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Abelian and Tauberian theorems for a new trigonometric method of summation

G DAS and B K RAY*

Department of Mathematics, Utkal University, Vani Vihar, Bhubaneswar 751 004, India
* Plot No. 102, Saheed Nagar, Bhubaneswar 751 007, India

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Abstract. We first introduce a new trigonometric method of summation and then prove some Abelian and Tauberian theorems for this method.

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1. Introducing the method

We know ([5], vol. I, p. 186) that for $0<\varepsilon<1$,

$$\sum_{k=0}^{\infty} A_k^{-\varepsilon} e^{ikx} = \left(2\sin\frac{1}{2}x\right)^{\varepsilon-1} \exp\left\{\frac{1}{2}i(\pi-x)(1-\varepsilon)\right\}, \quad 0<x<2\pi \tag{1.1}$$

where A_n^α is defined by

$$\frac{1}{(1-r)^{\alpha+1}} = \sum_{n=0}^{\infty} A_n^\alpha r^n, \quad \alpha > -1. \tag{1.2}$$

Equating real part on either side of (1.1), we get

$$\sum_{k=0}^{\infty} A_k^{-\varepsilon}\cos kx = \left(2\sin\frac{1}{2}x\right)^{\varepsilon-1}\cos\frac{1}{2}(\pi-x)(1-\varepsilon)$$

$$= x^{\varepsilon-1}\sin\frac{1}{2}\pi\varepsilon + O(1) \text{ as } x\to 0^+. \tag{1.3}$$

Integrating (1.3) twice, we obtain

$$A_0^\varepsilon\frac{x^2}{2} + \sum_{k=1}^{\infty} A_k^{-\varepsilon}\frac{1-\cos kx}{k^2} = \frac{x^{\varepsilon+1}}{\varepsilon(\varepsilon+1)}\sin\frac{1}{2}\pi\varepsilon + O(x^2). \tag{1.4}$$

We can rewrite (1.4) as

$$\frac{x^{1-\varepsilon}}{D(\varepsilon)}\sum_{k=1}^{\infty} A_k^{-\varepsilon}\left(\frac{\sin(1/2)kx}{(1/2)kx}\right)^2 = 1 + O(x^{1-\varepsilon}), \tag{1.5}$$

where

$$D(\varepsilon) = \frac{\pi}{(1+\varepsilon)}\left(\frac{\sin(1/2)\pi\varepsilon}{(1/2)\pi\varepsilon}\right). \tag{1.6}$$

The expression (1.5) motivates to a new method of summation as follows:

Given an infinite series $\Sigma_{n=0}^{\infty} a_n$ with the sequence of partial sums (s_n), we define a trigonometric mean of the sequence (s_n) by

$$F(h) = \frac{h^{1-\varepsilon}}{D(\varepsilon)} \sum_{k=1}^{\infty} A_k^{-\varepsilon} \left(\frac{\sin(1/2)kh}{(1/2)kh} \right)^2 s_k \tag{1.7}$$

provided the right side series is convergent for all small h. if

$$\lim_{h \to 0} F(h) = s$$

then we say that (s_n) is $(R_{2+\varepsilon})$ summable to s, where $0 < \varepsilon < 1$. We may take $h = 2/n$ and write

$$F\left(\frac{2}{n}\right) = \frac{2^{1-\varepsilon}}{D(\varepsilon) n^{1-\varepsilon}} \sum_{k=1}^{\infty} A_k^{-\varepsilon} \left(\frac{\sin(k/n)}{k/n} \right)^2 s_k \tag{1.8}$$

which is the discrete version of $(R_{2+\varepsilon})$ mean and the same will be denoted by $(R^*_{2+\varepsilon})$ mean. Though continuous method always implies its discrete version, the converse is not necessarily true.

The choice of the symbol $(R_{2+\varepsilon})$ is due to the fact that when $\varepsilon \to 0$ then

$$D(\varepsilon) \to \pi$$

and the method (1.7) in the limiting case reduces to the method defined by

$$t(h) = \frac{4}{\pi h} \sum_{k=0}^{\infty} \frac{\sin^2(1/2)kh}{k^2} s_k \tag{1.9}$$

and this is known as (R_2) method ([3], p. 89). Thus (R_2) method can be considered as a limiting case of $(R_{2+\varepsilon})$ method when $\varepsilon = 0$, though for the definition of $(R_{2+\varepsilon})$ method it is assumed that $0 < \varepsilon < 1$.

