**L^p** inequalities for polynomials with restricted zeros

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**Abstract.** Let \( P(z) \) be a polynomial of degree \( n \) which does not vanish in the disk \( |z| < k \). It has been proved that for each \( p > 0 \) and \( k \geq 1 \),

\[
\left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |P^{(s)}(e^{i\theta})|^p d\theta \right\}^{1/p} \leq n(n-1) \cdots (n-s+1) B_p \\
\times \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{1/p},
\]

where \( B_p = \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |k^s + e^{i\theta}|^p d\theta \right\}^{-1/p} \) and \( P^{(s)}(z) \) is the \( s \)th derivative of \( P(z) \). This result generalizes well-known inequality due to De Bruijn. As \( p \to \infty \), it gives an inequality due to Govil and Rahman which as a special case gives a result conjectured by Erdős and first proved by Lax.

**Keywords.** Derivative of a polynomial; Zygmund’s inequality; \( L^p \) norm of a polynomial.

1. **Introduction and statement of results**

Let \( P(z) \) be a polynomial of degree \( n \) and \( P'(z) \) its derivative, then for each \( p \geq 1 \)

\[
\left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |P'(e^{i\theta})|^p d\theta \right\}^{1/p} \leq n \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{1/p}.
\]

Inequality (1) is due to Zygmund [6] who proved it for all trigonometric polynomials of degree \( n \) and not only for those which are of the form \( P(e^{i\theta}) \). Arestov [1] proved that (1) remains true for \( 0 < p < 1 \) as well.

Inequality (1) can be sharpened if we restrict ourselves to the class of polynomials having no zeros in \( |z| < k \), where \( k \geq 1 \). In the case where \( k = 1 \), it was found out by De Bruijn [2] that (1) can be replaced by

\[
\left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |P'(e^{i\theta})|^p d\theta \right\}^{1/p} \leq n C_p \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{1/p},
\]

where

\[
C_p = \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |1 + e^{i\varphi}|^p d\varphi \right\}^{-1/p}.
\]

For \( k > 1 \) Govil and Rahman [4] have shown that if \( P(z) \) does not vanish in \( |z| < k \), then for every \( p \geq 1 \),

\[
\left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |P'(e^{i\theta})|^p d\theta \right\}^{1/p} \leq n E_p \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{1/p},
\]

where

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\]

where

\[
F_p = \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |1 + e^{i\varphi}|^p d\varphi \right\}^{-1/p}.
\]
where

\[ E_p = \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} \frac{|k + e^{i\alpha}|^p}{|k + e^{i\alpha}|^p} d\alpha \right\}^{-1/p}. \]

Recently, Rahman and Schmeisser \[5\] have proved that (2) remains true for \(0 < p < 1\) also. In case \(p = 2\), Dewan and Govil \[3\] have extended (3) to the \(s\)th derivative of \(P(z)\) by showing that

\[
\int_{0}^{2\pi} |P^{(s)}(e^{i\theta})|^2 d\theta \leq \frac{n^2(n-1)^2 \cdots (n-s+1)^2}{1 + k^{2s}} \int_{0}^{2\pi} |P(e^{i\theta})|^2 d\theta,
\]

where \(P(z)\) is a polynomial of degree \(n\) which does not vanish in \(|z| < k\), \(k \geq 1\).

In this paper we extend inequality (3) to the \(s\)th derivative of a polynomial having no zeros in \(|z| < k\) where \(k \geq 1\) and thereby present a generalization of (4). In particular we show that (3) remains true for \(0 < p < 1\) as well. More precisely we prove the following theorem.

**Theorem.** Let \(P(z) = \sum_{j=0}^{n} a_j z^j\) be a polynomial of degree \(n\) which does not vanish in \(|z| < k\), \(k \geq 1\). Then for every \(p > 0\),

\[
\left\{ \int_{0}^{2\pi} |P^{(s)}(e^{i\theta})|^p d\theta \right\}^{1/p} \leq \frac{n(n-1) \cdots (n-s+1) B_p}{1 + k^{2s}} \left\{ \int_{0}^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{1/p},
\]

where

\[ B_p = \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} \frac{|k + e^{i\alpha}|^p}{|k + e^{i\alpha}|^p} d\alpha \right\}^{-1/p}.
\]

