Proc. Indian Acad. Sci. (Math. Sci.), Vol. 107, No. 4, November 1997, pp. 405–409.

Dirichlet problem for some hypoelliptic operators

S RAMASWAMY
Department of Mathematics, Pondicherry University, Pondicherry 605 014, India

MS received 24 February 1997

Abstract. In this paper, the Dirichlet problem for hypoelliptic operators verifying Hörmander condition and the maximum principle is considered.

Keywords. Hypoelliptic operators; maximum principle; Dirichlet problem.

1. Introduction

Let

$$L = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) D_{ij} + \sum_{i=1}^{n} b_i(x) D_i + c(x)$$

be a degenerate elliptic operator of 2nd order with C^∞ coefficients with $c \leqslant 0$ defined on an open set Ω of $\mathbb{R}^n$. Let us suppose that there exist vector fields $X_1, X_2, \ldots, X_r, Y$ such that L can be written in the form

$$L = \sum_{k=1}^{k=r} X_k^2 + Y + c.$$

Further let us assume that the rank of the Lie algebra generated by $X_1, X_2, \ldots, X_r, Y$ at any point is of rank n.

Under the above assumption, Hörmander in [2] has proved that the operator is hypoelliptic, in the sense of the following definition.

DEFINITION

L is said to be *hypoelliptic* if every distribution u is C^∞ in an open set ω if Lu is C^∞ in ω.

For these operators, with the additional assumption that $\exists\, \lambda < 0$ such that $c(x) \leqslant \lambda$, $\forall x \in \Omega$, Bony [1] has proved the existence and uniqueness of the Dirichlet problem.

$$Lu = f \text{ in } \omega,$$

$$u = \phi \text{ on } \partial\omega,$$

where f and ϕ are continuous and ω is a relatively compact open subset of Ω, $\bar{\omega} \subset \Omega$, such that $\partial\omega$ admits a special kind of barrier to be specified later on.

In this article we remove the above restriction on c and assume just that $c \leqslant 0$, but assume that L satisfies the maximum principle as defined below and prove the existence and uniqueness of the Dirichlet problem when $f = 0$.

DEFINITION 1

Let ω be a relatively compact open subset of Ω such that $\bar{\omega} \subset \Omega$. The maximum principle for subsolutions of $L = 0$ is said to hold if $\forall u \in C^2(\omega)$, upper semi continuous in $\bar{\omega}$,

$$Lu \geqslant 0 \text{ in } \omega, \limsup u(y)_{y \to X,\ y \in \omega,\ X \in \partial\omega} \leqslant 0, \quad \forall X \in \partial\omega,$$

$$\Rightarrow u \leqslant 0 \text{ in } \omega.$$

DEFINITION 2

A vector v is called an *exterior normal* to a closed set F at a point $x_0 \in F$ if there exists an open ball contained in $\Omega \setminus F$ centered at a point x_1 such that x_0 belongs to the closure of this ball and such that $v = \lambda(x_1 - x_0)$ with $\lambda > 0$.

It is easy to see by reducing the radius if necessary that x_0 is the only point of F that belongs to the closure of the ball.

Theorem. *Let L be an operator of the type described above. Let further L satisfy the maximum principle for subsolutions of $L = 0$. Let ω be a relatively compact open subset of Ω, $\bar{\omega} \subset \Omega$, such that $\forall X \in \partial\omega$, $\exists$ an exterior normal v such that*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(X) v_i v_j > 0.$$

Then $\forall \phi$ continuous on $\partial\omega$, $\exists$ a unique continuous function u on *$\bar{\omega}$ such that*

$$Lu = 0 \textit{ in the sense of distributions in } \omega$$

$$u = \phi \textit{ on } \partial\omega.$$

Proof. The uniqueness comes from the fact that L is hypoelliptic and that the maximum principle is valid for subsolutions of $L = 0$.

Therefore it is sufficient to show the existence.

Let $\phi \in C(\partial\omega)$ be given. Let L_n be the operator $L - (1/n)$.

Therefore by the theorem 5.2 in Bony [1], $\forall n \in \mathbb{N}$, $\exists$ a function $u_n \in C^\infty(\omega)$ such that

$$L_n u_n = \left(L - \frac{1}{n}\right) u_n = 0 \text{ in } \omega$$

$$u_n = \phi \text{ on } \partial\omega.$$

Since the maximum principle is valid for $L_n \forall n$ we have $\forall x \in \bar{\omega}$,

$$|u_n(x)| \leqslant \sup_{x \in \partial\omega} |\phi(x)|.$$

Hence the u_n's are all uniformly bounded in $\bar{\omega}$. Therefore (u_n) has a subsequence which converges weakly in $L^2(\bar{\omega})$ to a function $u \in L^2(\bar{\omega})$. Without loss of generality, we shall suppose that (u_n) itself converges weakly to u in $L^2(\omega)$.

Let $\psi \in \mathscr{D}(\omega)$. Then,

$$\begin{aligned}\int_\Omega (Lu)\psi \,\mathrm{d}x &= \int_\Omega uL^*(\psi)\mathrm{d}x \\ &= \lim_{n\to\infty} \int u_n L^*(\psi)\mathrm{d}x \\ &= \lim_{n\to\infty} \int (Lu_n)\psi \,\mathrm{d}x \\ &= \lim_{n\to\infty} \frac{1}{n}\int u_n \psi \,\mathrm{d}x \\ &= 0.\end{aligned}$$

Therefore $Lu = 0$ in the sense of distributions. Since L is hypoelliptic, $u \in C^\infty(\omega)$.

We shall now show that $u(x) \to \phi(X) \forall X \in \partial\omega$, when $x \in \omega \to X$.

For this we suppose that ϕ admits an extension as a function of class C^2 in a neighbourhood of $\bar{\omega}$. We denote by ϕ the extension also.

