On the neutrix convolution product of $x_\infty \ln x_\infty$ and $x_\infty$}

EMIN ÖZÇAĞ
Department of Mathematics, University of Hacettepe, 06532-Beytepe, Ankara, Turkey
E-mail: ozcagl@eti.cc.hun.edu.tr

Abstract. The existence of the neutrix convolution product of distribution $x_\infty \ln x_\infty$ and $x_\infty$ is proved and some convolution products are evaluated.

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Let $\mathcal{D}$ be the space of infinitely differentiable functions on the real line $\mathbb{R}$ with compact support and $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. The convolution product $f \ast g$ of two distributions $f$ and $g$ in $\mathcal{D}'$ may be defined by

\[ \langle (f \ast g)(x), \phi \rangle = \langle f(y), \langle g(x), \phi(x+y) \rangle \rangle \]

for arbitrary $\phi$ in $\mathcal{D}$ if $f$ and $g$ satisfy either of the conditions:

(B1) either $f$ or $g$ has bounded support,

(B2) the supports of $f$ and $g$ are bounded on the same side, (see [5]).

It follows from the definition that if the convolution products $f \ast g$ and $f \ast g'$ (or $f' \ast g$) exist then

\[ f \ast g = g \ast f; \]

\[ (f \ast g)' = f' \ast g' = f' \ast g. \]

Now let $f$ and $g$ be two distributions on $\mathbb{R}$ such that $A = \text{Supp}(f)$ and $B = \text{Supp}(g)$ satisfying the following conditions:

(i) for every bounded set $K \subset \mathbb{R}$, the set $(A \times B) \cap K^A$ is bounded in $\mathbb{R}^2$;

(ii) for every bounded set $K \subset \mathbb{R}$, the set $A \cap (K - B)$ is bounded in $\mathbb{R}$;

(iii) for every bounded set $K \subset \mathbb{R}$, the set $(K - A) \cap B$ is bounded in $\mathbb{R}$;

(iv) if $x_n \in A, y_n \in B$ and $|x_n| + |y_n| \to \infty$, then $|x_n + y_n| \to \infty$,

where $K^A = \{(x, y) \in \mathbb{R}^2 : x + y \in K\}$, then the convolution product $f \ast g$ of $f$ and $g$ exists and is defined as in (1).

The conditions (i)–(v) are well known (see [4, 7]). Condition (iv) was introduced by Mikusinski in [1]. If the supports of distributions $f$ and $g$ satisfy conditions (B1) or (B2), then they fulfill conditions (i)–(iv). In [4], Fisher and Kaminski gave two pairs of distributions $F, G$ and $f, g$ which did not satisfy the conditions (B1) and (B2), but the convolution products $F \ast G$ and $f \ast g$ exist and the conditions (i)–(iv) were satisfied.

The convolution product of distributions may be defined in a more general way without any restriction on the supports. The most known are given by Vladimirov and
Jones (see [8, 10]). However, there still exist many pairs of distributions such that the convolution products do not exist in the sense of these definitions. In order to extend the convolution product to larger class of distributions Fisher developed the method, which is very similar to Hadamard's method used to define the pseudo-functions, regarded as a particular application of the so called neutrix calculus developed by Van der Corput [2]. His method is based on discarding of unwanted infinite quantities from asymptotic expansions and has been widely exploited in connection with the problem of distributional multiplication and convolution, see [6]. To introduce Fisher's definition of convolution product, we need the following.

Let $f$ and $g$ be distributions and let $f_n = f \tau_n (n = 1, 2, \ldots)$ where $\tau$ is an infinitely differentiable function satisfying the following conditions:

(a) $\tau(x) = \tau(-x)$,
(b) $0 \leq \tau(x) \leq 1$,
(c) $\tau(x) = 1$ for $|x| \leq \frac{1}{n}$,
(d) $\tau(x) = 0$ for $|x| \geq 1$,

and the function $\tau_n$ is defined by

$$
\tau_n(x) = \begin{cases} 
1, & |x| \leq n, \\
\tau(n^x - n^{x+1}), & x > n, \\
\tau(n^x + n^{x+1}), & x < -n,
\end{cases}
$$

for $n = 1, 2, \ldots$.

