Remarks on Banaschewski–Fomin–Shanin extensions

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Abstract. The notion of $B^*$-continuous and $B^{*\prime}$-continuous maps is introduced. The problem of epireflection of Banaschewski–Fomin–Shanin extension for a general Hausdorff space is investigated with the help of $pB^*$ and $pB^{*\prime}$-continuous maps.

Keywords. BFS-extension; $H$-closed extension; minimal Hausdorff extension.

1. Introduction

The concept of $H$-closed spaces is introduced by Alexandroff and Urysohn [1]. On the line of Stone–Cech compactification the reflective properties of $H$-closed spaces are studied by many researchers ([6, 8, 9, 10, 13]). Herrlich and Strecker [6] have shown that Katetov extension $k$ is not a functor from the category of Hausdorff spaces and continuous maps to the full subcategory of $H$-closed spaces. Harris [5] showed that if we restrict the class of continuous maps to $p$-maps then $k$ is the required reflective functor from the category of Hausdorff spaces and $p$-maps to the full subcategory of $H$-closed spaces. Between 1960 to 1980, a good amount of work was done on categorical aspects to $H$-closed extensions. But, in case of minimal Hausdorff extensions, work carried out by researchers is of local nature, i.e., certain properties had to hold at each point. Problems related with categorical aspects of minimal Hausdorff spaces remain largely open. Banaschewski [2] proved that a Hausdorff space has a minimal Hausdorff extension if and only if it is semiregular. Each semiregular Hausdorff space $X$ has largest minimal Hausdorff extension (cf. [14, 15]) denoted by $\mu X$ and called the Banaschewski–Fomin–Shanin Hausdorff extension of $X$. We do not have a suitable answer regarding epireflection of the category of semiregular Hausdorff spaces to the full subcategory of minimal Hausdorff spaces, $\mu$ as a reflection. In this paper we give an answer to this problem with the help of some new class of maps.

2. Preliminaries

Throughout this paper all spaces assumed are to be Hausdorff. If $A$ is a subset of a space $X$, then $\text{int}_X(A)$ (resp. $\text{cl}_X(A)$) will denote the interior (resp. closure) of $A$ in $X$. A space $X$ is called $H$-closed if every homeomorphic image of $X$ which is a subset of a Hausdorff space $Y$, is a closed subset of $Y$.

For a Hausdorff space, $X$ the semiregularization of $X$, denoted by $X_s$, is the space generated by the open basis $\{\text{int}(\text{cl} U)|U \text{ open in } X\}$. $X$ is semiregular iff $X = X_s$.

A space $Y$ is an extension of $X$ if $X$ is a dense subspace of $Y$; if $Y$ possesses some topological property $P$, then $Y$ is a $P$-extension of $X$. Let $Y$ be a Hausdorff extension of $X$. For $y \in Y$, $O^y$ (sometimes written as $O^y$) denotes $\{U \cap X|y \in U, U \text{ open in } Y\}$;
\( \{O_{y} | y \in Y \} \) is called the neighborhood filter trace of \( Y \) on \( X \). Then \( O_{x} \) is an open filter on \( X \), and for \( x \in X \), \( O_{x} \) is precisely the open neighborhood system \( \eta_{x} \) of \( x \). For an open subset \( U \) of \( X \), let \( oU \) denote \( \{y \in Y | U \in O_{y}\} \). The \( \{U \cup \{y\} | U \in O_{y}, y \in Y \} \) and \( \{oU | U \) open in \( X \} \) form bases for topologies on \( Y \), and the resulting new spaces are denoted by \( Y^{+} \) and \( Y^{*} \), respectively. Now \( Y^{+} \) and \( Y^{*} \) are also Hausdorff extensions of \( X \), and the topology of \( Y^{+} \) (resp. \( Y^{*} \)) is coarser than the topology of \( Y \) (resp. \( Y^{*} \)) \[3\]; in fact, \( Y^{+} \) and \( Y^{*} \) are \( H \)-closed if and only if \( Y \) is \( H \)-closed. \( Y \) is called simple (resp. strict) extension if \( Y = Y^{+} \) (resp. \( Y^{*} \)).

