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# $L^p$ inequalities for polynomials with restricted zeros

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**Abstract.** Let $P(z)$ be a polynomial of degree $n$ which does not vanish in the disk $|z| < k$. It has been proved that for each $p > 0$ and $k \geqslant 1$,

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} |P^{(s)}(e^{i\theta})|^p d\theta\right\}^{1/p} \leqslant n(n-1)\cdots(n-s+1)B_p \times \left\{\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{1/p},$$

where $B_p = \left\{\frac{1}{2\pi}\int_0^{2\pi} |k^s + e^{i\alpha}|^p d\alpha\right\}^{-1/p}$ and $P^{(s)}(z)$ is the $s$th derivative of $P(z)$. This result generalizes well-known inequality due to De Bruijn. As $p \to \infty$, it gives an inequality due to Govil and Rahman which as a special case gives a result conjectured by Erdös and first proved by Lax.

**Keywords.** Derivative of a polynomial; Zygmund's inequality; $L^p$ norm of a polynomial.

## 1. Introduction and statement of results

Let $P(z)$ be a polynomial of degree $n$ and $P'(z)$ its derivative, then for each $p \geqslant 1$

$$\left\{\int_0^{2\pi} |P'(e^{i\theta})|^p d\theta\right\}^{1/p} \leqslant n\left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{1/p}. \tag{1}$$

Inequality (1) is due to Zygmund [6] who proved it for all trigonometric polynomials of degree $n$ and not only for those which are of the form $P(e^{i\theta})$. Arestov [1] proved that (1) remains true for $0 < p < 1$ as well.

Inequality (1) can be sharpened if we restrict ourselves to the class of polynomials having no zeros in $|z| < k$, where $k \geqslant 1$. In the case where $k = 1$, it was found out by De Bruijn [2] that (1) can be replaced by

$$\left\{\int_0^{2\pi} |P'(e^{i\theta})|^p d\theta\right\}^{1/p} \leqslant nC_p\left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{1/p}, \tag{2}$$

where

$$C_p = \left\{\frac{1}{2\pi}\int_0^{2\pi} |1 + e^{i\alpha}|^p d\alpha\right\}^{-1/p}.$$

For $k \geqslant 1$ Govil and Rahman [4] have shown that if $P(z)$ does not vanish in $|z| < k$, then for every $p \geqslant 1$,

$$\left\{\int_0^{2\pi} |P'(e^{i\theta})|^p d\theta\right\}^{1/p} \leqslant nE_p\left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{1/p}, \tag{3}$$

where

$$E_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |k + e^{i\alpha}|^p d\alpha \right\}^{-1/p}.$$

Recently, Rahman and Schmeisser [5] have proved that (2) remains true for $0 < p < 1$ also. In case $p = 2$, Dewan and Govil [3] have extended (3) to the $s$th derivative of $P(z)$ by showing that

$$\int_0^{2\pi} |P^{(s)}(e^{i\theta})|^2 d\theta \leqslant \frac{n^2(n-1)^2 \cdots (n-s+1)^2}{1+k^{2s}} \int_0^{2\pi} |P(e^{i\theta})|^2 d\theta, \tag{4}$$

where $P(z)$ is a polynomial of degree $n$ which does not vanish in $|z| < k$, $k \geqslant 1$.

In this paper we extend inequality (3) to the $s$th derivative of a polynomial having no zeros in $|z| < k$ where $k \geqslant 1$ and thereby present a generalization of (4). In particular we show that (3) remains true for $0 < p < 1$ as well. More precisely we prove the following theorem.

**Theorem.** *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n which does not vanish in $|z| < k$, $k \geqslant 1$. Then for every $p > 0$,*

$$\left\{ \int_0^{2\pi} |P^{(s)}(e^{i\theta})|^p d\theta \right\}^{1/p} \leqslant n(n-1) \cdots (n-s+1) B_p \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{1/p}, \tag{5}$$

*where*

$$B_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |k^s + e^{i\alpha}|^p d\alpha \right\}^{-1/p}.$$

*Remark* 1. Inequality (4) of Dewan and Govil [3] is a special case of our theorem, when $p = 2$.

