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Finite dimensional imbeddings of harmonic spaces

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Abstract. For a noncompact harmonic manifold M we establish finite dimensionality of the eigensubspaces V_λ generated by radial eigenfunctions of the form $\cosh r + c$. As a consequence, for such harmonic manifolds, we give an isometric imbedding of M into (V_λ, B), where B is a nondegenerate symmetric bilinear indefinite form on V_λ (analogous to the imbedding of the real hyperbolic space H^n into $\mathbb{R}^{n+1}$ with the indefinite form $Q(x, x) = -x_0^2 + \Sigma x_i^2$). This imbedding is minimal in a 'sphere' in (V_λ, B). Finally we give certain conditions under which M is symmetric.

Keywords. Harmonic manifolds; eigen spaces, imbeddings; symmetric spaces.

1. Introduction

A Riemannian manifold (M, g) is said to be harmonic if its volume density function $\omega_m = \sqrt{|\det(g_{ij})|}$ is a spherically symmetric function around m. In polar coordinates this density can be written as

$$\theta_m = r_m^{d-1}\,\omega_m,$$

where $r_m(n) = r(m, n)$ is the geodesic distance between m and n. Thus (M, g) is harmonic if θ_m is a radial function around m (see [8] or [1] for details).

Let (M, g) be a harmonic manifold and let Δ be the Laplacian on M. Consider a nonzero eigenvalue λ and the corresponding eigenspace V_λ. If M is compact, V_λ is known to be finite dimensional. In this case Besse [1] constructed isometric immersion of M into V_λ. Moreover the image of M in this case is a minimal submanifold of some sphere in V_λ. This immersion was crucially used by Szabo [10] in his proof of the Lichnerowicz conjecture, which asserts that *harmonic manifolds are symmetric*. Szabo proved the conjecture for compact harmonic manifolds with finite fundamental group. In his paper he generalized Besse's imbedding theorem to give isometric immersions of any harmonic manifold M into $L^2(M, g)$, an infinite dimensional space. In particular he recovers Besse's imbedding for compact harmonic manifold.

The eigenspaces V_λ of noncompact manifolds, in particular of noncompact harmonic manifolds, need not be finite dimensional. Nevertheless the problem of imbedding M into a finite dimensional vector space albeit with an indefinite metric is important and has not been addressed (analogous to the imbedding of the real hyperbolic space H^n into $\mathbb{R}^{n+1}$ with the indefinite metric $Q(x, x) = -x_0^2 + \Sigma x_i^2$). We prove a partial result in this direction. Before we state our result let us recall that the action of the Laplacian on radial functions f is

$$f''(r) + \frac{\theta'(r)}{\theta(r)} f' + \lambda f = 0,$$

where $\theta(r)$ is the density function of M. We now state our theorem.

Theorem 1.1. *Let* (M^d, g) *be a noncompact harmonic manifold with density function* $\theta(r)$. *Let* (f, λ) *be a solution of the equation*

$$f''(r) + \frac{\theta'(r)}{\theta(r)} f' + \lambda f = 0 \tag{1.1}$$

with $\lambda \neq 0$. *Assume that* f *is of the form*

$$f(r) = \cosh r + D$$

where D *is a constant. Let* V_λ *be the eigensubspace generated by radial eigenfunctions. Then*

1. V_λ *is finite dimensional and*
2. *there exists a symmetric nondegenerate bilinear form* B *on* V_λ *and an isometric imbedding of* M *into* $(V_\lambda, -B)$. *Moreover the image of* M *is a minimal submanifold of a 'sphere' in* V_λ.

Moreover in this case the volume density is

$$\theta(r) = 2^{d-1}\left(\sinh\frac{r}{2}\right)^{d-1}\left(\cosh\frac{r}{2}\right)^{d-1-4/3(k+d-1)}$$

We can also calculate λ *and* D *in terms of the Ricci curvature. They are given by the formulas*

$$D = \frac{k+d-1}{\frac{d}{2}+1-k}$$

and

$$\lambda = -\frac{2}{3}\left(\frac{d}{2}+1-k\right)$$

where Ricci $(M) = k$. *Finally* $f(r) = \cosh r + D$ *has no zeroes on* M.