Let \( X^{*} = \{ \mathcal{U} | \mathcal{U} \) is a free open ultra filter on \( X \} \). For each open subset \( U \) of \( X \), let \( O_{U} = U \cup \{ \mathcal{U} \in X^{*} - X | U \in \mathcal{U} \} \ \ \ X^{*} \) with the topology generated by the open basis \( \{O_{U} | U \) open in \( X \} \) is an \( H \)-closed extension \[4\] of \( X \) denoted by \( \sigma X \) and called Fomin extension of \( X \); \( X^{*} \) with the topology generated with the open basis \( \{U | U \) open in \( X \} \cup \{U \cup \{\mathcal{U}\} | U \) open in \( X, U \in \mathcal{U}, \mathcal{U} \in X^{*} - X \} \) is an \( H \)-closed extension \[8\] of \( X \) denoted by \( kX \) and called Katetov extension of \( X \).

If \( X \) is \( H \)-closed, then \( X_{s} \) is minimal Hausdorff and the topology of \( X_{s} \) is the smallest of the Hausdorff topology coarser than the topology of \( X \). We recall that a space \( X \) is said to be minimal Hausdorff if its topology does not contain any coarser Hausdorff topology on \( X \).

The following results are in \[16\].

1) For a Hausdorff space \( X \) the following are equivalent:

(i) \( X \) is minimal Hausdorff,

(ii) \( X \) is semiregular and \( H \)-closed, and

(iii) every open filter with a unique accumulation point converges.

2) (a) A space can be densely embedded in a minimal Hausdorff space if and only if it is semiregular.

(b) Any space can be embedded as a closed nowhere dense subspace of a minimal Hausdorff space.

3. BFS-extension for a general Hausdorff space

If \( X \) is semiregular, then its minimal Hausdorff extension is denoted by \( \mu X \) and is called the Banaschewski–Fomin–Shanin extension (BFS-extension) of \( X \). \( \sigma X \), and \( \mu X \) are both minimal Hausdorff extension of \( X \) and we also have \( \mu X = \sigma X \). BFS-extension has been studied by many researchers for a semiregular space \( X \) ([11, 14, 15]). Tikoo [17] constructed an extension of the type \( \mu X \) for a general Hausdorff space \( X \).

Let \( X \) be a Hausdorff space and let

\[ \mathcal{X} = X \cup \{ \mathcal{U} | \mathcal{U} \) is a free open ultra filter on \( X \}, \]

where \( \mathcal{U} \) is generated by the filter base \( \{int(cl U) | U \in \mathcal{U} \} \).

For each \( G \) open in \( X \), let

\[ \delta(G) = G \cup \{ \mathcal{U} | \mathcal{U} \in X - X, G \in \mathcal{U} \}, \]

the family \( \{\delta(G) | G \) open in \( X \} \) form a base for a topology \( \mathcal{G}^{+} \) on \( X \). A routine verification shows that \( (\mathcal{X}, \mathcal{G}^{+}) \) is a strict \( H \)-closed extension of \( X \).

We define a topology \( \mathcal{G}^{+} \) on \( \mathcal{X} \) by declaring that \( X \) is open in \( \mathcal{X} \), and for \( \mathcal{U} \in \mathcal{X} - X \)

a \( \mathcal{G}^{+} \)-basic neighborhood of \( \mathcal{U} \) is \( U \cup \{ \mathcal{U} \} \) where \( U \) is open in \( X \) and \( U \in \mathcal{U} \). Then \( (\mathcal{X}, \mathcal{G}^{+}) \) is a simple \( H \)-closed extension of \( X \).
Remarks on BFS extensions

Note that, if \( a: \sigma X \to X \) is defined by
\[
a(x) = x \quad \text{if} \quad x \in X,
\]
\[
a(U) = U \quad \text{if} \quad U \in \sigma X - X,
\]
then \( a \) is a bijection.