*Remark* 2. Another special case of our theorem is a result due to Govil and Rahman [5, Theorem 4], which follows from inequality (5) by letting $p \to \infty$ and noting that

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{1/p} = \max_{|z|=1} |P(z)|.$$

## 2. Lemmas

For $\delta = (\delta_0, \delta_1, \ldots, \delta_n) \in c^{n+1}$ and $P(z) = \sum_{j=0}^{n} a_j z^j$, we define

$$\Lambda_\delta P(z) = \sum_{j=0}^{n} \delta_j a_j z^j.$$

The operator $\Lambda_\delta$ is said to be admissible if it preserves one of the following properties:

(i) $P(z)$ has all its zeros in $\{z \in c : |z| \leqslant 1\}$

(ii) $P(z)$ has all its zeros in $\{z \in c : |z| \geqslant 1\}$.

*Lemma* 1 [1, *Theorem* 4]. *Let $\phi(x) = \psi(\log x)$, where $\psi$ is a convex non-decreasing function on R. Then for all polynomials of degree at most n and each admissible operator $\Lambda$,*

$$\int_0^{2\pi} \phi(|\Lambda_\delta P(e^{i\theta})|) d\theta \leqslant \int_0^{2\pi} \phi(c(\delta, n) |P(e^{i\theta})|) d\theta, \tag{6}$$

*where $C(\delta, n) = \max\{|\delta_0|, |\delta_n|\}$.*

In particular, lemma 1 applies with $\phi: x \mapsto x^p$ for every $p \in (0, \infty)$ and with $\phi: x \mapsto \log x$ as well. Therefore, we have

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} |\Lambda_\delta P(e^{i\theta})|^p d\theta\right\}^{1/p} \leqslant c(\delta, n)\left\{\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{1/p}, \tag{7}$$

where $0 < p < 1$.

We also need the following Lemma.

*Lemma* 2. *If* $P(z) = \sum_{j=0}^n a_j z^j$ *is a polynomial of degree n which does not vanish in* $|z| < k$, $k \geqslant 1$ *and if* $Q(z) = z^n \overline{P(1/\bar{z})}$, *then*

$$k^s |P^{(s)}(z)| \leqslant |Q^{(s)}(z)| \quad \text{for} \quad |z| = 1, 1 \leqslant s \leqslant n. \tag{8}$$

This lemma is implicit in [4, p. 511]; however for the sake of completeness here we present a simple and an independent proof of this result.

*Proof of Lemma* 2. By hypothesis all the zeros of the polynomial

$$P(z) = a_n z^n + \cdots + a_s z^s + \cdots + a_1 z + a_0$$

lie in $|z| \geqslant k \geqslant 1$. Therefore, all the zeros of the polynomial

$$F(z) = P(kz) = a_n k^n z^n + \cdots + a_s k^s z^s + \cdots + a_1 kz + a_0 \tag{9}$$

lie in $|z| \geqslant 1$. If we take

$$\begin{aligned} G(z) &= z^n \overline{F(1/\bar{z})} = z^n \overline{P(k/\bar{z})} \\ &= \bar{a}_n k^n + \bar{a}_{n-1} k^{n-1} z + \cdots + \bar{a}_{n-s} k^{n-s} z^s + \cdots + \bar{a}_1 k z^{n-1} + \bar{a}_0 z^n, \end{aligned} \tag{10}$$

then all the zeros of $G(z)$ lie in $|z| \leqslant 1$. Thus the function $f(z) = G(z)/F(z)$ is analytic in $|z| \leqslant 1$ and

$$|f(z)| = |G(z)/F(z)| = |G(z)|/|F(z)| = 1 \quad \textit{for} \quad |z| = 1.$$

By the maximum modulus principle, we have

$$|G(z)/F(z)| = |f(z)| \leqslant 1 \quad \text{for} \quad |z| \leqslant 1. \tag{11}$$

Replacing $z$ by $1/\bar{z}$ in (11), we conclude that

$$|F(z)| \leqslant |G(z)| \quad \text{for} \quad |z| \geqslant 1. \tag{12}$$

Hence for every real or complex number $\lambda$ with $|\lambda| > 1$, the polynomial $T(z) = F(z) + \lambda G(z)$ has all its zeros in $|z| \leqslant 1$, because if this is not true, then there is a point $z = z_0$ with $|z_0| > 1$ such that $T(z_0) = 0$. This gives

$$0 = T(z_0) = F(z_0) + \lambda G(z_0), \quad \text{where } |z_0| > 1.$$

This implies

$$|F(z_0)| = |\lambda||G(z_0)| > |G(z_0)|, \quad \text{where } |z_0| > 1,$$

which is a contradiction to (12). Thus all the zeros of $F(z) + \lambda G(z)$ lie in $|z| \leqslant 1$ for every complex number $\lambda$ with $|\lambda| > 1$. By Gauss–Lucas theorem, it follows that all the zeros of $F^{(s)}(z) + \lambda G^{(s)}(z)$ lie in $|z| \leqslant 1$. This implies that

$$|F^{(s)}(z)| \leqslant |G^{(s)}(z)| \quad \text{for } |z| \geqslant 1. \tag{13}$$