Note. $D + 1 > 0$ and V_λ is not the full eigenspace. Moreover the fact that the density function is

$$\theta(r) = 2^{d-1}\left(\sinh\frac{r}{2}\right)^{d-1}\left(\cosh\frac{r}{2}\right)^{d-1-(4/3)(k+d-1)}$$

is equivalent to the fact that there exists an eigenfunction of the form

$$f(r) = \cosh r + D,$$

where

$$D = \frac{k+d-1}{(d/2)+1-k}.$$

In the next section we prove our main theorem. In the last section we prove symmetry of M under some assumptions.

2. The imbedding of M into $(V_\lambda, -B)$

Before proving our main theorem we first show that all known examples of noncompact harmonic spaces satisfy the hypothesis of our theorem. Thus all known noncompact harmonic manifolds admit the desired imbedding. Nevertheless we strongly believe that all harmonic manifolds admit the desired imbedding. Notice that all the noncompact rank one symmetric spaces are NA spaces [3, 4].

Examples

1. *Real hyperbolic space* $\mathbb{R}H^d$. We know that the density function of the real hyperbolic space is

$$\theta(r) = \sinh^{d-1} r.$$

Thus the equation we wish to solve is

$$f''(r) + (d-1)\coth r f'(r) + \lambda f(r) = 0.$$

Put $f(r) = \cosh r$ in the above equation to get

$$f''(r) + (d-1)\coth r f'(r) = d\cosh r.$$

Hence $(\cosh r, -d)$ is the required solution.

2. *Complex hyperbolic space* $\mathbb{C}H^d$. In this case the density function is

$$\theta(r) = 2^{2d-1}\left(\sinh\frac{r}{2}\right)^{2d-1}\cosh\frac{r}{2}.$$

The function $f(r) = \cosh r + (d-1)/(d+1)$ satisfies the equation

$$f''(r) + \frac{\theta'(r)}{\theta(r)} f'(r) - (d+1)f = 0.$$

Thus $(\cosh r + ((d-1)/(d+1)), -(d+1))$ is the required solution.

3. *The NA spaces.* Recall that the density of the NA spaces ([3]) is of the form

$$\theta(r) = 2^{m+l}\left(\sinh\frac{r}{2}\right)^{m+l}\left(\cosh\frac{r}{2}\right)^{l}.$$

Hence

$$\frac{\theta'(r)}{\theta(r)} = \frac{1}{2}\left\{(m+l)\coth\frac{r}{2} + l\tanh\frac{r}{2}\right\}.$$

Assume $f(r) = \cosh r + D$, then

$$f''(r) + \frac{\theta'(r)}{\theta(r)} f'(r) = \frac{m+2l+2}{2}\cosh r + \frac{m}{2}.$$

Thus $(\cosh r + (m/(m+2l+2)), -(m+2l+2)/2)$ is the required solution. Let us make a simple but useful observation.

Observation. Let V be a vector space, $S \subseteq V$ a set such that $V = \operatorname{span} S$. Let $\tilde{B}: S \times V \to R$ be a map such that

1. $\tilde{B}$ is linear in the second variable, i.e., $\tilde{B}(s, .)$ is linear on V for all s in S.
2. $\tilde{B}(s, t) = \tilde{B}(t, s)$ for all $s, t \in S$.

Then there exists a unique bilinear form $B: V \times V \to R$ which extends $\tilde{B}$.

Proof. Define

$$B\left(\sum_i a_i s_i, v\right) = \sum_i a_i \tilde{B}(s_i, v).$$

Proof of Theorem 1.1. Let m be any point of M and let B_m be the unit ball around m. Let $C^\infty(B_m)$ be the space of smooth functions on B_m with the sup norm.

Consider the restriction map

$$\Psi: V_\lambda \to C^\infty(B_m); \quad f \mapsto f|_{B_m}.$$

Due to analyticity Ψ is injective. We will show that $\operatorname{Im}\Psi$ is finite dimensional. This will prove that V_λ is finite dimensional.