Then the neutrix convolution product $f \circ g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, providing the limit $h$ exists in the sense that

$$
N - \lim_{n \to \infty} \langle f_n * g, \phi \rangle = \langle h, \phi \rangle,
$$

for all $\phi$ in $\mathcal{D}$, where $N$ is the neutrix (see [2]), having domain $N = \{1, 2, \ldots, n, \ldots\}$ and range $N'$ the real numbers, with negligible functions finite linear sums of the functions

$$
n^\lambda \ln^{r-1} n, \ln^r n \ (\lambda > 0, r = 1, 2, \ldots)
$$

and all functions which converge to zero in the usual sense as $n$ tends to infinity.

In this definition the convolution product $f_n * g$ exists since the distribution $f_n$ is having bounded support. Because the definition in nonsymmetric, the convolution product $f \circ g$ is not always commutative.

The essential use of the neutrix limit is to extract the finite part from a divergent quantity. In the neutrix calculus each limit, if properly defined, always exists. The neutrix convolution product depends on the negligible functions and so depends on the sequence $\tau_n$. The negligible functions in the neutrix $N$ given above are selected because these are the functions that occur in mathematics and physics.

In [3], Fisher proved that his definition was a generalization of the convolution product in the sense of Gel'fand and Shilov. Indeed, if $f$ and $g$ be distributions in $\mathcal{D}'$ satisfying either conditions (B1) or (B2), then the neutrix convolution product $f \circ g$ exists and

$$
f \circ g = f \ast g.
$$
He also proved that if the neutrix convolution product $f \circledast g'$ exists, then

$$(f \circledast g)' = f \ast g'.$$  \hfill (3)

Note however that equation (2) does not necessarily hold for the neutrix convolution product and that $(f \circledast g')$ is not necessarily equal to $f' \circledast g$. In this paper, using Fisher's definition we get some results. First of all we give the following theorem.

**Theorem 1.** The neutrix convolution product $(x'_- \ln x_-) \circledast \ln x_+$ exists and

$$(x'_- \ln x_-) \circledast \ln x_+ = \begin{cases} \frac{-(r+1)^2}{2} x'_- \ln^2 x_- & \text{if } r \leq 0 \\ \frac{(-1)^{r+1}}{2(r+1)} x'_- \ln^2 x_- & \text{if } r > 0 \end{cases}$$

for $r = 1, 2, \ldots$ where

$$\psi(r) = \begin{cases} 0, & r = 0 \\ \sum_{i=1}^{r-1} i^{-1}, & r \geq 1. \end{cases}$$

$$\xi(2) = \sum_{i=1}^{\infty} i^{-2}.$$  \hfill (4)

**Proof.** Putting $(x'_- \ln x_-)_n = (x'_- \ln x_-) \tau_n(x)$, we have

$$\langle (x'_- \ln x_-)_n \ast \ln x_+, \phi(x) \rangle = \langle (y'_- \ln y_-)_n, \langle \ln x_+, \phi(x+y) \rangle \rangle$$

$$= \int_a^b \int_{-n}^{b} (y)_+ \ln(-y) \tau_n(y) \int_a^b \ln(x-y)_+ \phi(x) dx dy$$

$$+ \int_a^b \phi(x) \int_{-n}^{0} (y)_+ \ln(-y) \ln(x-y) + dy dx$$

$$+ \int_a^b \phi(x) \int_{-n}^{a} (y)_+ \ln(-y) \tau_n(y) \ln(x-y) dy dx$$

$$= \int_a^b \phi(x) \int_{-n}^{0} (y)_+ \ln(-y) \ln(x-y) + dy dx$$

$$+ \int_a^b \phi(x) \int_{-n}^{a} (y)_+ \ln(-y) \tau_n(y) \ln(x-y) dy dx$$

for $n > -a$ and arbitrary $\phi$ in $\mathcal{O}$ with compact support contained in $[a, b]$. When $x < 0$, we have on making the substitution $y = xu^{-1}$,