**Remark 1.** Inequality (4) of Dewan and Govil \[3\] is a special case of our theorem, when \(p = 2\).

**Remark 2.** Another special case of our theorem is a result due to Govil and Rahman \[5, Theorem 4\], which follows from inequality (5) by letting \(p \to \infty\) and noting that

\[
\left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{1/p} = \max_{|z|=1} |P(z)|.
\]

### 2. Lemmas

For \(\delta = (\delta_0, \delta_1, \ldots, \delta_n) \in \mathbb{C}^{n+1}\) and \(P(z) = \sum_{j=0}^{n} a_j z^j\), we define

\[ \Lambda_{\delta} P(z) = \sum_{j=0}^{n} \delta_j a_j z^j. \]

The operator \(\Lambda_{\delta}\) is said to be admissible if it preserves one of the following properties:

(i) \(P(z)\) has all its zeros in \(\{z \in \mathbb{C} : |z| \leq 1\}\).

(ii) \(P(z)\) has all its zeros in \(\{z \in \mathbb{C} : |z| \geq 1\}\).

**Lemma 1** \[1, Theorem 4\]. Let \(\psi(x) = \psi(\log x)\), where \(\psi\) is a convex non-decreasing function on \(\mathbb{R}\). Then for all polynomials of degree at most \(n\) and each admissible operator \(\Lambda_{\delta}\),

\[
\int_{0}^{2\pi} \phi(|\Lambda_{\delta} P(e^{i\theta})|) d\theta \leq \int_{0}^{2\pi} \phi(c(\delta, n)|P(e^{i\theta})|) d\theta,
\]

where \(C(\delta, n) = \max\{|\delta_0|, |\delta_n|\}\).
In particular, lemma 1 applies with $\phi : x \mapsto x^p$ for every $p \in (0, \infty)$ and with $\phi : x \mapsto \log x$ as well. Therefore, we have

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta \right\}^{1/p} \leq c(\delta, n) \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta \right\}^{1/p},$$

where $0 < p < 1$.

We also need the following Lemma.

**Lemma 2.** If $P(z) = \sum_{i=0}^n a_i z^i$ is a polynomial of degree $n$ which does not vanish in $|z| < k$, $k \geq 1$ and if $Q(z) = z^n P(1/z)$, then

$$k^s |P^{(s)}(z)| \leq |Q^{(s)}(z)| \quad \text{for} \quad |z| = 1, 1 \leq s \leq n.$$  (8)

This lemma is implicit in [4, p. 511]; however for the sake of completeness here we present a simple and an independent proof of this result.

**Proof of Lemma 2.** By hypothesis all the zeros of the polynomial

$$P(z) = a_n z^n + \cdots + a_2 z^2 + a_1 z + a_0$$

lie in $|z| \geq k \geq 1$. Therefore, all the zeros of the polynomial

$$F(z) = P(kz) = a_n k^n z^n + \cdots + a_2 k^2 z^2 + a_1 k z + a_0$$

lie in $|z| \geq 1$. If we take

$$G(z) = z^n F(1/z) = z^n P(k/z)$$

$$= \tilde{a}_n k^n + \tilde{a}_{n-1} k^{n-1} z + \cdots + \tilde{a}_2 k^2 z^2 + \tilde{a}_1 k z + \tilde{a}_0,$$  (10)

then all the zeros of $G(z)$ lie in $|z| \leq 1$. Thus the function $f(z) = G(z)/F(z)$ is analytic in $|z| \leq 1$ and

$$|f(z)| = |G(z)/F(z)| = |G(z)|/|F(z)| = 1 \quad \text{for} \quad |z| = 1.$$  (11)

By the maximum modulus principle, we have

$$|G(z)/F(z)| = |f(z)| \leq 1 \quad \text{for} \quad |z| \leq 1.$$  (12)

Replacing $z$ by $1/z$ in (11), we conclude that

$$|F(z)| \leq |G(z)| \quad \text{for} \quad |z| \geq 1.$$  (12)

Hence for every real or complex number $\lambda$ with $|\lambda| > 1$, the polynomial $T(z) = F(z) + \lambda G(z)$ has all its zeros in $|z| \leq 1$, because if this is not true, then there is a point $z = z_0$ with $|z_0| > 1$ such that $T(z_0) = 0$. This gives

$$0 = T(z_0) = F(z_0) + \lambda G(z_0), \quad \text{where} \quad |z_0| > 1.$$  (13)