Let ϕ be an eigenfunction for the eigenvalue λ and let ϕ_m be the corresponding radial eigenfunction around m. Then ϕ_m satisfies the differential equation (1.1). Thus $\phi_m \in V_\lambda$. Let $V_\lambda^+ = \{\sum_m a_m \phi_m : a_m \geqslant 0, m \in M\}$ where only finite sums are taken. V_λ^+ is the cone generated by $\{\phi_m, m \in M\}$.

Claim 1. $V_\lambda \cap C^\infty(B_m)$ is finite dimensional, i.e. $\operatorname{Im}\Psi$ is finite dimensional.

Proof. Let $h = \sum_1^k a_{m_i} \phi_{m_i} \in V_\lambda^+$. By the hypothesis of the theorem,

$$\phi_{m_i} = \cosh d(m_i, .) + D.$$

Thus

$$\nabla h = \sum_1^k a_{m_i} \sinh d(m_i, .) \frac{\partial}{\partial r_i},$$

where r_i is the distance function from the point m_i. Hence

$$|\nabla h|^2 = \sum_{i=1}^k a_{m_i}^2 (\sinh d(m_i, .))^2$$

$$+ 2 \sum_{i=1}^k a_{m_i} a_{m_j} \sinh d(m_i, .) \sinh d(m_j, .) \left\langle \frac{\partial}{\partial r_i}, \frac{\partial}{\partial r_j} \right\rangle.$$

But by Cauchy–Schwartz inequality $\langle \partial/\partial r_i, \partial/\partial r_j \rangle \leqslant 1$. Moreover $D > -1$ and $a_{m_i} \geqslant 0$. Hence we get

$$|\nabla h|^2$$

$$\leqslant \sum_{i=1}^k a_{m_i}^2 \sinh^2 d(m_i, .)$$

$$+ 2 \sum_{i=1}^k a_{m_i} a_{m_j} \sinh d(m_i, .) \sinh d(m_j, .)$$

$$= \left(\sum_{i=1}^k a_{m_i} \sinh d(m_i, .) \right)^2$$

$$\leqslant C \left(\sum_{i=1}^k a_{m_i} \cosh d(m_i, .) + D \right)^2 \quad \text{where } C \text{ is some constant}$$

$$= C|h|^2.$$

Now we consider the cone $X = C^\infty(B_m) \cap V_\lambda^+$. Since $|\nabla h|_{B_m}| \leqslant |\nabla h|$, the above inequality shows that for $h \in X$, $|\nabla h| \leqslant C|h|$ holds. This shows the following: If W is an open ball in V_λ^+ such that $W \cap C^\infty(B_m)$ is a bounded family then it is a pointwise bounded equicontinuous family in X. Hence by the Ascoli–Arzela theorem it is relatively compact in X.

To prove the claim we now argue as follows: let U be the open ball around the origin in V_λ. Choose a g in V_λ such that $U \subset V_\lambda^+ - g$. The above argument shows that $(U + g) \cap C^\infty(B_m)$ is relatively compact in $V_\lambda^+ \cap C^\infty(B_m)$. Hence U is relatively compact in V_λ. Thus $V_\lambda \cap C^\infty(B_m)$ is finite dimensional and Claim 1 is proved.

Let $S = \{\phi_m : m \in M\}$ then $V_\lambda = \text{Spans } S$. Define

$$\tilde{B} : S \times V_\lambda \to \mathbb{R}$$

by

$$\tilde{B}(\phi_m, \textstyle\sum a_n \phi_n) = \sum a_n \phi_n(m).$$

Then by the above observation there exists a unique $B : V_\lambda \times V_\lambda \to \mathbb{R}$, which extends $\tilde{B}$.

Claim 2. B is symmetric and nondegenerate.

Proof. Clearly B is symmetric. Nondegeneracy of B can be established as follows. Let $h \in \operatorname{Ker} B$ then

$$B(h, \phi_n) = 0, \text{ for all } n \in M.$$

Hence

$$h(n) = 0 \text{ for all } n \in M,$$

i.e., $h \equiv 0$ on M and hence B is nondegenerate. This completes the proof of Claim 2.

