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Transformation of chaotic nonlinear polynomial difference systems through Newton iterations

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Abstract. Chaotic sequences generated by nonlinear difference systems or 'maps' where the defining nonlinearities are polynomials, have been examined from the point of view of the sequential points seeking zeroes of an unknown function f following the rule of Newton iterations. Following such nonlinear transformation rule, alternative sequences have been constructed showing monotonic convergence. Evidently, these are maps of the original sequences. For second degree systems, another kind of possibly less chaotic sequences have been constructed by essentially the same method. Finally, it is shown that the original chaotic system can be decomposed into a fast monotonically convergent part and a principal oscillatory part showing sharp oscillations. The methods are exemplified by the well-known logistic map, delayed-logistic map and the Hénon map.

Keywords. Chaos; nonlinear polynomial difference systems; Newton iterations; convergent sequences; decomposition; principal oscillatory part.

1. Introduction

Nonlinear discrete dynamical systems described by a variety of difference equations have been studied in depth as iterative maps and the different associated (qualitative) aspects are described in several books, such as, Baker and Gollub [2], Lauwerier [4], Lichtenberg and Lieberman [5], the review article by Whitney [11] specially for bifurcations and Smale [10] for diffeomorphism of manifolds where many other references can be found. Chaos, the main objective of these studies has also been given stochastic interpretation (cf. [4, 5]).

Herein, we consider such systems as sequences defined by nonlinear equations, which must converge to some point, diverge to infinity or oscillate finitely or infinitely. Chaos belongs to the last category when the oscillations are finite aperiodic oscillations. We look into ways of constructing nonlinear transformations, which are maps of the original but masking chaos by way of reduced chaotic oscillations or even monotonic convergence. Due to such behaviour of the new sequences, the error in long iterative computer simulations is well contained, while it is not so in an original given chaotic sequence. Handling and storage of the generated data is also accomplished more efficiently. Thus in a stored form, the new sequences will be better working tools and when the original sequence is needed, the inverse of the original trnsformation can be invoked. The plotted form of the new sequences can also possess their own geometrical appeal. These aspects in a general sense, can also be of significance in the treatment of difficult nonlinear differential equations by finite differences.

For construction of such transformations, we assume that the nonlinear difference equation is given in the form

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n), \tag{1}$$

where $\mathbf{x} = (x^1, x^2, \ldots, x^m)$, m usually $\leqslant 3$ and $\mathbf{F} = (F^1, F^2, \ldots, F^m)$ has polynomial forms as its components, and the oscillating sequence seeks certain set in a domain containing a fixed point of the map. Transferring origin to the fixed point, we can assume the form

$$F^i = x_n^i - \sum_{j=1}^{m} a_j^i x_n^j - \sum g^i_{i_1 i_2 \ldots i_m} (x_n^1)^{i_1} (x_n^2)^{i_2} \cdots (x_n^m)^{i_m}, \tag{2}$$

$2 \leqslant i_1 + i_2 + \cdots + i_m \leqslant p_i$, where p_i is the degree of the polynomial and the origin becomes the fixed point of interest. It is shown in § § 2 and 3 that convergence property depends solely on the coefficients a_j^i. We assume here the presence of chaotic oscillations. Now, in the Newton iteration of zeroes of a single function $f(\mathbf{x})$ [7] the sequence of points is

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \{f'(\mathbf{x}_n)\}^{-1} \cdot f(\mathbf{x}_n),$$

where $f'(\mathbf{x}_n)$ is the Jacobian matrix, which in this case is the diagonal matrix $\mathrm{diag}\{\nabla f(\mathbf{x}_n)\}$. This is of the form (1). Thus we can interpret that the dynamics (1) seeks zeroes of $f(\mathbf{x})$ according to the above rule which can be written as

$$(\mathbf{x}_{n+1} - \mathbf{x}_n) \cdot \nabla f(\mathbf{x}_n) = -f(\mathbf{x}_n). \tag{3}$$

Hence, the sequence of points lie on the solutions of

$$[\mathbf{F}(\mathbf{x}) - \mathbf{x}] \cdot \nabla f = -f, \tag{4}$$

which is a Lagrange linear first order P.D.E [3]. The characteristics of the equation completely determining the general solution of (4) are given by

$$\frac{\mathrm{d}x^i}{F^i(\mathbf{x}) - x^i} = \frac{\mathrm{d}f}{-f}, \quad i = 1, 2, \ldots, m. \tag{5}$$

Interestingly, the zeroes of f are the fixed points of (1), if ∇f is finite. Because of this, one may say, the chaotic sequence (1), seeking a zero of f, has a strange attraction caused by the fixed point.