It may be remarked that (R_2) method is closely connected with Riemann method of summation $(R, 2)$ ([3], p. 89). It has been proved by Marcinkiewicz, and also Kuttner that (R_2) and $(R, 2)$ are not comparable. In fact it has been shown by Hardy and Rogosinski that the methods are incomparable even for Fourier series, that is, there are two examples of Fourier series of which one is summable $(R, 2)$, but not (R_2), and the other (R_2), but not $(R, 2)$. See Hardy ([3], p. 93) where relevant works of Marcinkiewicz, Kuttner, Hardy and Rogosinski are mentioned.

In what follows, we state two known methods of summation which we need in the sequel.

Let (p_n) be a sequence of constants. The Nörlund (N, p) mean of (s_n) is given by

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k \tag{1.10}$$

where $P_n = p_0 + p_1 + \cdots + p_n$, $p_{-1} = P_{-1} = 0$ and $P_n \neq 0$, for $n \geqslant 0$.

The sequence (s_n) is (N, p)-summable to s when $t_n \to s$. The Cesàro method (C, α) $(\alpha > 0)$ is the special case $p_n = A_n^{\alpha-1}$ (as defined by (1.2)).

Associated with (p_n) we formally write the power series

$$p(x) = \sum_{n=0}^{\infty} p_n x^n$$

and define the sequence of constants (C_n) by

$$\sum_{n=0}^{\infty} C_n x^n = \left(\sum_{n=0}^{\infty} p_n x^n \right)^{-1}. \tag{1.11}$$

We write $(p_n) \in \mathscr{M}$, if

$$p_n > 0, \quad \frac{p_{n+1}}{p_n} \leqslant \frac{p_{n+2}}{p_{n+1}} \leqslant 1 \ (n = 0, 1, 2, \ldots). \tag{1.12}$$

The power series method (A_α) of (s_n) is defined by the mean

$$A(x) = (1-x)^{\alpha+1} \sum_{n=0}^{\infty} A_n^\alpha s_n x^n, \quad 0 < x < 1 \tag{1.13}$$

where A_n^α is defined as in (1.2).

If $A(x) \to s$ as $x \to 1-$ then we say that (s_n) is summable by (A_α) method to s. It is obvious that (A_0) is the same as the Abel method (A). It is further known that

$$(A_\delta) \subset (A) \subset (A_\alpha)$$

for $\delta > 0$, $-1 < \alpha < 0$ ([1]).

In § 2, we consider Abelian theorems and examine the regularity and Fourier effectiveness of $(R_{2+\varepsilon})$ method. In § 3, we establish a Tauberian theorem.

2. Abelian theorems

In this section we prove the following two abelian theorems:

Theorem 1. *Let $(p_n) \in \mathscr{M}$ and $0 \leqslant \varepsilon < 1$. Then*

$$(N, p) \subset (R_{2+\varepsilon}).$$

In particular, for $0 < \alpha \leqslant 1$,

$$(C, \alpha) \subset (R_{2+\varepsilon}).$$

Theorem 2. *Let $0 \leqslant \varepsilon < 1$. Then*

$$(R_{2+\varepsilon}) \subset (A_{-\varepsilon}).$$

In connection with Theorem 1 we may remark that since (C, α) method is regular and Fourier effective for $\alpha > 0$, it follows that

(i) $(R_{2+\varepsilon})$ is regular

(ii) $(R_{2+\varepsilon})$ is Fourier effective.

Regularity of (R_2^*) method has been established by Carmichael and Herwitz (see [4]). Wiener [4] has shown that (R_2^*) method is Fourier effective.

Theorem 2 generalizes a result of Kuttner who proved $(R_2) \subset (A)$ (see Hardy [3], p. 369 for a proof).

For a proof of Theorem 1, we need the following lemmas:

Lemma 1 ([3], Theorem 22). *Let $(p_n)\in\mathscr{M}$. Then with the notation of* (1.11)

(i) $C_0>0,\ C_n\leqslant 0\quad (n=1,2,3,\ldots)$

(ii) $\sum_{n=0}^{\infty} C_n x^n$ *is absolutely convergent for* $|x|\leqslant 1$

(iii) $\sum_{n=0}^{\infty} C_n>0$ *except when* $\Sigma\, p_n=\infty$ *in which case* $\sum_{n=0}^{\infty} C_n=0.$