Now define the map

$$\Phi : M \to (V_\lambda, -B)$$

by

$$\Phi(m) := \phi_m = \cosh d(m, .) + D.$$

Let $\gamma(t)$ be a geodesic in M then

$$B(\phi_{\gamma(t)}, \phi_{\gamma(t)}) = \phi_{\gamma(t)}(\gamma(t)) = 1 + D.$$

This shows that $\Phi(M)$ is contained in $S = \{B(h, h) = 1 + D\}$, a 'sphere' in (V_λ, B). Further

$$B(\phi_{\gamma(t)}, \phi_{\gamma(t)}) = 1 + D$$

gives on twice differentiating,

$$B(\phi'_{\gamma(t)}, \phi'_{\gamma(t)})|_{t=0} + B(\phi''_{\gamma(t)}, \phi_{\gamma(t)})|_{t=0} = 0, \tag{2.1}$$

where $\phi'_{\gamma(t)} = \mathrm{d}/\mathrm{d}t(\phi_{\gamma(t)})$ etc. But

$$\phi_{\gamma(t)} = \cosh d(\gamma(t), .) + D.$$

Hence

$$B(\phi''_{\gamma(t)}|_{t=0}, \phi_{\gamma(0)}) = 1.$$

Therefore eq. (2.1) gives

$$B(\phi'_{\gamma(t)}, \phi'_{\gamma(t)})|_{t=0} = -1.$$

This shows that Φ is an isometric immersion of M into $(V_\lambda, -B)$. Finally, since $\cosh r + D$ is a monotone function, $\phi_m \neq \phi_n$ for $m \neq n$. Thus Φ is an imbedding of M into $(V_\lambda, -B)$.

Claim 3. $\Phi(M)$ is minimal in S.

Proof. Note that the Levi–Civita connection on $(V_\lambda, -B)$ is the flat Euclidean connection. Moreover the restriction of B to $T_p(\Phi(M))$, the tangent space of $\Phi(M)$, is positive definite. Thus $T_p(\Phi(M))^\perp$ is complementary to $T_p(\Phi(M))$. Now any vector can be uniquely decomposed into tangential and normal components so that the second fundamental form can be defined. Hence it makes sense to define the mean curvature of $\Phi(M)$ as the trace of the second fundamental form. Now since the imbedding is given by eigenfunctions we conclude that $\Phi(M)$ is minimal in S by a result of [6], pp. 340.

We now compute the densities of these harmonic manifolds. Recall that the function $f(r) = \cosh r + D$ satisfies the equation

$$f'' + \frac{\theta'}{\theta} f' + \lambda f = 0.$$

Hence we get

$$\frac{\theta'}{\theta}(r) \sinh r = -(1+\lambda)\cosh r - \lambda D$$

which gives

$$\theta(r) = 2^{-(1+\lambda)}\left(\sinh\left(\frac{r}{2}\right)\right)^{-(1+\lambda+\lambda D)}\left(\cosh\left(\frac{r}{2}\right)\right)^{-(1+\lambda-\lambda D)}$$

But as $r \to 0$, $\theta(r) \to (\sinh(r/2))^{d-1}$, hence we get

$$1 + \lambda + \lambda D = 1 - d, \quad \text{and} \quad \theta(r) = 2^{(d-1)}\left(\sinh\left(\frac{r}{2}\right)\right)^{d-1}\left(\cosh\left(\frac{r}{2}\right)\right)^{(d-1+2\lambda D)}$$

Let us now use the fact that harmonic manifolds are Einstein. Let Ricci $(M) = k$. Now

$$\theta(r) = r^{d-1}\omega(r)$$

gives

$$\frac{\theta'(r)}{\theta(r)} = \frac{d-1}{r} + \frac{\omega'(r)}{\omega(r)}.$$

Using the expression for θ obtained above, we get

$$\frac{\omega'(r)}{\omega(r)} = -(1+\lambda)\coth r - \frac{\lambda D}{\sinh r} - \frac{n-1}{r}.$$