In § § 2 and 3, it is shown that in the general solution of (4) with (2), fractional power functions of the arguments are present in the neighbourhood of the fixed point, namely the origin. The solution in the general m-dimensional case in § 3, is achieved after a linear transformation of the arguments. Suggestive from the forms, new arguments which are power functions of the old ones are then introduced which in effect clear the fractional powers. By subjecting them to the law (5) and the Newton's iterative rule (3) in the new arguments, we obtain a new sequence monotonically convergent in nature. Thus the new sequence even though convergent, is a map of the original obtained by these nonlinear rules. As examples we consider the treatment of the well known logistic map and in two dimensions, the delayed-logistic and the Hénon maps.

In the next section, we consider quadratic systems and look into the effect of stretching by sending the nonzero fixed point to infinity by exponential transformation. As before new sequences are constructed, which in one dimension are oscillatory, confirmed by the logistic map. In two dimensions, the delayed-logistic map is oscillatory but the Hénon map is slowly divergent. Chaotic oscillations in the oscillatory cases is very much reduced as is expected from the nature of transformation. Here, some interesting plots have been obtained.

It is noticed in §§ 2 and 3 that the convergence of the sequence (1), (2) depends on the linear part only. Hence, by introducing an under relaxation factor ω in damped Newton method we can construct fast monotonically convergent sequences. Rearranging the right hand side of (2) of the original sequence on the same basis, the defining equation (1) can be split into a fast monotonically convergent part and a principal highly oscillatory part which is really responsible for the chaotic oscillations. This is discussed in § 5. A consequence is that it should be possible to construct new chaotic sequences by combining the principal oscillatory parts with other monotonically convergent sequences.

2. One-dimension

In one dimension the difference equation (1) is of the form

$$x_{n+1} = (1-a)x_n - \sum g_i x_n^i, \quad 2 \leqslant i \leqslant p. \tag{6}$$

The fixed point $x^* = 0$ is under consideration. By iteration the solution of (6) is

$$x_n = (1-a)^n x_0 + (1-a)^{n-1} O(x_0^2), \tag{7}$$

which converges to the fixed point (or in other words the fixed point is attracting or stable) if $|1-a| < 1$, i.e. $0 < a < 2$. The convergence is monotonic for $0 < a \leqslant 1$ and oscillatory for $1 < a < 2$.

For increasing $a \geqslant 2$, the oscillatory behaviour may persist with bifurcations, chaos and divergence. For such an eventuality, we seek alternative sequence, map of the original one viz. (6) masking these features and converging to the fixed point. If equation (6) defines Newton's iteration for the zeroes of a function $f(x)$, then (3) is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{8}$$

and by comparison, f satisfies the differential equation

$$\frac{\mathrm{d}x}{-ax - \sum g_i x^i} = -\frac{\mathrm{d}f}{f}, \tag{9}$$

which is a one parameter (c) family of curves. In the neighbourhood of the fixed point $x = x^* = 0$, dropping the $O(x^2)$ terms,

$$f = c|x|^{1/a}, \tag{10}$$

which is an ath degree parabola through the fixed point. The form of f explains oscillations in the sequence for $a > 1$ seeking the zero by the tangent method of Newton. It is suggestive of transformation of f defined by (9) according to

$$x = |X|^k \operatorname{sgn}(X), \quad k \geqslant a. \tag{11}$$

Equation (9) then becomes

$$\frac{k|X|^{k-1}\mathrm{d}X}{-aX|X|^{k-1} - \sum g_i X^i |X|^{i(k-1)}} = -\frac{\mathrm{d}f}{f}. \tag{12}$$

The corresponding Newton iteration is

$$X_{n+1} = X_n + \frac{1}{k}[-aX_n - \sum g_i X_n^i |X_n|^{(i-1)(k-1)}]. \tag{13}$$

As in the case of (6), this sequence (13) converges monotonically in as much as $0 \leqslant 1 - a/k < 1$. The sequence will be closely packed near $X^* = 0$ for large enough n. If we want to obtain the original sequence (6) from the monotonically convergent sequence (13), then we retrace the steps by invoking the transformation rule (12), (11), (9) and (8).

2.1 *Example*

The logistic map: This 'map' is the iteratively defined sequence

$$\xi_{n+1} = a\xi_n(1 - \xi_n), \quad a \geqslant 1. \tag{14}$$

The fixed points of the iteration are $\xi^* = 0, (a-1)/a$. The former is unstable and not of much interest. For the latter, the transformation $\xi = x + (a-1)/a$ converts (14) to

$$\begin{aligned} x_{n+1} &= -(a-2)x_n - ax_n^2 \\ &= \{1 - (a-1)\}x_n - ax_n^2. \end{aligned} \tag{15}$$

Thus, as in the general discussion, the sequence is monotonic convergent to the fixed point $x^* = 0$ for $1 \leqslant a < 2$ and oscillatory for $2 \leqslant a < 3$. It is for $a \geqslant 3$, much has been written in the earlier literature on chaos. It is known [6] that for $3 \leqslant a \leqslant 1 + \sqrt{6} \approx 3{\cdot}54$, the sequence exhibits bifurcation with a cycle of period 2. For $1 + \sqrt{6} < a \leqslant 3{\cdot}57$, cascade of bifurcations of cycles of periods $2, 4, 8, \ldots, 2^k$ take place and for $3{\cdot}57 < a < 4$ the oscillations become unstable, forming what is called the chaotic régime. For $a = 4$, $x_n = \sin^2(2^n c)$, beyond which there is oscillatory divergence.