Lemma 2 ([2], Lemmas 3 and 4). *Let $(p_n)\in\mathscr{M}$. Then*

(i) $C_n^{(1)}=C_0+C_1+\cdots+C_n\geqslant 0$ *and monotonic non-increasing,*

(ii) $\sum_{n=M+1}^{\infty} |C_{n-\nu}|\leqslant C_{M-\nu}^{(1)}.$

(iii) $\sum_{\nu=0}^{\infty} P_\nu \sum_{n=M+1}^{\infty} |C_{n-\nu}|\leqslant M+1.$

Lemma 3. *Let $(p_n)\in\mathscr{M}$ and let $M=[2\delta/h]$ for some $\delta: 0<\delta<\pi/4$. Then for $h>0$*

$$\sum_{n=\nu}^{M} C_{n-\nu}A_n^{-\varepsilon}\left(\frac{\sin(1/2)nh}{(1/2)nh}\right)^2\geqslant 0.$$

Proof. We note that for $0<\theta<\delta$, $(\sin\theta/\theta)$ is positive and $(\sin\theta/\theta)^2$ is decreasing. By Abel's transfomation

$$\sum_{n=\nu}^{M} A_n^{-\varepsilon}C_{n-\nu}\left(\frac{\sin(1/2)nh}{(1/2)nh}\right)^2$$

$$=\sum_{n=\nu}^{M-1}\Delta_n\left[A_n^{-\varepsilon}\left(\frac{\sin(1/2)nh}{(1/2)nh}\right)^2\right]C_{n-\nu}^{(1)}+A_M^{-\varepsilon}\left(\frac{\sin(1/2)Mh}{(1/2)Mh}\right)^2 C_{M-\nu}^{(1)}. \quad (2.1)$$

By lemma 2, the second term of (2.1) is non-negative. Also

$$(A_n^{-\varepsilon}-A_{n+1}^{-\varepsilon})\left(\frac{\sin(1/2)nh}{(1/2)nh}\right)^2\geqslant 0$$

and

$$A_{n+1}^{-\varepsilon}\left[\left(\frac{\sin(1/2)nh}{(1/2)nh}\right)^2-\left(\frac{\sin(1/2)(n+1)h}{(1/2)(n+1)h}\right)^2\right]\geqslant 0.$$

It follows that the first term of (2.1) is also non-negative. This proves Lemma 3.

Proof of Theorem 1. Using the well-known inverse transformation of (1.11)

$$s_n=\sum_{\nu=0}^{n} C_{n-\nu}P_\nu t_\nu,$$

We obtain from (1.7) that

$$F(h)=\sum_{\nu=0}^{\infty}\mathrm{d}_\nu(h)t_\nu,$$

where

$$\mathrm{d}_\nu(h) = \frac{h^{1-\varepsilon}}{D(\varepsilon)} P_\nu \sum_{n=\nu}^{\infty} A_n^{-\varepsilon} C_{n-\nu} \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2 \tag{2.2}$$

To prove Theorem 1, we establish the regularity of the transformation (2.2). For this we have to prove that (see Hardy [3], p. 49)

(i) $\sum_{\nu=0}^{\infty} d_\nu(h) \to 1$ as $h \to 0^+$,

(ii) $d_\nu(h) \to 0$ as $h \to 0^+$ (fixed ν),

(iii) $\sum_{\nu=0}^{\infty} |d_\nu(h)| = O(1)$ as $h \to 0^+$.