Differentiating twice and after simplification, we get

$$-\frac{1}{3}\,\mathrm{Ricci}\,(M) = \frac{d-1}{3} + \frac{\lambda D}{2}$$

which gives

$$\lambda D = -\frac{2}{3}(k + d - 1), \quad \text{but } 1 + \lambda + \lambda D = 1 - d$$

hence

$$\lambda = -\frac{2}{3}\left(1 - k + \frac{d}{2}\right) \quad \text{and} \quad D = \frac{k+d-1}{1-k+(d/2)}.$$

Finally the above computations show that

$$\begin{aligned} f(r) &= \cosh r + D \\ &= \cosh r - 1 + \frac{3d/2}{1-k+(d/2)} \\ &= 2\sinh^2\left(\frac{r}{2}\right) + \frac{3d/2}{1-k+(d/2)} \\ &\geqslant \frac{3d/2}{1-k+(d/2)}. \end{aligned}$$

Thus f has no zeroes on M. This completes the proof of the theorem. ■

3. Symmetry of M

Let M satisfy the hypothesis of Theorem 1.1. We prove the symmetry of M under some assumptions. Two cases arise.

Case 1. $D = 0$

In this case M is symmetric, since the Sturm–Liouville equation

$$f'' + \frac{\theta'}{\theta} f' + \lambda f = 0$$

shows that

$$\theta(r) = \sinh^{d-1} r$$

and Theorem (4.2.9) shows that M is isometric to the real hyperbolic space with constant curvature -1.

Case 2. $D \neq 0$

Let $\gamma(t)$ be geodesic of M and ϕ be an eigenfunction for the eigenvalue λ. Let V_γ be the subspace of V_λ generated by $\{\phi_{\gamma(t)} : t \in \mathbb{R}\}$.

Lemma 3.1. *Let M satisfy the hypothesis of Theorem* (1.1). *Assume in addition that B restricted to the subspace V_γ is nondegenerate. Then M is symmetric.*

Proof. Fix an imbedding $\Phi : M \to V_\lambda; \Phi(m) = \phi_m$. Let $\gamma(t)$ be a geodesic on M. Then

$$B(\phi_{\gamma(t)}, \phi_{\gamma(0)}) = \cosh t + D.$$

Differentiating we get

$$B(\phi'''_{\gamma(t)} - \phi'_{\gamma(t)}, \phi_{\gamma(0)}) = 0.$$

Now $\gamma(0)$ is an arbitrary point of M and $\phi'''_{\gamma(t)} - \phi'_{\gamma(t)} \in V_\gamma$. Since by our assumption B is nondegenerate on V_γ, the geodesic γ satisfies

$$\phi'''_{\gamma(t)} - \phi'_{\gamma(t)} = 0.$$

Take $m \in M$. Note that $B|_{T_m(M)}$ is positive, hence we can take the orthogonal complement N_m with respect to B, of $T_m(M)$ to M. Define

$$\tau_m : V_\lambda \to V_\lambda$$

to be the reflection with respect to the subspace N_m. τ_m is an isometry of V_λ, it fixes m and reverses the geodesics through m. Moreover it also leaves M invariant. Hence it induces an isometry on M which is obviously the geodesic involution of M. Thus M is a symmetric space. ■

The next lemma shows that V_γ is 3-dimensional if and only if B is nondegenerate on it.

Lemma 3.2. *Let M satisfy the hypothesis of our theorem. Let γ be a geodesic, then V_γ is 3-dimensional if and only if B is nondegenerate on V_γ.*

Proof. Fix an imbedding $\Phi: M \to V_\lambda : \Phi(m) = \phi_m$ where ϕ is an eigenfunction for the eigenvalue λ. Let $s_i = -1, 0, 1$ be such that $\{\phi_{\gamma(s_i)}\}_1^3$ generate V_γ. Let $g \in V_\gamma$ satisfy $B(g, \phi_{\gamma(t)}) = 0$ for all t. Put