From the general theory, the alternative convergent sequence is generated by (cf. equations (9), (11), (12))

$$-\frac{\mathrm{d}f}{f} = \frac{\mathrm{d}x}{-(a-1)x - ax^2} = \frac{k|X|^{k-1}\mathrm{d}X}{-(a-1)X|X|^{k-1} - aX^2|X|^{2(k-1)}}, \quad k \geqslant a - 1. \tag{16}$$

Thus, the new Newton scheme yields

$$X_{n+1} = X_n - \frac{X_n}{k}[a - 1 + a|X_n|^k sgn(X_n)], \tag{17}$$

which as in the general theory converges monotonically to the fixed point $X^* = 0$. The sequence can be generated on a computer and the points lie on a smooth trajectory.

3. Multi-dimensions

In m-dimensions with coordinates $(x^1, x^2, \ldots, x^m)$, equation (1) is

$$\begin{aligned} &x_{n+1}^i = x_n^i - \sum_{j=1}^{m} a_j^i x_n^j - \sum g_{i_1, i_2, \ldots, i_m}^i (x_n^1)^{i_1}(x_n^2)^{i_2} \cdots (x_n^m)^{i_m}, \\ &2 \leqslant i_1 + i_2 + i_m \leqslant p_i, \end{aligned} \tag{18}$$

with the fixed point $x^{1*} = x^{2*} = \cdots = x^{m*} = 0$ under consideration. Writing the equation in vector form

$$\mathbf{x}_{n+1} = A\mathbf{x}_n + O(|\mathbf{x}_n|^2)I, \quad A = [\delta_j^i - a_j^i], \tag{19}$$

iteration yields the solution

$$\mathbf{x}_n = A^n \mathbf{x}_0 + O(|\mathbf{x}_0|^2) A^{n-1} I. \tag{20}$$

Thus the vector sequence converges to $\mathbf{x}^* = 0$ if the spectral radius of A is less than unity, that is, the eigenvalues λ_k satisfy $|\lambda_k| < 1$, $k = 1, 2, \dots, m$. The fixed point is then attracting or stable. The proof follows from representing a vector $\xi = \Sigma_{k=1}^m c_k \xi^k$ in terms of eigenvectors ξ^k. Thus

$$A^n \xi = \sum_{k=1}^{m} c_k A^n \xi^k = \sum_{k=1}^{m} c_k \lambda_k^n \xi^k.$$

It also follows that if for any k, $-1 < \lambda_k < 0$, the sequence will oscillate and converge. For decreasing $\lambda_k \leqslant -1$, the oscillatory behaviour may persist leading to chaotic motion and divergence.

As in one dimension, if (18) defines Newton's iteration for the zeroes of $f(x^1, x^2, \dots, x^m)$, then (3) is

$$\sum_{i=1}^{m} (x_{n+1}^i - x_n^i) \frac{\partial f}{\partial x_n^i} = -f, \tag{21}$$

so that from (18), f satisfies the linear differential equation

$$\sum_{i=1}^{m} \left[-\sum_{j=1}^{m} a_j^i x_n^j - \sum g_{i_1, i_2 \dots i_m}^i (x_n^1)^{i_1} (x_n^2)^{i_2} \cdots (x_n^m)^{i_m} \right] \frac{\partial f}{\partial x_n^i} = -f. \tag{22}$$

In the neighbourhood of the trivial fixed point, we can drop the nonlinear terms and to facilitate the solution of the resulting equation, perform the linear transformation

$$x^i = \sum_{j=1}^{m} b_j^i y^j; \quad y^j = \frac{1}{\Delta} \sum_{i=1}^{m} B_i^j x^i, \tag{23}$$

where $\Delta = \det(b_j^i)$ and $B_i^j =$ cofactor of b_j^i in Δ. The transformed differential equation for f is

$$\sum_{k=1}^{m} \left[\sum_{l=1}^{m} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{m} B_i^k a_j^i b_l^j \right\} y^l \right] \frac{\partial f}{\partial yk} = \Delta f. \tag{24}$$