To see the characterization of the extension \( X \) for a semiregular space \( X \), we have the following [17].

3.1 PROPOSITION

The following statements are equivalent for a Hausdorff space \( X \).

(i) \( X \) is semiregular,
(ii) \( X \) is semiregular,
(iii) \( X = (\sigma X)_\theta = (kX)_\theta \).

In view of the above proposition we denote \( X \) by \( \mu X \), the BFS-extension of \( X \).

The idea of \( \theta \)-continuity is due to Fomin [4]. These maps have particular importance in our study on BFS-extension. We recall the following:

3.1 DEFINITION

A map \( f: X \to Y \) is said to be \( \theta \)-continuous if for each \( x \in X \) and each open neighborhood \( V \) of \( f(x) \), there exists an open neighborhood \( U \) of \( x \) such that \( f(\text{cl}_X U) \subseteq \text{cl}_Y V \).

Of course, continuity implies \( \theta \)-continuity.

The converse is true if we restrict our space, e.g., if \( Y \) is regular. Note that the identity map \( i: X_{\text{sr}} \to Y \), where \( X_{\text{sr}} \) is the semiregularization of a non semiregular space \( X \), is an example of \( \theta \)-continuous map which is not continuous.

Composition of two \( \theta \)-continuous maps is \( \theta \)-continuous. This fact helps us to construct categories with \( \theta \)-continuous maps.

In categorical point of view, it is important to find out the condition when a \( \theta \)-continuous map with semiregular codomain is continuous. Following proposition gives an answer to this.

3.3 PROPOSITION

Let \( f: X \to Y \) be a \( \theta \)-continuous map into a semiregular space. If \( f(A) \subseteq \text{int}_Y (f(\text{cl}_X A)) \) for all \( A \) open in \( X \), then \( f \) is continuous.

Proof. The proof is straightforward and is omitted.

Another important map in connection with \( H \)-closed extension are \( p \)-maps. An intensive application of these maps can be found in Harris [5].

3.4 DEFINITION

A \( p \)-cover of a space is an open cover such that the union of some finite subcollection is dense, a \( p \)-map is a continuous map such that the inverse of a \( p \)-cover of the codomain is a \( p \)-cover of the domain.

Every map with \( H \)-closed domain is a \( p \)-map.

Now we are giving the definition of \( B^* \)-continuous and \( B^*_c \)-continuous maps. Necessary application of these maps can be seen in the next section. But in order to maintain continuity with subject matter we are defining these here.
Let $X^*$ and $Y^*$ be the extensions of $X$ and $Y$ respectively. Here $*$ stands for the extension under consideration (e.g. $*$ may be a Hausdorff extension [4], a Katétov extension, a Fomin extension, or a BFS-extension).

3.5 DEFINITION

A continuous function $f:X \rightarrow Y$ is called $B^*$-continuous if $f$ has an extension (not necessarily continuous) $f^*:X^* \rightarrow Y^*$ such that $f^* (A) \subseteq \text{int}_{Y^*} \text{cl}_{Y^*} (f^* (\text{cl}_X \ast A))$ for all $A$ open in $X^*$. $f$ is said to be $B^*$-continuous if $f^*$ is also continuous.

The $B^*$-continuous maps are of real importance. In order to characterize these maps we have the following.

3.6 PROPOSITION

If $f:X \rightarrow Y$ is $B^*_c$-continuous, then $f(A \cap X) \subseteq \text{int}_{Y} \text{cl}_{Y} (f (A \cap X))$ for all $A$ open in $X^*$.

Proof. Since $f:X \rightarrow Y$ is $B^*_c$-continuous, there exists a unique continuous extension $f^*:X^* \rightarrow Y^*$ of $f$ such that $f^* (A) \subseteq \text{int}_{Y^*} \text{cl}_{Y^*} (f^* (\text{cl}_X \ast A))$ for all $A$ open in $X^*$.