Since by lemma 1, $\Sigma\,|C_n| < \infty$, it follows that

$$d_\nu(h) = O(\Sigma |C_n|) h^{1-\varepsilon} \to 0 \text{ as } h \to 0^+.$$

Since

$$\sum_{\nu=0}^{n} P_\nu C_{n-\nu} = 1 \text{ (all } n) \tag{2.3}$$

it follows from (1.5) that

$$\sum_{\nu=0}^{\infty} d_\nu(h) = 1 + O(h^{1-\varepsilon}) \to 1 (h \to 0^+).$$

We have now only to prove (iii). Now

$$\begin{aligned}
\Sigma &= \sum_{\nu=0}^{\infty} |d_\nu(h)| \\
&= \frac{h^{1-\varepsilon}}{D(\varepsilon)} \sum_{\nu=0}^{\infty} P_\nu \left| \sum_{n=\nu}^{\infty} A_n^{-\varepsilon} C_{n-\nu} \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2 \right| \\
&= \frac{h^{1-\varepsilon}}{D(\varepsilon)} \sum_{\nu=0}^{\infty} \left| P_\nu \left(\sum_{n=\nu}^{M} + \sum_{n=M+1}^{\infty} \right) A_n^{-\varepsilon} C_{n-\nu} \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2 \right| \\
&\leqslant \frac{h^{1-\varepsilon}}{D(\varepsilon)} \sum_{\nu=0}^{\infty} P_\nu \left| \sum_{n=\nu}^{M} A_n^{-\varepsilon} C_{n-\nu} \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2 \right| \\
&\quad + \frac{h^{1-\varepsilon}}{D(\varepsilon)} \sum_{\nu=0}^{\infty} P_\nu \left| \sum_{n=M+1}^{\infty} A_n^{-\varepsilon} C_{n-\nu} \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2 \right| \\
&= \Sigma_1 + \Sigma_2, \text{ say.}
\end{aligned} \tag{2.4}$$

Now in Σ_1, $\nu \leqslant M$ and so considering $\nu \geqslant M+1$ terms as empty, we have, by Lemma 3

$$\begin{aligned}
\Sigma_1 &= \frac{h^{1-\varepsilon}}{D(\varepsilon)} \sum_{\nu=0}^{M} P_\nu \left| \sum_{n=\nu}^{M} C_{n-\nu} A_n^{-\varepsilon} \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2 \right| \\
&= \frac{h^{1-\varepsilon}}{D(\varepsilon)} \sum_{\nu=0}^{M} P_\nu \sum_{n=\nu}^{M} C_{n-\nu} A_n^{-\varepsilon} \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2
\end{aligned}$$

$$= \frac{h^{1-\varepsilon}}{D(\varepsilon)} \sum_{n=0}^{M} A_n^{-\varepsilon} \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2 \sum_{\nu=0}^{n} P_\nu C_{n-\nu}$$

$$= \frac{h^{1-\varepsilon}}{D(\varepsilon)} \sum_{n=0}^{M} A_n^{-\varepsilon} \left(\frac{\sin(1/2)nh}{(1/2)nh} \right)^2$$

$$= 1 + O(h^{1-\varepsilon}), \tag{2.5}$$

by use of the identity (2.3) and the estimate (1.5). Using Lemma 2 (iii), we have

$$\Sigma_2 \leqslant \frac{h^{1-\varepsilon}}{D(\varepsilon)} \sum_{\nu=0}^{\infty} P_\nu \sum_{n=M+1}^{\infty} |C_{n-\nu}| A_n^{-\varepsilon}$$

$$\leqslant \frac{h^{1-\varepsilon}}{D(\varepsilon)} A_{M+1}^{-\varepsilon} \sum_{\nu=0}^{\infty} P_\nu \sum_{n=M+1}^{\infty} |C_{n-\nu}|$$

$$\leqslant \frac{h^{1-\varepsilon}(M+1)}{D(\varepsilon)(M+1)^{\varepsilon}\Gamma(1-\varepsilon)}$$

$$= \frac{[h(M+1)]^{1-\varepsilon}}{D(\varepsilon)\Gamma(1-\varepsilon)} = O(1).$$

This completes the proof of Theorem 1.

Proof of Theorem 2. We need the following lemma.

Lemma 4 ([3], p. 366, Theorem 258). *If* $\Sigma c_n(1-\cos nx)$ *is convergent for all* x *in an interval* (α, β), *then* Σc_n *is convergent.*

We are given that $s_n \to s(R_{2+\varepsilon})$, and we may suppose that $s_0 = 0$, $s = 0$. Then with the notation of (1.7)

$$T(h) = \frac{h^{1+\varepsilon}D(\varepsilon)}{4} F(h) = \sum_{n=1}^{\infty} A_n^{-\varepsilon} \frac{\sin^2(1/2)nh}{n^2} s_n \tag{2.6}$$

converges for small h, and

$$T(h) = o(h^{1+\varepsilon}); \tag{2.7}$$

and we have to prove that

$$(1-r)^{1-\varepsilon} \sum_{n=1}^{\infty} A_n^{-\varepsilon} s_n r^n = o(1) \text{ as } r \to 1- \tag{2.8}$$

By Lemma 4 convergence of (2.6) for small h ensures the convergence of the series $\Sigma_{n=1}^{\infty} (A_n^{-\varepsilon} s_n)/n^2$ and hence we can write series for $T(h)$ as

$$\frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos nh, \tag{2.9}$$

where

$$\alpha_n = -\frac{1}{2} A_n^{-\varepsilon} n^{-2} s_n (n > 0),$$

$$\alpha_0 = \sum_{n=1}^{\infty} A_n^{-\varepsilon} n^{-2} s_n = -2 \sum_{n=1}^{\infty} \alpha_n. \tag{2.10}$$

The series (2.9) converges to $T(h)$ for small h whenever $s_n \to s(R_{2+\varepsilon})$. We shall only consider two particular cases where

(I) the series (2.9) is a Fourier series, or
(II) the series (2.9) converges uniformly to zero in a neighbourhood of h that includes the origin.