We now assume that the unknown coefficients b_j^i satisfy the $m(m-1)$ nonlinear algebric equations

$$\sum_{i=1}^{m} \sum_{j=1}^{m} B_i^k a_j^i b_l^j = 0, \quad l \neq k, \tag{25}$$

leaving m of the unknowns arbitrary. The roots of these equations in general may be complex. So may be the nature of y^j. For $l = k$ we define the sum

$$\sum_{i=1}^{m} \sum_{j=1}^{m} B_i^k a_j^i b_k^j \stackrel{\text{def}}{=} c_k, \tag{26}$$

and the differential equation (24) reduces to

$$\sum_{k=1}^{m} c_k y^k \frac{\partial f}{\partial yk} = \Delta f. \tag{27}$$

The characteristics of this linear differential equation are given by

$$\frac{\mathrm{d}y^k}{c_k y^k} = \frac{\mathrm{d}f}{\Delta f}, \quad k = 1, 2, \ldots, m. \tag{28}$$

The general solution of (27) is thus given by

$$f^{1/\Delta} = (|y^k|^{1/c_k}\phi\{|y^1|^{1/c_1}/|y^k|^{1/c_k}, |y^2|^{1/c_2}/|y^k|^{1/c_k}, \ldots, |y^m|^{1/c_m}/|y^k|^{1/c_k}\}, \tag{29}$$

where k is fixed and ϕ is an arbitrary function.

The presence of powers of coordinates as in one dimension in (29) is suggestive of power law transformation of coordinates

$$y^k = (Y^k)^{d_k}, \tag{30}$$

in the differential equation for f, or equation of its characteristics in the y^k system

$$\frac{1}{\Delta}\sum_{k=1}^{m}\left[-c_k y^k - \sum_{i=1}^{m} B_i^k \Sigma g^i_{i_1 i_2 \ldots i_m} \prod_{p=1}^{m}\left(\sum_{q=1}^{m} b_q^p y^q\right)^{i_p}\right]\frac{\partial f}{\partial yk} = -f, \tag{31}$$

$$\frac{\mathrm{d}_k (Y^k)^{d_k - 1}\mathrm{d}Y^k}{\frac{1}{\Delta}[-c_k(Y^k)^{d_k} - \Sigma_{i=1}^m B_i^k \Sigma g^i_{i_1 i_2 \ldots i_m}\prod_{p=1}^m\{\Sigma_{q=1}^m b_q^p (Y^q)^{d_q}\}^{i_q}]} = -\frac{\mathrm{d}f}{f}. \tag{32}$$

Thus the alternative sequence following Newton's iteration of the type (21) in these new coordinates is given by

$$Y_{n+1}^k = Y_n^k - \frac{1}{\Delta \mathrm{d}_k}\left[c_k Y_n^k + (Y_n^k)^{1-d_k}\sum_{i=1}^{m} B_i^k \Sigma g^i_{i_1 i_2 \ldots i_m}\prod_{p=1}^{m}\left\{\sum_{q=1}^{m} b_q^p (Y_n^q)^{d_q}\right\}^{i_q}\right]. \tag{33}$$

This sequence which in general is complex will converge (to zero, the fixed point) as before if $|1 - c_k/\Delta d_k| < 1$. Choosing the powers d_k in this manner, we get alternative convergent sequences which are the maps of the original chaotic or even divergent sequences. If we want to obtain the original sequence (18), we have to retrace the steps beginning from (33) through (32), (30), (31), (23), (22) and finally (21).

In the important two dimensional case $m = 2$, the two quadratic equations of (25) yield

$$b_1^2 = \beta_1 b_1^1, \quad b_2^2 = \beta_2 b_2^1, \tag{34}$$

where

$$\beta_1, \beta_2 = \frac{1}{2a_2^1}[a_2^2 - a_1^1 \pm \{(a_2^2 - a_1^1)^2 + 4a_2^1 a_1^2\}^{1/2}]. \tag{35}$$

In this case,

$$c_1 = b_1^1 b_2^1[\beta_2 a_1^1 + \beta_1\beta_2 a_2^1 - a_1^2 - \beta_1 a_2^2], \tag{36a}$$

$$c_2 = b_1^1 b_2^1[-\beta_1 a_1^1 - \beta_1\beta_2 a_2^1 + a_1^2 + \beta_2 a_2^2], \tag{36b}$$

$$\Delta = b_1^1 b_2^1(\beta_2 - \beta_1). \tag{36c}$$

Evidently b_1^1 and b_2^1 can be chosen arbitrarily say 1. By selecting them small, the alterative sequence (33) can be made convergent comparatively rapidly.