We have $f^* (A) \subseteq \text{int}_{Y^*} \text{cl}_{Y^*} (f^* (\text{cl}_X \ast A))$, for all $A$ open in $X^*$. This gives

$$f^* (A \cap X) \subseteq \text{int}_{Y^*} \text{cl}_{Y^*} (f^* (\text{cl}_X \ast (A \cap X)))$$

or

$$f (A \cap X) \cap Y \subseteq \text{int}_{Y} \text{cl}_{Y} (f (A \cap X)) \cap Y$$

$$= \text{int}_{Y} \text{cl}_{Y} (f^* (A \cap X))$$

$$= \text{int}_{Y} \text{cl}_{Y} (f (A \cap X)).$$

To see the converse, we have

3.7 PROPOSITION

Let $f:X \rightarrow Y$ be continuous and let $f^*:X^* \rightarrow Y^*$ be the unique continuous extension of $f$. If $\text{cl}_{Y^*} f (A \cap X) \subseteq \text{int}_{Y^*} \text{cl}_{Y^*} (f (\text{cl}_X (A \cap X)))$ for all $A$ open in $X^*$. Then $f^* (A) \subseteq \text{int}_{Y^*} \text{cl}_{Y^*} (f^* (\text{cl}_X \ast A))$ i.e. $f$ is $B^*_c$-continuous.

Proof. We have $\text{cl}_{Y^*} f (A \cap X) \subseteq \text{int}_{Y^*} \text{cl}_{Y^*} (f (\text{cl}_X (A \cap X)))$. This gives

$$\text{cl}_{Y^*} f (A \cap X) \subseteq \text{int}_{Y^*} \text{cl}_{Y^*} (f (\text{cl}_X (A \cap X))),$$

$$\text{cl}_{Y^*} f (A \cap X) \subseteq \text{int}_{Y^*} \text{cl}_{Y^*} (f (\text{cl}_X \ast A)),$$

$$f^* (\text{cl}_X \ast (A \cap X)) \subseteq \text{int}_{Y^*} \text{cl}_{Y^*} (f (\text{cl}_X \ast A)),$$

$$f^* (\text{cl}_X \ast A) \subseteq \text{int}_{Y^*} \text{cl}_{Y^*} (f (\text{cl}_X \ast A)),$$

$$f^* (A) \subseteq \text{int}_{Y^*} \text{cl}_{Y^*} (f (\text{cl}_X \ast A)).$$

A $p$-map which is also $B^*$-continuous ($B^*_c$-continuous) said to be $pB^*$-continuous ($pB^*_c$-continuous).

Following proposition is due to Banaschevski [2], which shows that to get a continuous extension from $\mu X$ to minimal Hausdorff extension $Y$ of $X$, we have to restrict our continuous maps from $X$ to $Y$. 
3.8 PROPOSITION

For each minimal Hausdorff extension $Y$ of a space $X$ there exists a $\theta$-continuous map completing the following diagram:

\[ \begin{array}{ccc}
X & \xrightarrow{i} & Y \\
& \searrow & \downarrow \mu X \\
& & (\ast)
\end{array} \]

An example given in [2], shows that $\mu X \rightarrow Y$ in $\ast$ in general, is not continuous.

We strengthen Banaschewski's result in the following proposition.

3.9 PROPOSITION

If $f: X \rightarrow Y$ is a p-map into a H-closed space, then there exists a $\theta$-continuous map completing the following diagram:

\[ \begin{array}{ccc}
X & \xrightarrow{f} & Y \\
& \searrow & \downarrow \mu X
\end{array} \]