$$\int_{-n}^{0} (y)_+ \ln(-y) \ln(x-y)_+ dy = \int_{-n}^{x} (y)_+ \ln(-y) \ln(x-y) dy$$

$$= (-x)^{r+1} \int_{-x/n}^{1} \frac{1}{u^{r+2}} [\ln^2(-x) + \ln(-x) \ln(1-u) - 2 \ln(-x) \ln u$$

$$- 2 \ln u \ln(1-u) + \ln^2 u] du$$

$$= I_{1n} + I_{2n} - I_{3n} - I_{4n} + I_{5n}.$$  \hfill (5)
In [9], the neutrix limits of $I_{1n}, I_{2n}, I_{3n}, I_{4n}$ and $I_{5n}$ were evaluated as

\[
N \lim_{n \to \infty} I_{1n} = -\frac{(-x)^{r+1} \ln^2(-x)}{r+1},
\]

\[
N \lim_{n \to \infty} I_{2n} = -\frac{(-x)^{r+1} \ln(-x)}{(r+1)^2} + \frac{(-x)^{r+1} \ln^2(-x)}{r+1} \psi(r) \frac{1}{r+1}(-x)^{r+1} \ln(-x),
\]

\[
N \lim_{n \to \infty} I_{3n} = -\frac{2(-x)^{r+1} \ln(-x)}{(r+1)^2},
\]

\[
N \lim_{n \to \infty} I_{4n} = -\frac{(-x)^{r+1}}{(r+1)^3} + \frac{\xi(2)(-x)^{r+1}}{r+1} + \frac{\psi(r)(-x)^{r+1}}{(r+1)^2} \frac{1}{r+1} \sum_{i=1}^{r} i^{-2}(-x)^{r+1},
\]

\[
N \lim_{n \to \infty} I_{5n} = -\frac{2(-x)^{r+1}}{(r+1)^3}.
\]

It follows that

\[
N \lim_{n \to \infty} \int_{-x}^{0} (-y)^{r} \ln(-y) \ln(x-y) \, dy = -\frac{x^{r+1} \ln^2 x}{2(r+1)} \psi(r+1) \frac{1}{r+1} x^{-1} \ln x - \left[ \frac{\psi(r+1)}{r+1} + \xi(2) + \sum_{i=1}^{r} i^{-2} \right] \frac{x^{r+1}}{r+1}. \tag{7}
\]

When $x > 0$, we have on making the substitution $y = x(1-u^{-1})$,

\[
\int_{-x}^{0} (-y)^{r} \ln(-y) \ln(x-y) \, dy = x^{r+1} \int_{x/(x+n)}^{1} u^{r-2} \left[ \ln^2 x - 2 \ln x \ln u + \ln x \ln(1-u) - \ln u \ln(1-u) \right] + \ln^2 u] (1-u)^{r} \, du
\]

\[
= J_{1n} - J_{2n} - J_{3n} - J_{4n} + J_{5n}. \tag{8}
\]

Since

\[
\int_{x/(x+n)}^{1} u^{r-2} (1-u)^{r} \, du = \sum_{i=1}^{r} (-1)^{i} \binom{r}{i} \int_{x/(x+n)}^{1} u^{-r-i+2} \, du
\]

\[
= \sum_{i=1}^{r} (-1)^{i} \binom{r}{i} [(1+n/x)^{-i+1}-1],
\]

it follows that

\[
N \lim_{n \to \infty} J_{1n} = 0. \tag{9}
\]
Neutrix convolution product

Now, integrating by parts we have

\[
\int_{x/(x+n)}^1 u^{-r-2} \ln(1-u)^r \, du = -\frac{1}{r+1} \int_{x/(x+n)}^1 (1-u)^r \ln u \, du^{r-1}
\]

\[
= \frac{(x+n) \ln x - \ln n - \ln(1+x/n)}{(r+1)x^{r+1}}
\]

\[
+ \frac{1}{r+1} \int_{x/(x+n)}^1 u^{-r-2} (1-u)^r \, du - \frac{r}{r+1} \int_{x/(x+n)}^1 u^{-r-1}(1-u)^r \ln u \, du.
\]