3.1 *Examples*

(i) *The delayed-logistic map*: This 'map' is a two dimensional extension of (14):

$$\xi_{n+1} = a\xi_n(1-\eta_n), \quad a \geqslant 1,$$
$$\eta_{n+1} = \xi_n. \tag{37}$$

The fixed points of the iteration are $\xi^* = \eta^* = 0, (a-1)/a$. The map was first proposed by Maynard Smith and subsequently investigated by several authors (Pounder and Rogers [8], Aronson *et al* [1], Rogers and Clarke [9]). It has been shown that the sequence of points (ξ_n, η_n) are attracted to the nontrivial fixed point for $1 \leqslant a < 2$. For $2 \leqslant a < a^*$, $a^* \approx 2{\cdot}27$, the sequence is oscillatory as in limit cycles of convex polygons which degenerate at a^* and the points, in the limit, describe an invariant curve of complicated shape. For $a > a^*$ the sequence diverges.

For our analysis, the transformation of (37) according to $\xi = x + (a-1)/a$, $\eta = y + (a-1)/a$, yields

$$x_{n+1} = x_n - (a-1)y_n - ax_n y_n, \quad a \geqslant 1,$$
$$y_{n+1} = x_n. \tag{38}$$

Here, $a_1^1 = 0$, $a_2^1 = a - 1$, $a_1^2 = -1$, $a_2^2 = 1$. For the alternative convergent sequence

$$\beta_1, \beta_2 = \frac{1 \pm i\sqrt{4a-5}}{2(a-1)}, \tag{39}$$

with

$$c_1 = -\bar{c}_2 = \frac{\sqrt{4a-5}}{2(a-1)}[\sqrt{4a-5} - i], \quad \Delta = -\frac{\sqrt{4a-5}}{a-1}i. \tag{40}$$

The alternative sequence showing convergence, namely equation (33) is in this case

$$X_{n+1} = X_n - \frac{1}{\Delta d_1}[c_1 X_n + a\beta_2 X_n^{1-d_1}(X_n + Y_n)(\beta_1 X_n + \beta_2 Y_n)], \tag{41a}$$

$$Y_{n+1} = Y_n - \frac{1}{\Delta d_2}[c_2 Y_n - a\beta_1 Y_n^{1-d_2}(X_n + Y_n)(\beta_1 X_n + \beta_2 Y_n)], \tag{41b}$$

where for convergence it is required that $d_1, d_2 > a - 1$. This is also observed in computer simulations. The sequence is complex even though the original is real.

(ii) *The Hénon map*: This well known 'map' can be represented in the standard form

$$x_{n+1} = ax_n + by_n - x_n^2, \tag{42a}$$

$$y_{n+1} = x_n. \tag{42b}$$

For $a = 2{\cdot}1678$, $b = 0{\cdot}3$, it reveals a 'strange attractor', that is, in large number of iterations, the map has a self-similarity and Cantor set-like cross-section. Both the fixed points $(0,0)$ and $(1-a-b, 1-a-b)$ are unstable.

In seeking mapped alternative convergent sequence, we have $a_1^1 = -(a-1)$, $a_2^1 = -b$, $a_1^2 = -1$, $a_2^2 = 1$ and from equations (35) and (36)

$$\beta_1, \beta_2 = -\frac{1}{2b}(a \pm \sqrt{a^2 + 4b}), \tag{43a}$$

$$c_1 = 2 + \frac{1}{2b}[a^2 - (a-2)\sqrt{a^2+4b}], \tag{43b}$$

$$c_2 = -2 - \frac{1}{2b}[a^2 + (a-2)\sqrt{a^2+4b}], \tag{43c}$$

$$\Delta = \sqrt{a^2+4b}/b. \tag{43d}$$

The alternative sequence (33) becomes in this case

$$X_{n+1} = X_n - \frac{1}{\Delta d_1}[c_1 X_n + \beta_2 X_n^{(1-d_1)}(X_n + Y_n)^2] \tag{44a}$$

$$Y_{n+1} = Y_n - \frac{1}{\Delta d_2}[c_2 Y_n - \beta_1 Y_n^{(1-d_2)}(X_n + Y_n)^2] \tag{44b}$$

and is a real sequence. Since $c_1 > 0$ and $c_2 < 0$, we have to select for convergence $d_1 > c_1/2\Delta$ and $d_2 < c_2/2\Delta$. This may be verified in computer simulations to generate a smooth trajectory of the points.

4. Second degree systems and the exponential transformation

In one dimension, we represent the second degree system as

$$x_{n+1} = (1-a)x_n - gx_n^2. \tag{45}$$

Both the fixed points $x^* = 0$ and $-a/g$ are unstable if $a > 2$. We also assume $g > 0$ for the parabola to be convex upwards in the (x_n, x_{n+1})-plane. The Newton scheme (9) in this case is

$$\frac{\mathrm{d}x}{-x(a+gx)} = -\frac{\mathrm{d}f}{f}. \tag{46}$$

If we perform the transformation $x + a/g = \mathrm{e}^{-X}$, it sends the 'other' fixed point to infinity and the only finite fixed point is $X^* = -\ln(a/g)$. Inserting in (46) and comparing with the form (8), we generate the new sequence

$$X_{n+1} = X_n - (a - g\mathrm{e}^{-X_n}), \tag{47}$$

of which the only fixed point is unstable. Starting near the fixed point the sequence oscillates and cannot diverge. For if $a - g\mathrm{e}^{-X_n} < 0$, $X_{n+1} > X_n$ and in the next iterate, the exponential term diminishes making $X_{n+2} < X_{n+1}$.