By an argument similar to those used in ([3], p. 368) it can be proved that the theorem is true generally if it holds in cases (I) and (II). Thus, it is enough to prove the theorem in two special cases.

(I) Since in the present case (2.9) is a Fourier series there exists an even periodic function $\phi(h)$ with period 2π so that

$$\phi(h) \sim \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos nh. \tag{2.11}$$

As (2.11) converges to $T(h)$ for small h

$$\phi(h) = T(h)$$

for almost all such h, and we may suppose that

$$\phi(h) = T(h)$$

for all such h. By the hypothesis $T(h) = o(h^{1+\varepsilon})$ and hence

$$\phi(h) = o(h^{1+\varepsilon}) \text{ as } h \to 0. \tag{2.12}$$

We write

$$\Delta(r,\theta) = 1 - 2r\cos\theta + r^2,$$

$$P(r,\theta) = \frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n\theta = \frac{1}{2}\frac{1-r^2}{\Delta(r,\theta)}, \quad 0 < r < 1.$$

It is easily seen that

$$P_{22}(r,\theta) = \frac{\partial^2}{\partial\theta^2}P(r,\theta) = -\frac{r(1-r^2)[(1+r^2)\cos\theta - 2r(1+\sin^2\theta)]}{\Delta^3(r,\theta)}. \tag{2.13}$$

In the present context $r \to 1-$ and so we may assume without loss of generality that $1/4 \leqslant r < 1$. As $\sin\theta \geqslant (2/\pi)\theta$ for $0 \leqslant \theta \leqslant \pi/2$, we have

$$\Delta(r,\theta) = (1-r)^2 + 4r\sin^2\frac{1}{2}\theta \geqslant \pi^{-2}[(1-r)^2 + \theta^2]$$

which further ensures that

$$\Delta^{-1}(r,\theta) = O(1)\begin{cases}(1-r)^{-2}\\ \theta^{-2}\end{cases} \tag{2.14}$$

Also we have

$$(1+r^2)\cos\theta - 2r(1+\sin^2\theta) = O(\Delta(r,\theta)),\ 0<r<1,\ 0<\theta<\pi. \tag{2.15}$$

Using (2.14) and (2.15), we obtain

$$P_{22}(r,\theta)=O(1)\begin{cases}(1-r)^{-3} & (2.16)\\ (1-r)\theta^{-4}. & (2.17)\end{cases}$$

For $0<r<1$, we have

$$\frac{1}{2}\alpha_0+\sum_{n=1}^{\infty}\alpha_n r^n\cos nh$$
$$=\frac{2}{\pi}\int_0^{\pi}\phi(\theta)\left\{\frac{1}{2}\sum_{n=1}^{\infty}r^n\cos n\theta\cos nh\right\}\mathrm{d}\theta$$
$$=\frac{1}{\pi}\int_0^{\pi}\phi(\theta)\left\{1+\sum_{n=1}^{\infty}r^n(\cos n(\theta-h)+\cos n(\theta+h))\right\}\mathrm{d}\theta$$
$$=\frac{1}{\pi}\int_0^{\pi}\phi(\theta)\{P(r,\theta-h)+P(r,\theta+h)\}\mathrm{d}\theta$$
$$=\frac{2}{\pi}\int_0^{\pi}\phi(\theta)P(r,\theta-h)\mathrm{d}\theta.$$

Differentiating twice with respect to h, we get

$$-\sum_{n=1}^{\infty}n^2\alpha_n r^n\cos nh=\frac{2}{\pi}\int_0^{\pi}\phi(\theta)\frac{\partial^2}{\partial h^2}P(r,\theta-h)\mathrm{d}\theta$$
$$=\frac{2}{\pi}\int_0^{\pi}\phi(\theta)P_{22}(r,\theta-h)\mathrm{d}\theta.$$