For $X \in \partial\omega$, $\exists$ by assumption a point $x^0 \in \Omega\backslash\bar{\omega}$, and a $r > 0$, such that

$$\overline{B(x_0;\ r)} \cap \partial\omega = X$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(X)(X_i - x_i^0)(X_j - x_j^0) > 0.$$

From this property it is easy to see that if w is the function

$$\mathrm{e}^{-k(|x-x^0|)^2} - \mathrm{e}^{-k(|X-x^0|)^2}$$

then, for k large, $L(w(X)) > 0$. Let us fix one such k. $L(w) > 0$ in a neighbourhood V of X.

$$L(u_n - \phi) = \frac{1}{n}u_n - L\phi.$$

As the u_n's are uniformly bounded and as $L\phi$ is bounded, $L(u_n - \phi)$ is also uniformly bounded.

Therefore, we can choose M_1 large such that $\forall n$,

$$L(M_1 w \pm (u_n - \phi)) > 0, \quad \text{in } V \cap \omega.$$

As $w(X) = 0$, and as $w < 0$ in $\overline{(V \cap \omega)} \backslash X$, and as the u_n's are all uniformly bounded and $u_n = \phi$ on $\partial\omega$, we can choose $M_2 \geqslant M_1$ such that $\forall n \in \mathbb{N}$,

$$M_2 w \pm (u_n - \phi) \leqslant 0 \quad \text{on} \quad \partial(V \cap \omega).$$

Therefore we have,

$$L(M_2 w \pm (u_n - \phi)) > 0 \quad \text{on} \quad V \cap \omega$$

and

$$M_2 w \pm (u_n - \phi) \leqslant \text{ on } \partial(V \cap \omega).$$

Therefore by the maximum principle,

$$M_2 w \pm (u_n - \phi) \leqslant 0 \text{ on } V \cap \omega.$$

Therefore $|u_n - \phi| \leqslant M_2 |w|$ on $V \cap \omega$.
As u_n converges weakly to u in L^2, we have for any measurable set $E \subseteq V \cap \omega$,

$$\begin{aligned}\int_E |u - \phi| \mathrm{d}x &= \int_{E \cap (u-\phi) \geqslant 0} (u - \phi) + \int_{E \cap (u-\phi) < 0} -(u - \phi) \mathrm{d}x \\ &= \lim_{n \to \infty} \int_{E \cap (u-\phi) \geqslant 0} (u_n - \phi) \mathrm{d}x - \int_{E \cap (u-\phi) < 0} (u_n - \phi) \mathrm{d}x \\ &\leqslant \int_E M_2 |w| \mathrm{d}x.\end{aligned}$$

This implies that

$$|(u - \phi)| \leqslant M_2 |w| \text{ a.e. in } V \cap \omega.$$

As u, ϕ and w are continuous, we have the inequality

$$|(u - \phi)| \leqslant M_2 |w| \text{ everywhere in } V \cap \omega.$$

From this it is easy to see that $u(x) \to \phi(X)$ as $x \in \omega \to X$ as $w(X) = 0$, w and ϕ are continuous at X.

For a general ϕ, we approximate uniformly by a sequence of functions ϕ_n which admit an extension as a function of class C^∞ in $\bar{\omega}$. If u_n is the corresponding sequence such that

$$Lu_n = 0 \text{ on } \omega,$$

$$u_n = \phi_n \text{ on } \partial\omega,$$

then by the maximum principle, we see that the sequence u_n converges uniformly on $\bar{\omega}$ to a continuous function u such that

$$Lu = 0 \text{ on } \omega,$$

$$u = \phi \text{ on } \partial\omega.$$

Thus the proof of the theorem is complete.

In the theorem we assumed that the subsolutions of $L = 0$ satisfy the maximum principle. It is not true in general that the subsolutions of $L = 0$ where L is an hypoelliptic operator satisfy the maximum principle as the following example shows.

Example. Let Ω be the ball of center 0 and radius 3 in $\mathbb{R}^2$. Let L be the operator

$$\frac{\partial^2}{\partial\theta^2} + \alpha(r)\frac{\partial^2}{\partial r^2} + \beta(r)\frac{\partial}{\partial r},$$

where α and β do not vanish simultaneously, $\beta > 0$ in the interval $[0, 1]$, $\beta < 0$ in the interval $[2, 3]$, support of $\alpha \subseteq [1, 2]$, $0 \leqslant \alpha \leqslant 1$. It is clear that L is hypoelliptic as the condition of Hörmander is verified at each point.

Let u be a function of r such that $u(3)$ is 0, u is increasing in $[0,1]$, u is a positive constant in $[1,2]$, u is decreasing in $[2,3]$. Then it is easy to see that $Lu = \beta(r)(\partial u/\partial r)$ which is > 0. Hence u is a subsolution of $L = 0$. But the maximum principle is not verified.

We can take for example for u the following function:

$$\begin{aligned} u(r) &= r^5 \text{ in } [0,1] \\ &= 1 \text{ in } [1,2] \\ &= (3-r)^5 \text{ in } [2,3]. \end{aligned}$$

Acknowledgement

The author takes great pleasure in thanking J M Bony for several helpful discussions and for suggesting the idea behind the construction of the above example. The author thanks Colette Guillopé for inviting him to visit the Department of Mathematics, Paris 12 University which helped in carrying out the work in a friendly atmosphere, and also the French Embassy in India for awarding a three month French Government Fellowship to support the stay in Paris.

References

[1] Bony J M, Principe du Maximum, Inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés, *Ann. Inst. Fourier 19* **1** (1969) 277–304

[2] Hörmander L, Hypoelliptic second order differential equations, *Acta Math. Uppsala* **119** (1967) 147–171