In two dimenesions, we suppose that the system is

$$x_{n+1}^1 = x_n^1 - \sum_{j=1}^{2} a_j x_n^j - \sum_{j=1}^{2}\sum_{k=1}^{2} g_{jk} x_n^j x_n^k, \quad g_{jk} = g_{kj}, \tag{48a}$$

$$x_{n+1}^2 = x_n^1, \tag{48b}$$

to which many of the well known forms belong. The fixed points of the system are $x^{1*} = x^{2*} = 0$ and $x^{1*} = x^{2*} = -\Sigma_{j=1}^2 a_j / \Sigma_{j=1}^2 \Sigma_{k=1}^2 g_{jk}$. We assume that both the points are unstable with suitable range of values of the parameters with the transformation $x^j + \Sigma_{j=1}^2 a^j / \Sigma_{j=1}^2 \Sigma_{k=1}^2 g_{jk} = \pm\,\mathrm{e}^{-X^j}$ (positive or negative sign according to that of the Σ terms) the nontrivial fixed point is removed to infinity and the Newton scheme as in the derivation of (33) yields

$$X^j_{n+1} = X^j_n - \left[\frac{a^1(g_{11}-g_{22})+2a^2(g_{11}+g_{12})}{\Sigma^2_{j=1}\Sigma^2_{k=1} g_{jk}} + \frac{2a^1(g_{12}+g_{22})-a^2(g_{11}-g_{22})}{\Sigma^2_{j=1}\Sigma^2_{k=1} g_{jk}} e^{X^1_n - X^2_n} - g_{11}e^{-X^1_n} - 2g_{12}e^{-X^2_n} - g_{22}e^{X^1_n - 2X^2_n}\right], \tag{49a}$$

$$X^2_{n+1} = X^2_n - (e^{X^2_n - X^1_n} - 1). \tag{49b}$$

Evidently, the method may be generalized to higher dimensions of similar form. As examples we consider those of §§ 2.1 and 3.1. For the logistic map we get

$$X_{n+1} = X_n - (a - 1 - ae^{-X_n}). \tag{50}$$

Computer simulations show that the fixed point $X^* = -\ln(1 - 1/a)$ is an attractor for $a < 3$ and for $a \geqslant 3$ the sequence oscillates about X^*. In figure 1, we present the trajectory in the $X = X_n$, $Y = X_{n+1}$ plane of the sequential points for the case $a = 3{\cdot}9$–a value well within the chaotic regime of the original map. We note marked reduction of 'chaos'.

For the delayed-logistic map we get

$$X_{n+1} = X_n - (a - 1 - ae^{-Y_n}), \tag{51a}$$

$$Y_{n+1} = Y_n - (e^{-(X_n - Y_n)} - 1). \tag{51b}$$

In computer simulations, the sequence converges to the attractor $X^* = Y^* = -\ln(1 - 1/a)$ and oscillates about it for $a \geqslant 2$. In figure 2 its trajectory for $a = 2{\cdot}27$ which shows 'no chaos' is presented.

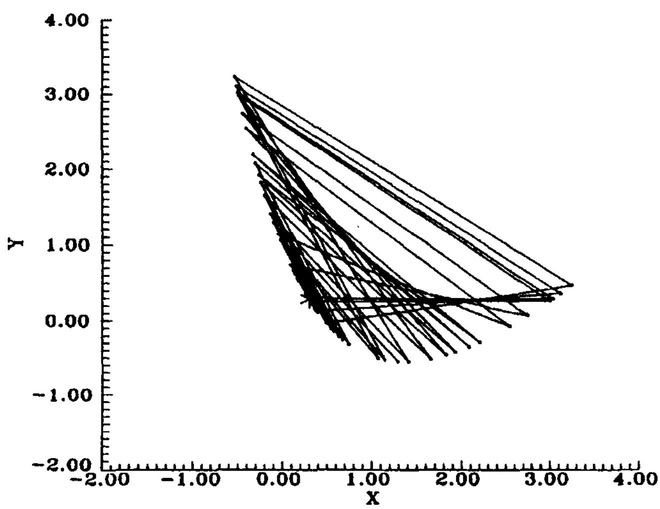

Figure 1. Transformed logistic map (50) for $a = 3{\cdot}9$ and 50 iterations. * is the fixed point $X^* = Y^* = -\ln(1 - 1/a)$.