At $h=0$, we obtain

$$-\sum_{n=1}^{\infty}n^2\alpha_n r^n=\frac{2}{\pi}\int_0^{\pi}\phi(\theta)P_{22}(r,\theta)\mathrm{d}\theta.$$

Therefore from (2.10)

$$\sum_{n=1}^{\infty}A_n^{-\varepsilon}s_n r^n=-2\sum_{n=1}^{\infty}n^2\alpha_n r^n$$
$$=\frac{4}{\pi}\int_0^{\pi}\phi(\theta)P_{22}(r,\theta)\mathrm{d}\theta. \qquad (2.18)$$

We choose a $\delta: 0<\delta<\pi$, such that by (2.12)

$$\phi(\theta)=o(\theta^{1+\varepsilon})\ \text{ for }\ |\theta|\leqslant\delta. \qquad (2.19)$$

Using (2.18), we write

$$\sum_{n=1}^{\infty}A_n^{-\varepsilon}s_n r^n=\frac{4}{\pi}\left[\int_0^{1-r}+\int_{1-r}^{\delta}+\int_{\delta}^{\pi}\right]\phi(\theta)P_{22}(r,\theta)\mathrm{d}\theta.$$

Now using (2.19), (2.16) and (2.17), we get

$$\sum_{n=1}^{\infty}A_n^{-\varepsilon}s_n r^n=o((1-r)^{-3})\int_0^{1-r}\theta^{1+\varepsilon}\mathrm{d}\theta+o(1-r)\int_{1-r}^{\delta}\theta^{\varepsilon-3}\mathrm{d}\theta$$

$$+O(1-r)\int_{\delta}^{\pi}\theta^{\varepsilon-3}\mathrm{d}\theta$$
$$=o((1-r)^{\varepsilon-1})+O(1-r)$$
$$=o((1-r)^{\varepsilon-1}) \text{ as } r\to 1-, \tag{2.20}$$

which ensures (2.8) and hence theorem holds in case (I).

(II) In this case the series (2.9) converges uniformly to zero in a neighbourhood of h that includes the origin. We have

$$\frac{2}{\pi}\int_0^{\pi}P(r,\theta)\cos n\theta\mathrm{d}\theta$$
$$=\frac{2}{\pi}\int_0^{\pi}\left(\frac{1}{2}+\sum_{m=1}^{\infty}r^m\cos m\theta\right)\cos n\theta\mathrm{d}\theta$$
$$=\frac{1}{\pi}\int_0^{\pi}(1+\cos 2n\theta)r^n\mathrm{d}\theta=r^n.$$

Thus, integrating by parts twice, we obtain

$$r^n=\frac{2}{\pi}\int_0^{\pi}P(r,\theta)\cos n\theta\mathrm{d}\theta,\quad n\geqslant 1$$
$$=-\frac{2}{\pi}\int_0^{\pi}P_{22}(r,\theta)\frac{1-\cos n\theta}{n^2}\mathrm{d}\theta,$$

from which it follows that

$$\sum_{n=1}^{\infty}A_n^{-\varepsilon}s_nr^n=-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{A_n^{-\varepsilon}s_n}{n^2}\int_0^{\pi}P_{22}(r,\theta)(1-\cos n\theta)\mathrm{d}\theta. \tag{2.21}$$

As $s_n\to s(R_{2+\varepsilon})$ the series $\Sigma_{n=1}^{\infty}A_n^{-\varepsilon}s_n/n^2$ is convergent and hence

$$\frac{1}{2}\alpha_0+\sum_{n=1}^{\infty}\alpha_n\cos n\theta=\frac{1}{2}\sum_{n=1}^{\infty}A_n^{-\varepsilon}\frac{s_n}{n^2}(1-\cos n\theta). \tag{2.22}$$

The last series in (2.22) converges uniformly to zero for small θ, say for $|\theta|\leqslant\theta_0$ and therefore

$$\sum_{n=1}^{\infty}A_n^{-\varepsilon}\frac{s_n}{n^2}\int_0^{\theta_0}(1-\cos n\theta)P_{22}(r,\theta)\mathrm{d}\theta=0. \tag{2.23}$$

