\infty} B_n t^n > 0, \tag{4.9}$$

where $B_1 = (2ab/(a+b)) - 1/2$ and, for $n \geqslant 2$,

$$B_n = \frac{(a, n-1)(b, n-1)}{2(a+b, n)(1, n)}[(n-1)(n-2) + (n-2)(a+b) + 2ab] - \frac{(-a, n)(b, n)}{(a+b, n)(1, n)}.$$

Suppose that $a \in (0,1)$. Then the coefficient of t^n for $n \geqslant 2$ is clearly positive and therefore (4.9) holds if $B_1 \geqslant 0$, which is equivalent to $b(4a-1) \geqslant a$. Since $0 < a \leqslant 1$, the last inequality requires that a has to be greater than 1/4 so that the condition on b becomes $b \geqslant a/(4a-1)$. This proves the sufficient part.

Next we prove the necessity part. In this case, dividing the inequality (4.9) by t and then taking the limit as $t \to 0$, it follows that the first coefficient B_1 has to be non-negative, i.e., $b(4a-1) \geqslant a$ is a necessary condition for the truth of the inequality (4.8). □□

Our next theorem considers the case $a > 1$.

4.10. Theorem. *Let $a \in (1, \infty)$ and $b > 0$. If a, b are related by any one of the following*

(i) *$a \in (1, (3+\sqrt{5})/4)$ and $b \in [a/(4a-1), \infty)$,*
(ii) *$a \in [(3+\sqrt{5})/4, \infty)$ and $b \in [a-1, \infty)$,*

then the inequality (4.8) holds.

Proof. The idea and the notation are as in the proof of Theorem 4.7. Let $a > 1$ and B_n be defined as in Theorem 4.7. Then after some computation we find that $B_n > 0$ for $n \geqslant 2$ is equivalent to the inequality

$$H(a) > 0, \tag{4.11}$$

where

$$H(a) = 2(a-1)(b+n-1)[(a+1,n-2) - (-a+2,n-2)] + n(a+1,n-1)(n+b-a-1).$$

(i) Suppose that the assumption (i) holds. Then the inequality

$$\frac{a}{4a-1} > a-1$$

holds and in particular $B_1 > 0$ and $b > a-1$. It is trivial to see that for $a > 1/2$ and $n \geqslant 2$, the inequality

$$(a+1,n-2) - (-a+2,n-2) > 0$$

holds. The condition $b > a-1$ and the fact that $a > 1$ imply that (4.11) holds so that $B_n > 0$ for all $n \geqslant 2$. This observation shows that if a and b are related by (i) then the inequality (4.8) holds.

(ii) Assume that (ii) holds. Then in this case we have

$$\frac{a}{4a-1} \leqslant a-1$$

and since $b \geqslant a-1$, we see that $B_n > 0$ for $n \geqslant 1$ and the conclusion follows similarly.

Acknowledgements

The authors are thankful to the referee for his useful comments. This work was completed during the second author (SP)'s visit to The Institute of Mathematical Sciences, Chennai and he would also like to thank Prof. Matti Vuorinen for helpful discussion during his visit to the University of Helsinki. This work was partially supported by National Board for Higher Mathematics, India.

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