Differentiating (10) $s$ times, $1 \leqslant s \leqslant n$, we get

$$G^{(s)}(z) = n(n-1)\cdots(n-s+1)\bar{a}_0 z^{n-s} + \cdots + s!\bar{a}_{n-s}k^{n-s}$$

and by Gauss–Lucas theorem all the zeros of $G^{(s)}(z)$ lie in $|z| \leqslant 1$. If

$$H(z) = z^{n-s}\overline{G^{(s)}(1/\bar{z})} = k^{n-s}a_{n-s}s!z^{n-s} + \cdots + n(n-1)\cdots(n-s+1)a_0,$$

then all the zeros of $H(z)$ lie in $|z| \geqslant 1$ and we see that

$$H(z/k) = a_{n-s}s!z^{n-s} + \cdots + n(n-1)\cdots(n-s+1)a_0$$
$$= z^{n-s}\overline{Q^{(s)}(1/\bar{z})},$$

where

$$Q(z) = z^n\overline{P(1/\bar{z})} = \bar{a}_0 z^n + \cdots + \bar{a}_{n-s}z^s + \cdots + \bar{a}_n.$$

Since $F^{(s)}(z) = k^s P^{(s)}(kz)$, from (13) we get

$$k^s|P^{(s)}(kz)| \leqslant |G^{(s)}(z)| = |z^{n-s}\overline{G(1/\bar{z})}| = |H(z)| \quad \text{for } |z| = 1.$$

By the maximum modulus principle,

$$k^s|P^{(s)}(kz)| \leqslant |H(z)| \quad \text{for } |z| \leqslant 1. \tag{14}$$

Taking in particular $z = e^{i\theta}/k$, $0 \leqslant \theta < 2\pi$, so that $|z| = 1/k \leqslant 1$ and from (14) we get

$$k^s|P^{(s)}(e^{i\theta})| \leqslant |H(e^{i\theta}/k)|.$$

Equivalently

$$k^s|P^{(s)}(z)| \leqslant |Q^{(s)}(z)| \quad \text{for } |z| = 1,$$

which is (8) and this completes the proof of the lemma.

## 3. Proof of the Theorem

First we supose that $k = 1$. Since $P(z)$ is a polynomial of degree $n - s + 1$, by repeated application of inequality (1), we have for every $p > 0$,

$$\left\{\int_0^{2\pi} |P^{(s)}(e^{i\theta})|^p d\theta\right\}^{1/p} \leqslant (n-s+1)\left\{\int_0^{2\pi} |P^{(s-1)}(e^{i\theta})|^p d\theta\right\}^{1/p}$$
$$\leqslant (n-s+1)(n-s+2)\left\{\int_0^{2\pi} |P^{(s-2)}(e^{i\theta})|^p d\theta\right\}^{1/p}$$
$$\cdots$$
$$\leqslant (n-s+1)\cdots(n-1)\left\{\int_0^{2\pi} |P'(e^{i\theta})|^p d\theta\right\}^{1/p}. \tag{15}$$

By hypothesis, $P(z)$ has all its zeros in $|z| \geqslant k = 1$. Using inequality (2) on the right hand side of (15), we get

$$\left\{\int_0^{2\pi} |P^{(s)}(e^{i\theta})|^p d\theta\right\}^{1/p} \leqslant \frac{n(n-1)\cdots(n-s+1)}{\left\{(1/2\pi)\int_0^{2\pi} |1 - e^{i\alpha}|^p d\alpha\right\}^{1/p}}\left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{1/p}$$

for every $p > 0$. This proves (5) for $k = 1$. Henceforth, we suppose $k > 1$. Since $P(z)$ has all its zeros in $|z| \geqslant k > 1$, the polynomial $Q(z) = z^n\overline{P(1/\bar{z})}$ has all its zeros in