This implies

$$|F(z_0)| = |\lambda| |G(z_0)| > |G(z_0)|,$$

which is a contradiction to (12). Thus all the zeros of $F(z) + \lambda G(z)$ lie in $|z| \leq 1$ for every complex number $\lambda$ with $|\lambda| > 1$. By Gauss–Lucas theorem, it follows that all the zeros of $F^{(s)}(z) + \lambda G^{(s)}(z)$ lie in $|z| \leq 1$. This implies that

$$|F^{(s)}(z)| \leq |G^{(s)}(z)| \quad \text{for} \quad |z| \geq 1.$$  (13)
Differentiating (10) $s$ times, $1 \leq s \leq n$, we get

$$G^{(s)}(z) = n(n-1) \cdots (n-s+1) \bar{a}_0 z^{n-s} + \cdots + s! \bar{a}_{n-s} k^{n-s}$$

and by Gauss–Lucas theorem all the zeros of $G^{(s)}(z)$ lie in $|z| \leq 1$. If

$$H(z) = z^{n-s} G^{(s)}(1/z) = k^{n-s} a_{n-s} z^{n-s} + \cdots + n(n-1) \cdots (n-s+1) a_0,$$

then all the zeros of $H(z)$ lie in $|z| \geq 1$ and we see that

$$H(z/k) = a_{n-s} \bar{s} z^{n-s} + \cdots + n(n-1) \cdots (n-s+1) a_0$$

$$= z^{n-s} Q^{(s)}(1/z),$$

where

$$Q(z) = z^n P(1/z) = \bar{a}_0 z^n + \cdots + \bar{a}_{n-s} z^s + \cdots + \bar{a}_n.$$

Since $F^{(s)}(z) = k^n P^{(s)}(kz)$, from (13) we get

$$k^n |P^{(s)}(kz)| \leq |H(z)|$$

for $|z| = 1$. By the maximum modulus principle,

$$k^n |P^{(s)}(kz)| \leq |H(z)|$$

(14)

Taking in particular $z = e^{i\theta}/k$, $0 \leq \theta < 2\pi$, so that $|z| = 1/k \leq 1$ and from (14) we get

$$k^n |P^{(s)}(e^{i\theta})| \leq |H(e^{i\theta}/k)|.$$

Equivalently

$$k^n |P^{(s)}(z)| \leq |Q^{(s)}(z)|$$

for $|z| = 1$,

which is (8) and this completes the proof of the lemma.

3. Proof of the Theorem

First we suppose that $k = 1$. Since $P(z)$ is a polynomial of degree $n-s+1$, by repeated application of inequality (1), we have for every $p > 0$,

$$\left\{ \int_0^{2\pi} |P^{(s)}(e^{i\theta})|^p d\theta \right\}^{1/p} \leq (n-s+1) \left\{ \int_0^{2\pi} |P^{(s-1)}(e^{i\theta})|^p d\theta \right\}^{1/p}$$

$$\leq (n-s+1)(n-s+2) \left\{ \int_0^{2\pi} |P^{(s-2)}(e^{i\theta})|^p d\theta \right\}^{1/p}$$

$$\cdots$$

$$\leq (n-s+1) \cdots (n-1) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{1/p}.$$ 

(15)

By hypothesis, $P(z)$ has all its zeros in $|z| \geq k = 1$. Using inequality (2) on the right hand side of (15), we get

$$\left\{ \int_0^{2\pi} |P^{(s)}(e^{i\theta})|^p d\theta \right\}^{1/p} \leq \frac{n(n-1) \cdots (n-s+1)}{(1/2\pi) \int_0^{2\pi} |1-e^{i\theta}|^p d\theta} \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{1/p}$$

for every $p > 0$. This proves (5) for $k = 1$. Henceforth, we suppose $k > 1$. Since $P(z)$ has all its zeros in $|z| \geq k > 1$, the polynomial $Q(z) = z^n P(1/z)$ has all its zeros in
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\[ |z| \leq 1/k < 1, \] which implies by Gauss–Lucas theorem that all the zeros of \( Q^{(\alpha)}(z) \) lie in \( |z| < 1 \). By lemma 2, we have

\[ |P^{(\alpha)}(z)| \leq \frac{1}{k^2} |Q^{(\alpha)}(z)| < |Q^{(\alpha)}(z)| \quad \text{for } |z| = 1. \quad (16) \]