For the proof of the theorem, it remains to show that

$$\sum_{n=1}^{\infty}A_n^{-\varepsilon}\frac{s_n}{n^2}\int_{\theta_0}^{\pi}(1-\cos n\theta)P_{22}(r,\theta)\mathrm{d}\theta=o\left(\frac{1}{(1-r)^{1-\varepsilon}}\right) \tag{2.24}$$

for fixed positive θ_0. As

$$\int_{\theta_0}^{\pi}P_{22}(r,\theta)\mathrm{d}\theta=O(1),$$

it is clear that

$$\sum_{n=1}^{\infty}A_n^{-\varepsilon}\frac{s_n}{n^2}\int_{\theta_0}^{\pi}P_{22}(r,\theta)\mathrm{d}\theta=O(1). \tag{2.25}$$

By simple calculation, it can be seen that $P_{22}(r,\theta)$, $P_{222}(r,\theta)$ and $P_{2222}(r,\theta)$ are uniformly bounded in the interval (θ_0,π) and hence integrating by parts twice, we obtain for $n\geqslant 1$

$$\begin{aligned} j_n &= \int_{\theta_0}^{\pi} \cos n\theta P_{22}(r,\theta)\mathrm{d}\theta \\ &= -\frac{P_{22}(r,\theta_0)\sin n\theta_0}{n} - \frac{P_{222}(r,\theta_0)\cos n\theta_0}{n^2} \\ &\quad + \frac{1}{n^2}\int_{\theta_0}^{\pi} P_{2222}(r,\theta)\cos n\theta \mathrm{d}\theta \\ &= -P_{22}(r,\theta_0)\frac{\sin n\theta_0}{n} + O(1/n^2). \end{aligned} \tag{2.26}$$

As $s_n = o(n^2/A_n^{-\varepsilon})$, we have

$$\sum_{n=1}^{\infty} s_n A_n^{-\varepsilon}\frac{j_n}{n^2} = -P_{22}(r,\theta_0)\sum_{n=1}^{\infty}\frac{A_n^{-\varepsilon}s_n\sin n\theta_0}{n^3} + \sum_{n=1}^{\infty} O\left(\frac{1}{n^2}\right).$$

Since $\Sigma_{n=1}^{\infty}$ $A_n^{-\varepsilon}/n^2 s_n\cos n\theta$ converges uniformly for $|\theta|\leqslant\theta_0$, the series $\Sigma_{n=1}^{\infty}(A_n^{-\varepsilon}s_n\sin n\theta_0)/n^3$ is convergent and hence

$$\sum_{n=1}^{\infty}\frac{A_n^{-\varepsilon}}{n^2}s_n j_n = O(1)$$

which together with (2.25) ensures that

$$\sum_{n=1}^{\infty} A_n^{-\varepsilon}\frac{s_n}{n^2}\int_{\theta_0}^{\pi}(1-\cos n\theta)P_{22}(r,\theta)\mathrm{d}\theta = O(1)$$

from which (2.24) follows and this completes the proof of the theorem.

3. Tauberian theorem

In this section we prove the following Tauberian theorem for $(R_{2+\varepsilon})$ summability which corresponds to a theorem on (R_2) summability due to Vijayaraghavan (see [3], p. 306). For this we first recall the definition.

DEFINITION ([3], p. 124)

A functon f, defined for $x>0$, is said to be slowly decreasing if it is real and

$$\liminf\{f(y)-f(x)\}\geqslant 0$$

whenever $x\to\infty$, $y>x$, $y/x\to 1$.

Theorem 3. *Let*

(i) $s_n = O(1)(R_{2+\varepsilon})$, $0<\varepsilon<1$
(ii) s_n *be real and slowly decreasing.*

Then $s_n = O(1)$.

Let a method of summation be defined by

$$J(x) = \sum_{n=0}^{\infty} c_n(x) s_n. \tag{3.1}$$

We suppose that

$$c_n(x) \geqslant 0,\ c_n(x) \to 0\ (x \to \infty),\ \sum_{n=0}^{\infty} c_n(x) = 1 \tag{3.2}$$

so that the method is totally regular. We need the following lemma.