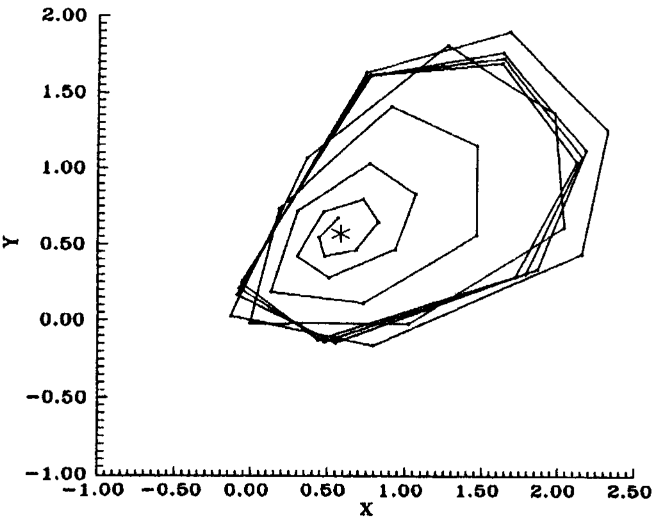

Figure 2. Transformed delayed logistic map (51) for $a = 2{\cdot}27$ and 50 iterations. * is the fixed point $X^* = Y^* = -\ln(1 - 1/a)$.

Transformation of the Hénon map leads to

$$X_{n+1} = X_n + (a + 2b - 1 - b\mathrm{e}^{X_n - Y_n} - \mathrm{e}^{-X_n}), \tag{52a}$$

$$Y_{n+1} = Y_n - (\mathrm{e}^{-(X_n - Y_n)} - 1). \tag{52b}$$

For $a = 2{\cdot}1678$, $b = 0{\cdot}3$ the sequence slowly diverges monotonically (in simulations).

5. Damped Newton and principal oscillatory part

If we consider chaotic oscillations in the general form (6) due to failure of convergence of Newton iterations (8), then we can think of a damped sequence

$$x_{n+1} = x_n - \omega(ax_n + \Sigma g_i x_n^i), \tag{53}$$

which fast converges to the fixed point. The fractional power of f, equation (10), implies that $|\omega| < 1$ (under relaxation). Fastest monotonic convergence takes place when the coefficient of the linear term is absent, that is when $\omega = a^{-1}$. Isolating this part in (6) we get

$$x_{n+1} = \left\{-(a-1)x_n\left(1 + \frac{1}{a}\Sigma g_i x_n^i\right)\right\} + \left\{-\frac{1}{a}\Sigma g_i x_n^i\right\}. \tag{54}$$

Thus the generating map of the sequence is decomposed into two; 1) the second expression within braces which is fast monotonically converging and 2) the first expression within braces. The latter must be highly oscillatory if the whole is chaotic. This part may be defined as the principal oscillatory part of the chaotic sequence.

The definition may be extended to two dimensional system (48) with the decomposition

$$x_{n+1}^1 = \left\{-(1-\omega)\left[\sum_{j=1}^{2} a_j x_n^j + \sum_{j=1}^{2}\sum_{k=1}^{2} g_{jk} x_n^j x_n^k\right]\right\}$$
$$+ \left\{x_n^1 - \omega\left[\sum_{j=1}^{2} a_j x_n^j + \sum_{j=1}^{2}\sum_{k=1}^{2} g_{jk} x_n^j x_n^k\right]\right\}, \tag{55a}$$

$$x_{n+1}^2 = x_n^1 \tag{55b}$$

where the under relaxation factor ω, $|\omega| < 1$, is so chosen that the larger of the two eigenvalues

$$\lambda = \frac{1}{2}[1 - \omega a_1 \pm \sqrt{(1 - \omega a_1)^2 - 4\omega a_2}], \tag{56}$$

in magnitude is smallest. The right hand side of (55a) thus splits into the fast convergent second part and the principal oscillatory part represented by the first expression within braces, in case (48) is chaotic

The principal oscillatory parts of the three examples of §§ 2.1 and 3.1 are drawn in figures 3, 4, and 5. The under relaxation ω in these cases is respectively

(i) $\omega_L = (a-1)^{-1}$,

(ii) $\omega_D = \frac{1}{4}(a-1)^{-1}$ + a very small fraction,

(iii) $\omega_H = -\dfrac{a-1+2b-\sqrt{(a-1+2b)^2-(a-1)^2}}{(a-1)^2}$ − a very small fraction

The necessity of a very small fraction arises from strict inequality required for ω and the three maps are respectively