If \( H(z) = z^{n-s}Q^{(\alpha)}(1/\bar{z}) \), then all the zeros of \( H(z) \) lie in \( |z| > 1 \) and from (16), we have for \( |z| = 1 \)

\[ |e^{i\alpha}z^s P^{(\alpha)}(z)| = |P^{(\alpha)}(z)| < |Q^{(\alpha)}(z)| = |z^{n-s}Q^{(\alpha)}(1/\bar{z})| \]

\[ = |H(z)|. \]

Using Rouche’s theorem, it follows that all the zeros of

\[ \Lambda_\delta P(z) = H(z) + e^{i\alpha}z^s P^{(\alpha)}(z) \]

\[ = n(n-1) \cdots (n-s+1)a_0 + (n-1) \cdots (n-s)a_1 z + \cdots + s!a_{n-s}z^{n-s} \]

\[ + e^{i\alpha}z^s \{ n(n-1) \cdots (n-s+1)a_0 z^{n-s} + (n-1) \cdots (n-s)a_1 z^{n-s-1} \}

\[ + \cdots + s!a_{s} \}

\[ = n(n-1) \cdots (n-s+1)a_0 + (n-1) \cdots (n-s)a_1 z + \cdots + n(n-1) \cdots (n-s+1)e^{i\alpha}a_s z^n \]

lie in \( |z| > 1 \). This shows that the operator \( \Lambda_\delta \) is admissible. Hence by (7), we have for every \( p > 0 \),

\[ \int_0^{2\pi} |H(e^{i\theta}) + e^{i\alpha}e^{i\theta}P^{(\alpha)}(e^{i\theta})|^p d\theta \leq n(n-1) \cdots (n-s+1) \int_0^{2\pi} |P(e^{i\theta})|^p d\theta. \quad (17) \]

Integrating both sides of (17) with respect to \( \alpha \) from 0 to 2\( \pi \), we get for \( p > 0 \),

\[ \int_0^{2\pi} \int_0^{2\pi} |H(e^{i\theta}) + e^{i\alpha}e^{i\theta}P^{(\alpha)}(e^{i\theta})|^p d\alpha d\theta \leq n(n-1) \cdots (n-s+1) \]

\[ \times 2\pi \int_0^{2\pi} |P(e^{i\theta})|^p d\theta. \quad (18) \]

Now for every real \( \alpha \) and \( A \geq B \geq 1 \), it can be easily verified that \( |A + e^{i\alpha}| \geq |B + e^{i\alpha}| \), which implies for every \( p > 0 \),

\[ \int_0^{2\pi} |A + e^{i\alpha}|^p d\alpha \geq \int_0^{2\pi} |B + e^{i\alpha}|^p d\alpha. \quad (19) \]

We take \( A = |H(e^{i\theta})/P^{(\alpha)}(e^{i\theta})| \) and \( B = k^2 > 1 \), then by (8), \( A \geq B \geq 1 \) and we get with the help of (19) that

\[ \int_0^{2\pi} |H(e^{i\theta}) + e^{i\alpha}e^{i\theta}P^{(\alpha)}(e^{i\theta})|^p d\alpha = |P^{(\alpha)}(e^{i\theta})|^p \int_0^{2\pi} \left| \frac{H(e^{i\theta})}{P^{(\alpha)}(e^{i\theta})} + e^{i\alpha} \right|^p d\alpha \]

\[ = |P^{(\alpha)}(e^{i\theta})|^p \int_0^{2\pi} \left| \frac{H(e^{i\theta})}{P^{(\alpha)}(e^{i\theta})} + e^{i\alpha} \right|^p d\alpha \]

\[ \geq |P^{(\alpha)}(e^{i\theta})|^p \int_0^{2\pi} |k^2 + e^{i\alpha}|^p d\alpha. \]
Using this in (18), we conclude for each \( p > 0 \),

\[
\int_0^{2\pi} |P^{(a)}(e^{i\theta})|^p d\theta \leq \frac{n(n-1) \cdots (n-s+1)}{(1/2\pi)} \int_0^{2\pi} |k^p + e^{i\alpha}|^p d\alpha
\]

from which (5) follows immediately and this completes the proof of the theorem.

References