Lemma 5 ([3], p. 306, Theorem 238). *Suppose that the following conditions are satisfied.*

(i) *$\phi(u)$ is positive and differentiable for $u \geqslant 1$*

$$\phi \to \infty, \quad 0 < \phi' < K, \tag{3.3}$$

where K is independent of u,

$$\Phi(u) = \int_1^u \frac{\mathrm{d}t}{\phi(t)} \tag{3.4}$$

(so that $\Phi \to \infty$ with u).
(ii) *The co-efficient $c_n(x)$ have, in addition to those already stated, the properties;*

$$\sum_{n=0}^{M} c_n(x) \to 0, \tag{3.5}$$

if $M \to \infty,\ x \to \infty,\ \Phi(x) - \Phi(M) \to \infty$;

$$\sum_{n=N}^{\infty} c_n(x) \to 0, \tag{3.6}$$

$$\sum_{n=N}^{\infty} c_n(x)(\Phi(n) - \Phi(N)) \to 0, \tag{3.7}$$

if $N \to \infty,\ x \to \infty,\ \Phi(N) - \Phi(x) \to \infty$.

(iii) *If $s(t) = s_n$ for $n \leqslant t < n + 1$, then*

$$\liminf(s(t) - s(u)) \geqslant 0 \tag{3.8}$$

when $u \to \infty,\ t > u,\ (t - u)/\phi(u) \to 0$.

(iv) *$\tau(x) = \Sigma c_n(x)\, s_n$ is bounded.*
Then s_n is bounded.

Proof of Theorem 3. We shall deduce Theorem 3 as a particular case of Lemma 5 by a special choice of $c_n(x)$ and $\phi(u)$.

We take

$$c_n(x) = \frac{x^{\varepsilon - 1}}{D(\varepsilon)} A_n^{-\varepsilon} \left(\frac{\sin(n/2x)}{n/2x} \right)^2$$

$$\phi(u) = u,\ \Phi(u) = \log u.$$

Clearly $c_n(x)$ corresponds to $(R_{2+\varepsilon})$ method and it satisfies condition (3.2) for total regularity. Condition (iv) of Lemma 5 is satisfied as $s_n = O(1)\,(R_{2+\varepsilon})$ by the hypothesis while condition (iii) follows from the fact that s_n is real and slowly decreasing. Now

$$\begin{aligned}\sum_{n=1}^{M} c_n(x) &= \frac{x^{\varepsilon-1}}{D(\varepsilon)} \sum_{n=1}^{M} A_n^{-\varepsilon}\left(\frac{\sin(n/2x)}{n/2x}\right)^2\\ &= O(1)x^{\varepsilon-1}\sum_{n=1}^{M}\frac{1}{n^{\varepsilon}}\\ &= O(1)(M/x)^{1-\varepsilon}\to 0\end{aligned}$$

as $M\to\infty$, $x\to\infty$, $M/x\to 0$ (i.e., $\log x - \log M\to\infty$); and

$$\begin{aligned}\sum_{n=N}^{\infty} c_n(x) &= \frac{x^{\varepsilon-1}}{D(\varepsilon)} \sum_{n=N}^{\infty} A_n^{-\varepsilon}\left(\frac{\sin(n/2x)}{n/2x}\right)^2\\ &= O(1)x^{\varepsilon-1}\sum_{n=N}^{\infty}\frac{1}{n^{\varepsilon}}\frac{4x^2}{n^2}\\ &= O(1)x^{\varepsilon+1}\sum_{n=N}^{\infty}\frac{1}{n^{2+\varepsilon}}\\ &= O(1)(x/N)^{1+\varepsilon}\to 0;\end{aligned}$$

if $N\to\infty$, $x\to\infty$, $N/x\to\infty$ (i.e. $\log N - \log x\to\infty$). Finally

$$\begin{aligned}&\sum_{n=N}^{\infty} c_n(x)(\Phi(n)-\Phi(N))\\ &= \frac{x^{\varepsilon-1}}{D(\varepsilon)}\sum_{n=N}^{\infty} A_n^{-\varepsilon}\left(\frac{\sin(n/2x)}{n/2x}\right)^2\log\frac{n}{N}\\ &= O(1)x^{\varepsilon+1}\sum_{n=N}^{\infty}\frac{1}{n^{2+\varepsilon}}\log\frac{n}{N}\\ &= O(1)x^{\varepsilon+1}\int_N^{\infty}\frac{\log t/N}{t^{2+\varepsilon}}\,\mathrm{d}t\\ &= O(1)(x/N)^{\varepsilon+1}\int_1^{\infty}\frac{\log u}{u^{2+\varepsilon}}\,\mathrm{d}u\\ &= O(1)(x/N)^{\varepsilon+1}\to 0;\end{aligned}$$

if $N\to\infty$, $x\to\infty$, $N/x\to\infty$.

Thus the remaining conditions (i) and (ii) of Lemma 5 are satisfied. This completes the proof of Theorem 3.

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