It can be easily seen that

\[
N \cdot \lim_{n \to \infty} \frac{(x+n) \ln x - \ln n - \ln(1+x/n)}{(r+1)x^{r+1}} = \frac{(-1)^r}{r+1} \left( \frac{1}{r} - \frac{1}{r+1} \right).
\]

If we assume that

\[
a_{r-1} = N \cdot \lim_{n \to \infty} \int_{x/(x+n)}^1 u^{-r-1}(1-u)^{r-1} \ln u \, du.
\]

exists, it follows that \(a_r\) exists and

\[(r+1)a_r + ra_{r-1} = (-1)^r \left( \frac{1}{r} - \frac{1}{r+1} \right)\]

for \(r = 1, 2, \ldots\). Since

\[
a_0 = N \cdot \lim_{n \to \infty} \int_{x/(x+n)}^1 u^{-2} \ln u \, du = \ln x - 1
\]

certainly exists, it follows easily by induction that

\[
a_r = \frac{(-1)^r}{r+1} \ln x - \frac{(-1)^r}{(r+1)^2}.
\]

Thus

\[
N \cdot \lim_{n \to \infty} (-J_{2a}) = \frac{2(-1)^{r+1}}{r+1} x^{r+1} \ln^2 x - \frac{2(-1)^{r+1}}{(r+1)^2} x^{r+1} \ln x.
\]

(10)

Next

\[
\int_{x/(x+n)}^1 u^{-r-2} \ln(1-u)(1-u)^r \, du = -\frac{(x+n)n^r}{(r+1)x^{r+1}} \ln(1+x/n)
\]

\[
- \frac{1}{r+1} \int_{x/(x+n)}^1 u^{-r-1}(1-u)^r \ln u \, du - \frac{r}{r+1} \int_{x/(x+n)}^1 u^{-r-1}(1-u)^r \ln(1-u)(1-u)^r \, du.
\]

Assuming that

\[
b_{r-1} = N \cdot \lim_{n \to \infty} \int_{x/(x+n)}^1 u^{-r-1}(1-u)^{r-1} \ln(1-u) \, du
\]

exists, we see with the same argument as above that \(b_r\) exists and

\[(r+1)b_r + rb_{r-1} = \frac{(-1)^r}{r+1} \left( \frac{1}{r} - \frac{1}{r+1} \right).
\]
Since \( b_0 = \ln x - 1 \), it follows that
\[
b_r = \frac{(-1)^r}{r+1} \ln x - \frac{(-1)^r}{(r+1)^2}
\]
and so
\[
N - \lim_{n \to \infty} J_{3n} = -\frac{(-1)^{r+1}}{r+1} x^{r+1} \ln^2 x + \frac{(-1)^{r+1}}{(r+1)^2} x^{r+1} \ln x.
\] (11)

Integrating by parts, we have
\[
\int_{x/(x+n)}^{1} u^{-r-2} \ln u \ln (1-u)(1-u)^y \, du
\]
\[
= -\frac{1}{r+1} \int_{x/(x+n)}^{1} \ln u \ln (1-u)(1-u)^y \, d(u^{-r-1})
\]
\[
= (x+n)^n \left[ \ln x - \ln n - \ln (1 + x/n) \right] \ln (1 + x/n)
\]
\[
+ \frac{1}{r+1} \int_{x/(x+n)}^{1} u^{-r-2} \ln (1-u)(1-u)^y \, du
\]
\[
- \frac{1}{r+1} \int_{x/(x+n)}^{1} u^{-r-1} \ln u(1-u)^{y-1} \, du
\]
\[
- \frac{r}{r+1} \int_{x/(x+n)}^{1} u^{-r-1} \ln u \ln (1-u)(1-u)^y \, du.
\]