(i) $$x_{n+1} = -(a-2)x_n\left(1 + \frac{a}{a-1}x_n\right), \tag{57}$$

(ii) $$x_{n+1} = -(1-\omega_D)y_n(a-1+ax_n), \tag{58a}$$

$$y_{n+1} = x_n, \tag{58b}$$

(iii) $$x_{n+1} = (1-\omega_H)[(a-1)x_n + by_n - x_n^2], \tag{59a}$$

$$y_{n+1} = x_n. \tag{59b}$$

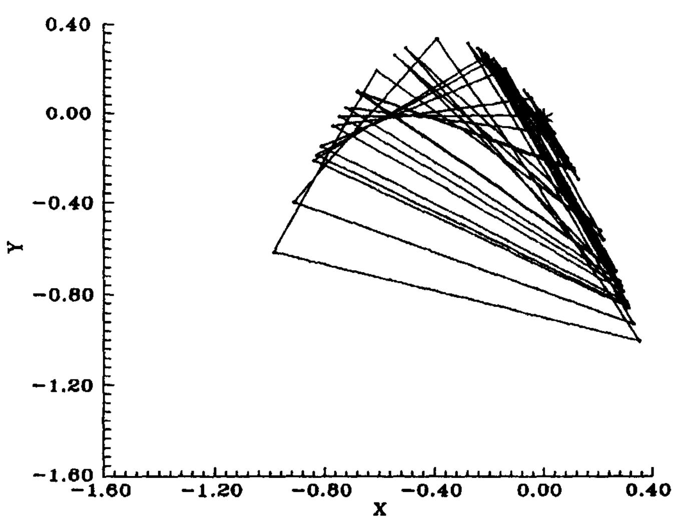

Figure 3. Principal oscillatory part of the logistic map (57) for $a = 3{\cdot}9$ and 50 iterations. * is the fixed point $X^* = Y^* = 0$.

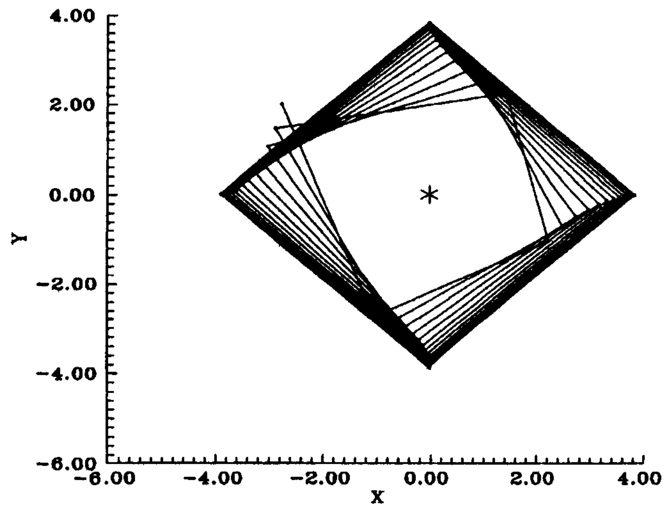

Figure 4. Principal oscillatory part of the delayed logistic map (58) (ten times magnified) for $a = 2{\cdot}27$ and 50 iterations. * is the fixed point $X^* = Y^* = 0$.

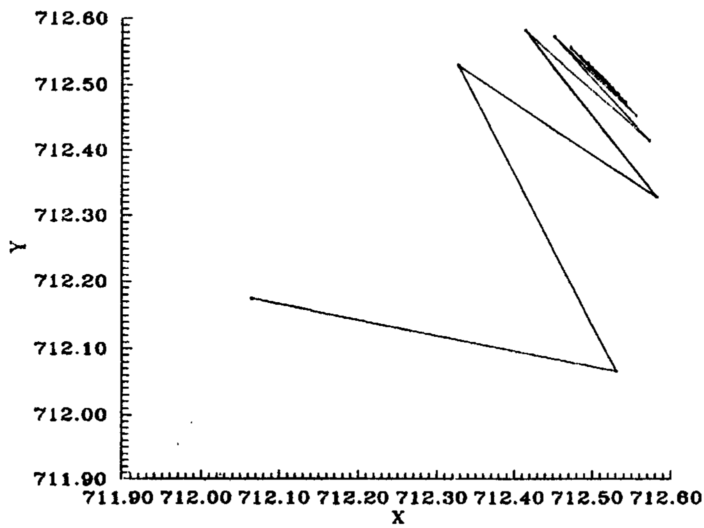

Figure 5. Principal oscillatory part of the Hénon map (59) (one thousand times magnified) for 12–50 iterations.

The extraction of the principal oscillatory part in the general m-dimensional case may be proceeded in a similar way. Finally, we note that since the principal oscillatory part is solely responsible for the chaotic oscillations, it should be possible to generate new chaotic sequences by combining these with other monotonically convergent sequences.

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