Proof. Let $f: X \rightarrow Y$ be a p-map and let $Y$ be H-closed. Since for each H-closed extension $Y$ of $X$ there is a unique continuous function from $kX$ onto $Y$ that leaves $X$ pointwise fixed, there exists a unique map $g: kX \rightarrow Y$ such that $g \circ k = f$. The identity map $i: \sigma X \rightarrow kX$ is $\theta$-continuous, therefore $g \circ i: \sigma X \rightarrow Y$ is $\theta$-continuous. Define $j: \mu X \rightarrow \sigma X$ as follows:

\[ j(x) = x \quad \text{if} \quad x \in X \]

\[ j(\mathcal{U}_s) = \mathcal{U}_s \quad \text{if and only if} \quad x(\mathcal{U}) = \mathcal{U}_s \text{ for all } \mathcal{U}_s \in X - X. \]

By a routine verification we can prove that $j$ is a $\theta$-continuous map. Since $g \circ i: \sigma X \rightarrow Y$ and $j: \mu X \rightarrow \sigma X$ are $\theta$-continuous maps therefore the composite map $j \circ g \circ i$ is the required $\theta$-continuous map.

One of the basic problems related to extension properties of topological spaces is that: a continuous map between two spaces can be extended to a map between their given extension spaces, such as their Katetov extensions. In the case of BFS-extension we have the following theorem (cf. [7] and use 3.3).

3.10 Theorem. Let $f: X \rightarrow Y$ be a p-map and let $f^*$ be a function on $\mu X$ to $\mu Y$ defined as follows

\[ f^*(x) = x \quad \text{if} \quad x \in X \]

\[ f^*(\mathcal{U}_s) = \begin{cases} 
\mathcal{N}_{f(s)} & \text{if } \mathcal{U}_s = \mathcal{N}_x \\
\mathcal{N}_{y_0} & \text{if } f^0(\mathcal{U}_s) \rightarrow y_0, \\
f^0(\mathcal{U}_s) & \text{otherwise}
\end{cases} \quad \text{if } \mathcal{U}_s \in \mu X - X, \]
where \( f^0(\mathcal{U}_s) = \{ V \in \{ S \subset Y \mid \text{int} \, \text{cl}(S) = S \} \mid f^{-1}(V) \in \mathcal{U}_s \} \) and \( \mathcal{N}_x \) be the open neighborhood system of \( x \).

In addition, if \( f^* \) satisfies the condition \( f^*(A) \subseteq \text{int}_Y \ast \text{cl}_Y \ast (f^*(\text{cl}_X \ast A)) \) for all \( A \) open in \( X^* \). Then \( f^*: \mu X \to \mu Y \) is a unique continuous extension of \( f \).

Theorem 3.10 can be restated as follows:

**3.11 Theorem.** If \( X \) and \( Y \) are semiregular and \( f: X \to Y \) is \( pB^* \)-continuous, then there exists a unique extension \( f^*: \mu X \to \mu Y \) of \( f \) which is also a \( pB^*_e \)-continuous.

### 4. Categorical aspects of BFS

A functor \( r \) from a category \( \mathcal{C} \) to a subcategory \( \mathcal{D} \) of \( \mathcal{C} \) is a reflective functor if there is a morphism \( f: C \to rC \) and every morphism \( g \) from \( C \) to an object \( D \) of \( \mathcal{D} \) factors uniquely through \( rC \) via \( f \) so that the following diagram commutes:

\[
\begin{array}{ccc}
C & \xrightarrow{f} & rC \\
\downarrow g & & \downarrow h \\
\downarrow f & & \downarrow h \\
D & \xrightarrow{\text{reflected}} & D \\
\end{array}
\]

If \( r: \mathcal{C} \to \mathcal{D} \) is a reflective functor, the subcategory \( \mathcal{D} \) is called a reflective subcategory.

The object \( rC \) of \( \mathcal{D} \) is called the reflection of \( C \) in \( \mathcal{D} \). If \( r \) is an epimorphism, the subcategory \( \mathcal{D} \) is called the epireflective subcategory of the category \( \mathcal{C} \).

The Stone–Cech compactification is an example of epireflective functor from the category of completely regular Hausdorff spaces to the category of compact Hausdorff spaces.