The neutrix limit of the first term on the right-hand side of the equation above is equal to
\[
N - \lim_{n \to \infty} \frac{(x+n)^n}{(r+1)x^{r+1}} \left[ \ln x - \ln n - \ln (1 + x/n) \right] \ln (1 + x/n)
\]
\[
= 2(-1)^r \left( \frac{1}{r} - \frac{1}{r+1} \right) \ln x + \frac{2(-1)^r}{r+1} \left[ \frac{\psi(r-1)}{r} - \frac{\psi(r)}{r+1} \right],
\]
where
\[
\ln^2 (1-x) = 2 \sum_{n=1}^{\infty} \frac{\psi(n)}{n+1} x^{n+1}.
\]

Similarly, if we assume that
\[
c_{r-1} = N - \lim_{n \to \infty} \int_{x/(x+n)}^{1} u^{-r-1} \ln u \ln (1-u)(1-u)^{y-1} \, du
\]
exists, it follows that \( c_r \) exists and
\[
(r+1) c_r + rc_{r-1} = (-1)^r \left[ \frac{2 \ln x}{r} - \frac{1}{r^2} - \frac{1}{(r+1)^2} \right] + 2(-1)^r \left[ \frac{\psi(r-1)}{r} - \frac{\psi(r+1)}{r+1} \right]
\]
for \( r = 1, 2, \ldots \). Since
\[
c_0 = N - \lim_{n \to \infty} \int_{x/(x+n)}^{1} u^{-2} \ln u \ln (1-u) \, du = \xi(2) - 1 + \frac{1}{2} \ln^2 x
\]
certainly exists, it can be shown by induction that
\[ c_r = \frac{2(-1)^r \psi(r)}{r+1} \ln x + \frac{2(-1)^r}{r+1} \sum_{i=1}^{r} \left[ \frac{\psi(i-1)}{i} - \frac{\psi(i+1)}{i+1} \right] \]
\[ + \frac{(-1)^r}{r+1} \left[ \xi(2) - 2 + \frac{1}{2} \ln^2 x \right]. \]

Thus
\[ N - \lim_{n \to \infty} (-J_{4n}) = \frac{2(-1)^{r+1} \psi(r)}{r+1} x^{r+1} \ln x + \frac{2(-1)^{r+1} x^{r+1}}{(r+1)^3} + \frac{2(-1)^{r+1}}{r+1} \]
\[ \times \sum_{i=1}^{r} \left[ \frac{\psi(i-1)}{i} - \frac{\psi(i+1)}{i+1} \right] x^{r+1} + \frac{(-1)^{r+1} \xi(2)}{r+1} x^{r+1} - \frac{2(-1)^{r+1}}{r+1} x^{r+1}. \] (12)

Finally, it has been shown in [9] that

\[ N - \lim_{n \to \infty} J_{5n} = \frac{2(-1)^r \psi(r)}{r+1} x^{r+1} \ln x + \frac{2(-1)^r}{r+1} \sum_{i=1}^{r} \left[ \frac{\psi(i-1)}{i} - \frac{\psi(i+1)}{i+1} \right] x^{r+1} \]
\[ + \frac{(-1)^r}{r+1} x^{r+1} \ln^2 x - \frac{2(-1)^r}{r+1} x^{r+1}. \] (13)

Equations (8)–(13) imply that when \( x > 0 \)
\[ N - \lim_{n \to \infty} \int_{-y}^{0} (-y)^r \ln(-y) \ln(x+y) \ln(x-y) dy = \frac{(-1)^{r+1} x^{r+1} \ln^2 x}{(2r+1)} + \]
\[ - \frac{(-1)^{r+1} x^{r+1} \ln x}{(r+1)^3} + \frac{(-1)^{r+1} \xi(2)}{r+1} x^{r+1}. \] (14)

Further, with \( a \leq x \leq b \) and \( n > -a \), we have
\[ \left| \int_{-n}^{-n-a} (-y)^r \ln(-y) \tau_n(y) \ln(x+y) dy \right| = O(n^{-a} \ln^2 n) \]
and so
\[ \lim_{n \to \infty} \int_{-n}^{-n-a} (-y)^r \ln(-y) \tau_n(y) \ln(x+y) dy = 0. \] (15)