$$g = \sum_1^3 a_i \phi_{\gamma(s_i)}$$

$$= \sum_1^3 a_i \cosh d(\gamma(s_i), .) + D \sum_1^3 a_i.$$

Now $B(g, \phi_{\gamma(t)}) = 0$ gives $g(\gamma(t)) = 0$ for all t, i.e

$$\sum a_i D + \sum a_i \cosh(s_i - t) = 0, \text{ for all } t.$$

Dividing by $\cosh t$ and letting $t \to \infty$ we see that

$$\sum a_i = 0,$$

hence

$$\sum a_i \cosh(s_i - t) = 0, \text{ for all } t.$$

Expanding and equating the coefficients of $\cosh t$ and $\sinh t$ to 0 we get

$$\sum a_i \cosh s_i = 0 = \sum a_i \sinh s_i.$$

Substituting the values of s_i and using the equation $\sum a_i = 0$ we obtain a system of three equations

$$a_0 \cosh 1 + a_1 + a_2 \cosh 1 = 0,$$

$$-a_0 \sinh 1 + 0 + a_2 \sinh 1 = 0,$$

$$a_0 + a_1 + a_2 = 0.$$

But the coefficient matrix of the above system is nonsingular, hence $a_i = 0$ for all i, i.e $g \equiv 0$ or B is nondegenerate.

Conversely let B be nondegenerate on V_χ. As in proof of lemma (5.3.6), we see that $\phi_{\gamma(t)}$ satisfies the differential equation

$$\phi'''_{\gamma(t)} - \phi'_{\gamma(t)} = 0.$$

Thus V_γ is 3 dimensional. Combining the above two lemmas one sees that M is symmetric if V_γ is 3-dimensional.

4. Remarks

For manifolds which satisfy the hypothesis of the theorem the density $\theta(r)$ is given by

$$\theta(r) = 2^{d-1}\left(\sinh\frac{r}{2}\right)^{d-1}\left(\cosh\frac{r}{2}\right)^{d-1-(4/3)(k+d-1)}$$

Put $b = d - 1 - (4/3)(k + d - 1)$. By the Bishop–Gromov comparison theorem ([5] or [7]), b satisfies $0 \leqslant b \leqslant d - 1$.

1. If $b = d - 1$, $\theta(r) = \sinh^{d-1} r$ and hence M is isometric to the real hyperbolic space. In this case $D = 0$.
2. If $b = 0$, $\theta(r) = 2^{d-1}(\sinh(r/2))^{d-1}$. Again M is isometric to the real hyperbolic space. But in this case $D \neq 0$ and the eigenfunction $f_p = \cosh r + D$ becomes the eigenfunction for the next eigenvalue.
3. All the other harmonic spaces which satisfy the hypothesis of the theorem, in particular the NA spaces have $0 < b < d - 1$.

Question. Is b an integer? It is for the NA spaces.

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References

[1] Besse A L, *Manifolds all of whose geodesics are closed*, Berlin: Springer (1978)

[2] Besse A L, *Einstein Manifolds*, (Berlin: Springer) (1987)

[3] Damek E and Riccı F, A class of nonsymmetric harmonic Riemmanıan spaces, *Bull. AMS* **27** (1992) 139–142

[4] Damek E and Ricci F, Harmonic analysis on solvable extension of H-type groups, *J. Geom. Anal.* **2** (1992) 213–247

[5] Gallot S, Hulin D and Lafontaine J, *Riemannian Geometry* (Berlin: Springer) (1990)

[6] Kobayashi S and Nomizu K, *Foundations of Differential Geometry, II* (New York: Wiley Inter-science) (1969) p. 340

[7] Kumaresan S, *A Course in Riemannian Geometry*, Lecture Notes, Instructional Workshop on Riemannian Geometry, T.I.F.R., Bombay 1990

[8] Ramachandran K and Ranjan A, *Curvature and helical immersions of harmonic manifolds*, Preprint, 1995

[9] Ramachandran K and Ranjan A, Harmonic manifolds with some specific volume densities, *Proc. Indian Acad. Sci.* **107** (1997) 251–261

[10] Szabo Z I, The Lichnerowicz Conjecture on Harmonic Manifolds, *J. Diff. Geom.* **31** (1990) 1–28

[11] Szabo Z I, Spectral theory for operator families on Riemannian manifolds, *Proc. Symp. Pure Math.* **54** (1993) 615–665