$|z| \leqslant 1/k < 1$, which implies by Gauss–Lucas theorem that all the zeros of $Q^{(s)}(z)$ lie in $|z| < 1$. By lemma 2, we have

$$|P^{(s)}(z)| \leqslant \frac{1}{k^s}|Q^{(s)}(z)| < |Q^{(s)}(z)| \quad \text{for } |z| = 1. \tag{16}$$

If $H(z) = z^{n-s}\overline{Q^{(s)}(1/\bar{z})}$, then all the zeros of $H(z)$ lie in $|z| > 1$ and from (16), we have for $|z| = 1$

$$|e^{i\alpha}z^s P^{(s)}(z)| = |P^{(s)}(z)| < |Q^{(s)}(z)| = |z^{n-s}\overline{Q^{(s)}(1/\bar{z})}|$$
$$= |H(z)|.$$

Using Rouche's theorem, it follows that all the zeros of

$$\begin{aligned}\Lambda_\delta P(z) &= H(z) + e^{i\alpha}z^s P^{(s)}(z)\\ &= n(n-1)\cdots(n-s+1)a_0 + (n-1)\cdots(n-s)a_1 z + \cdots + s!a_{n-s}z^{n-s}\\ &\quad + e^{i\alpha}z^s\{n(n-1)\cdots(n-s+1)a_n z^{n-s} + (n-1)\cdots(n-s)a_{n-1}z^{n-s-1}\\ &\quad + \cdots + s!a_s\}\\ &= n(n-1)\cdots(n-s+1)a_0 + (n-1)\cdots(n-s)a_1 z\\ &\quad + \cdots + n(n-1)\cdots(n-s+1)e^{i\alpha}a_n z^n\end{aligned}$$

lie in $|z| > 1$. This shows that the operator $\Lambda_\delta$ is admissible. Hence by (7), we have for every $p > 0$,

$$\int_0^{2\pi} |H(e^{i\theta}) + e^{i\alpha}e^{is\theta}p^{(s)}(e^{i\theta})|^p d\theta \leqslant n(n-1)\cdots(n-s+1)\int_0^{2\pi} |P(e^{i\theta})|^p d\theta. \tag{17}$$

Integrating both sides of (17) with respect to $\alpha$ from 0 to $2\pi$, we get for $p > 0$,

$$\int_0^{2\pi}\int_0^{2\pi} |H(e^{i\theta}) + e^{i\alpha}e^{is\theta}p^{(s)}(e^{i\theta})|^p d\alpha d\theta \leqslant n(n-1)\cdots(n-s+1)$$
$$\times 2\pi\int_0^{2\pi} |P(e^{i\theta})|^p d\theta. \tag{18}$$

Now for every real $\alpha$ and $A \geqslant B \geqslant 1$, it can be easily verified that $|A + e^{i\alpha}| \geqslant |B + e^{i\alpha}|$, which implies for every $p > 0$,

$$\int_0^{2\pi} |A + e^{i\alpha}|^p d\alpha \geqslant \int_0^{2\pi} |B + e^{i\alpha}|^p d\alpha. \tag{19}$$

We take $A = |H(e^{i\theta})/P^{(s)}(e^{i\theta})|$ and $B = k^s > 1$, then by (8), $A \geqslant B \geqslant 1$ and we get with the help of (19) that

$$\begin{aligned}\int_0^{2\pi} |H(e^{i\theta}) + e^{i\alpha}e^{is\theta}p^{(s)}(e^{i\theta})|^p d\alpha &= |P^{(s)}(e^{i\theta})|^p \int_0^{2\pi}\left|\frac{H(e^{i\theta})}{e^{is\theta}P^{(s)}(e^{i\theta})} + e^{i\alpha}\right|^p d\alpha\\ &= |P^{(s)}(e^{i\theta})|^p \int_0^{2\pi}\left|\left|\frac{H(e^{i\theta})}{P^{(s)}(e^{i\theta})}\right| + e^{i\alpha}\right|^p d\alpha\\ &\geqslant |P^{(s)}(e^{i\theta})|^p \int_0^{2\pi} |k^s + e^{i\alpha}|^p d\alpha.\end{aligned}$$

Using this in (18), we conclude for each $p > 0$,

$$\int_0^{2\pi} |P^{(s)}(e^{i\theta})|^p d\theta \leqslant \frac{n(n-1)\cdots(n-s+1)}{(1/2\pi)\int_0^{2\pi} |k^s + e^{i\alpha}|^p d\alpha} \int_0^{2\pi} |P(e^{i\theta})|^p d\theta,$$

from which (5) follows immediately and this completes the proof of the theorem.

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