It follows from equations (6), (7), (8), (14) and (15) that
\[ N - \lim_{n \to \infty} \langle x^r \ln x_n, * \ln x_n, \phi(x) \rangle \]
\[ = \left( \frac{(-1)^{r+1} x^{r+1} \ln^2 x}{(2r+1)} + \frac{(-1)^{r+1} x^{r+1} \ln x}{(r+1)^3} + \frac{(-1)^{r+1} \xi(2)}{r+1} x^{r+1} \right) \]
\[ + \left[ \frac{\psi(r+1)}{r+1} x^{r+1} \ln x - \frac{(-1)^{r+1} x^{r+1}}{(r+1)^3} + \frac{(-1)^{r+1} \xi(2)}{r+1} x^{r+1} \right] \]
\[ \phi(x) \right) \]
for \( r = 1, 2, \ldots \) and arbitrary \( \phi \) in \( \mathcal{D} \).
Theorem 2. The neutrix convolution product $(x_+^r \ln x_-) \otimes x_+^{-s}$ exist and

$$
(x_+^r \ln x_-) \otimes x_+^{-s} = \left(\begin{array}{c} r \\ s-1 \end{array}\right) \frac{x_+^{r-s+1} \ln^2 x_-}{2} + \frac{(-1)^{s-1} x_+^{r-s+1} \ln x_+}{2} - \psi(r-s+1) x_+^{r-s+1} \ln x_- + [\psi(r) - \psi(r-s+1)] x_+^{r-s+1} \ln x_+ + [\chi_s(r) + \Phi_s(r)] x_+^{r-s+1} + (-1)^{s-1} x_+^{r-s+1} \right]
$$

(16)

for $s=1,2,\ldots,r+1$ and $r=1,2,\ldots$ and

$$
(x_+^r \ln x_-) \otimes x_+^{-s} = \frac{r!(s-r-2)!}{(s-1)!} \left[ x_+^{r-s+1} \ln x_+ + (-1)^{s-1} x_+^{r-s+1} \ln x_- + \psi(s-r-2) \left[ x_+^{r-s+1} + (-1)^{s-1} x_+^{r-s+1} \right] - \psi(r) x_+^{r-s+1} \right]
$$

(17)

for $s=r+2,r+3,\ldots$ and $r=1,2,\ldots$ where

$$
\Phi_s(r) = \sum_{i=1}^{r-1} \frac{\psi(i)}{i+1} - \psi(r-s+1) \left[ \psi(r) - \psi(r-s+2) \right] + \zeta(2)
$$

and

$$
\chi_s(r) = \sum_{i=1}^{r-2} \psi(r) \left[ \psi(r) - \psi(r-s+1) \right].
$$

Proof. If we differentiate equation (4) $s$ times and using equation (3) and the following two equations

$$
(x_+^r \ln^2 x_+)^{(s)} = \frac{r!}{(r-s)!} \left[ x_+^{r-s} \ln^2 x_+ + 2 \left[ \psi(r) - \psi(r-s) \right] x_+^{r-s} \ln x_+ + 2 \sum_{i=r-s+1}^{r-1} \psi(i) x_+^{r-s-2} - 2 \psi(r-s) \left[ \psi(r) - \psi(r-s+1) \right] x_+^{r-s} \right]
$$

$$
(x_+^r \ln x_+)^{(s)} = \frac{r!}{(r-s)!} \left[ x_+^{r-s} \ln x_+ + \left[ \psi(r) - \psi(r-s) \right] x_+^{r-s} \right]
$$

we get (16). Equation (16) for $s = r + 1$ gives

$$
x_+^{-1} \ln x_- \otimes x_+^{-r-1} = \frac{1}{2} \ln^2 x_- - \frac{1}{2} \ln^2 x_+ + \psi(r) \ln x_+ + \chi(r)
$$

and so by equation (3)

$$-(r + 1)(x_+^{-1} \ln x_-) \otimes x_+^{-r-2} = -x_+^{-1} \ln x_+ + x_+^{-1} \ln x_- \psi(r) x_+^{-1}
$$

which proves (17) for the case $s = r + 2$. Equation (17) now follows easily by induction.

References