The study of epireflective subcategories of Hausdorff spaces is an important part of topological research. One of the well known result related to \( H \)-closed extension is due to Harris [5]. He proved that the category of \( H \)-closed spaces and maps is an epireflective subcategory of the category of Hausdorff spaces and \( p \)-maps, and the epireflection of a space is its Katetov extension. In the case of BFS-extension Hunsaker and Naimpally [7] have shown that the category of minimal Hausdorff spaces and continuous maps is an epireflective subcategory of the category of semiregular spaces and \( \lambda \)-perfect maps. But by theorem 4.4 of [7], composition of two \( \lambda \)-perfect maps is not \( \lambda \)-perfect map unless the \( \lambda \)-perfect map takes free open ultrafilters to free open ultrafilters. Therefore theorem 4.5 of [7] is required additional restriction on maps to get the respective categories. Also the way in which theorem 4.5 of [7] has been stated shows that maps of the subcategories do not belong to their respective categories.

In order to get a suitable answer we start with the following:
4.1 Lemma. If $\mathcal{M}_0$ denotes the collection of minimal Hausdorff spaces and $p$-maps having the property: for every $f$ in $\mathcal{M}_0$, $f(A) \subseteq \text{int} \ cl(f(clA))$ for all $A$ open in domain $f$. Then $\mathcal{M}_0$ forms a category.

Proof. It is enough to show that composition of two maps in $\mathcal{M}_0$ is again a map in $\mathcal{M}_0$.

Let $f$ and $g$ be two maps in $\mathcal{M}_0$ with codomain $f = \text{domain } g$, and let $A$ be open in the domain $f$.

$$gf(A) = g(f(A)) \subseteq g(\text{int}(\text{cl}(f(\text{cl}A))))$$

$$\subseteq \text{int}(\text{cl}(g(\text{int}(\text{cl}(f(\text{cl}A)))))$$

$$\subseteq \text{int}(\text{cl}(g(\text{cl}(f(\text{cl}A)))))$$

$$\subseteq \text{int}(\text{cl}(gf(\text{cl}A))).$$

This shows that $gf$ is a map in $\mathcal{M}_0$.

4.2 Lemma. If $\mathcal{S}$ denotes the collection of semiregular spaces and $pB^*_\xi$-continuous maps then $\mathcal{S}$ forms a category.

Proof. Use of the fact that composition of two $pB^*_\xi$-continuous maps is again $pB^*_\xi$-continuous.

4.3 Lemma. The category $\mathcal{M}$ of minimal Hausdorff spaces and $pB^*_\xi$-continuous maps is a full subcategory of $\mathcal{S}$.

By theorem 3.11 and the above lemmas we have the following.

4.4 Theorem. $\mathcal{M}$ is an epireflective subcategory of $\mathcal{S}$. Moreover, $\mathcal{M}$ is the largest subcategory of $\mathcal{S}$ in the sense of Porter [12].

Note that, a monomorphism $j$ is an extremal monomorphism if whenever we have the commutative diagram as illustrated so that $e$ is an epimorphism, then $e$ is an isomorphism. The object $X$ is said to be extremal subobject of $Y$.

Since the extremal subobjects in the category of Hausdorff spaces are the closed subspaces, and the epireflective subcategories are closed under extremal subobjects. Therefore it is interesting to see the extremal subobjects in the epireflective subcategories. In this connection we have the following.

4.5 Problem. Characterize the extremal subobjects of $\mathcal{M}$ in $\mathcal{S}$.

Since the result stated in theorem 3.10 is also true for strict extension of Fomin type, therefore theorem 3.11 can be restated as, if $f : X \to Y$ is $pB^*_\xi$-continuous, then there
exists a unique extension $f^*: \sigma X \to \sigma Y$ of $f$ which is also a $pB^*_\kappa$-continuous. On the line of theorem 4.4 we have the following theorem.

**4.6 Theorem.** The category of $H$-closed spaces and $pB^*_\kappa$-maps is an epireflective subcategory of Hausdorff spaces and $pB^*_\kappa$-maps, $\sigma$ as an epireflection.

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**References**